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The Geometry of the Sasaki Metric on the Sphere Bundles of Euclidean Atiyah Vector Bundles

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Abstract. Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold. It is well known that the Sasaki metric on TM is very rigid, but it has nice properties when restricted to $T^{(r)}M = \{u \in TM, |u| = r\}$. In this paper, we consider a general situation where we replace TM by a vector bundle $E \longrightarrow M$ endowed with a Euclidean product $\langle\ ,\ \rangle_E$ and a connection ∇^E which preserves $\langle\ ,\ \rangle_E$. We define the Sasaki metric on E and we consider its restriction h to $E^{(r)} = \{a \in E, \langle a, a\rangle_E = r^2\}$. We study the Riemannian geometry of $(E^{(r)}, h)$ generalizing many results first obtained on $T^{(r)}M$ and establishing new ones. We apply the results obtained in this general setting to the class of Euclidean Atiyah vector bundles introduced by the authors in Boucetta and Essoufi J Geom Phys 140:161–177, 2019). Finally, we prove that any unimodular three dimensional Lie group G carries a left invariant Riemannian metric, such that $(T^{(1)}G, h)$ has a positive scalar curvature.

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1. Introduction

Through this paper, a Euclidean vector bundle is a vector bundle $\pi_E : E \longrightarrow M$ endowed with $\langle\ ,\ \rangle_E \in \Gamma(E^* \otimes E^*)$ which is bilinear symmetric and positive definite in the restriction to each fiber.

Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold of dimension n, $\pi_E : E \longrightarrow M$ a vector bundle of rank m endowed with a Euclidean product $\langle\ ,\ \rangle_E$, and a linear connection ∇^E which preserves $\langle\ ,\ \rangle_E$. Denote by $K : TE \longrightarrow E$ the connection map of ∇^E locally given by:

$$K\left(\sum_{i=1}^{n} b_i \partial_{x_i} + \sum_{j=1}^{m} Z_j \partial_{\mu_j}\right) = \sum_{l=1}^{m}\left(Z_l + \sum_{i=1}^{n}\sum_{j=1}^{m} b_i \mu_j \Gamma_{ij}^{l}\right) s_l,$$

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where $(x_1,\ldots,x_n)$ is a system of local coordinates, $(s_1,\ldots,s_m)$ is a basis of local sections of E, (x_i,μ_j) the associated system of coordinates on E, and $\nabla^E_{\partial_{x_i}} s_j = \sum_{l=1}^m \Gamma_{ij}^l s_l$. Then:

$$TE = \ker d\pi_E \oplus \ker K.$$

The Sasaki metric g_s on E is the Riemannian metric given by:

$$g_s(A,B) = \langle d\pi_E(A), d\pi_E(B)\rangle_{TM} + \langle K(A), K(B)\rangle_E, \quad A,B\in T_aE.$$

For any $r>0$, the sphere bundle of radius r is the hypersurface $E^{(r)} = \left\{a\in E, \langle a,a\rangle_E = r^2\right\}$.

They are two classes of such Euclidean vector bundles naturally associated with a Riemannian manifold.

We refer to the first one as the classical case. It is the case where $E = TM$, $\langle\ ,\ \rangle_E = \langle\ ,\ \rangle_{TM}$, and ∇^E is the Levi–Civita connection of $(M,\langle\ ,\ \rangle_{TM})$.

The second case will be called the *Atiyah Euclidean vector bundle* associated with a Riemannian manifold. It has been introduced by the authors in Ref. [4]. It is defined as follows.

Let $(M,\langle\ ,\ \rangle_{TM})$ be a Riemannian manifold, $\mathrm{so}(TM) = \bigcup_{x\in M}\mathrm{so}(T_xM)$ where $\mathrm{so}(T_xM)$ is the vector space of skew-symmetric endomorphisms of T_xM and $k>0$. The Levi–Civita connection ∇^M of $(M,\langle\ ,\ \rangle_{TM})$ defines a connection on the vector bundle $\mathrm{so}(TM)$ which we will denote in the same way and it is given, for any $X\in\Gamma(TM)$ and $F\in\Gamma(\mathrm{so}(TM))$, by:

$$\nabla^M_X F(Y) = \nabla^M_X(F(Y)) - F(\nabla^M_X Y).$$

The *Atiyah Euclidean vector bundle* associated with $(M,\langle\ ,\ \rangle_{TM},k)$ is the triple $(E(M,k),\langle\ ,\ \rangle_k,\nabla^E)$ where $E(M,k) = TM\oplus\mathrm{so}(TM)\longrightarrow M$, $\langle\ ,\ \rangle_k$, and ∇^E are a Euclidean product and a connection on $E(M,k)$ given, for any $X,Y\in\Gamma(TM)$ and $F,G\in\Gamma(\mathrm{so}(TM))$, by:

$$\begin{aligned}\nabla^E_X Y = \nabla^M_X Y + H_X Y,\ \nabla^E_X F = H_X F + \nabla^M_X F,\\ \langle X+F, Y+G\rangle_k = \langle X,Y\rangle_{TM} - k\,\mathrm{tr}(F\circ G),\end{aligned}$$

where R^M is the curvature tensor of ∇^M given by: $R^M(X,Y) = \nabla^M_{[X,Y]} - \left(\nabla^M_X\nabla^M_Y - \nabla^M_Y\nabla^M_X\right)$,

$$H_X Y = -\frac{1}{2}R^M(X,Y) \quad\text{and}\quad \langle H_X F, Y\rangle_{TM} = -\frac{1}{2}k\,\mathrm{tr}(F\circ R^M(X,Y)). \tag{1}$$

The connection ∇^E preserves $\langle\ ,\ \rangle_k$ and its curvature R^{∇^E} plays a key role in the study of $(E^{(r)}(M,k)$ endowed with the Sasaki metric. Since R^{∇^E} depends only on $(M,\langle\ ,\ \rangle_{TM},k)$, we will call it the *supracurvature* of $(M,\langle\ ,\ \rangle_{TM},k)$. The origin of Atiyah Euclidean vector bundle and the justification of its name can be found in [4].

This paper has two goals:

1. The study of the Riemannian geometry of $E^{(r)}$ endowed with the Riemannian metric h restriction of g_s to generalize all the results obtained in the classical case. We refer to [3,6] for a survey on the geometry of $(T^{(r)}M,h)$. Our study has some similarities with [2].

2. The application of the results obtained in the general case to the Euclidean Atiyah vector bundle $E^{(r)}(M,k)$ endowed with the Sasaki metric. We will show that the geometry of $(E^{(r)}(M,k),h)$ is so rich, and by doing so, we open new horizons for further explorations.

Let us give now the organization of this paper. In Sect. 2, we give the different curvatures of $(E^{(r)},h)$. In Sect. 3, we derive sufficient conditions for which $(E^{(r)},h)$ has either non-negative sectional curvature, positive Ricci curvature, positive or constant scalar curvature. In Sect. 4, we first compute the supracurvature of different classes of Riemannian manifolds and we characterize those with vanishing supracurvature (see Theorem 4.1). Then, we perform a detailed study of $(E^{(r)}(M,k),h)$ having in mind the results obtained in Sect. 3. In Sect. 5, we prove that any unimodular three-dimensional Lie group G carries a left invariant Riemannian metric, such that $(T^{(1)}G,h)$ has a positive scalar curvature.

2. Sectional Curvature, Ricci Curvature, and Scalar Curvature of the Sasaki Metric on Sphere Bundles

Through this section, $(M,\langle\ ,\ \rangle_{TM})$ is a n-dimensional Riemannian manifold and $\pi_E : E \longrightarrow M$ a vector bundle of rank m endowed with a Euclidean product $\langle\ ,\ \rangle_E$ and a linear connection ∇^E for which $\langle\ ,\ \rangle_E$ is parallel. We shall denote by ∇^M the Levi–Civita connection of $(M,\langle\ ,\ \rangle_{TM})$, and by R^M and R^{∇^E} the curvature tensors of ∇^M and ∇^E, respectively. We use the convention:

$$R^M(X,Y) = \nabla^M_{[X,Y]} - \left(\nabla^M_X\nabla^M_Y - \nabla^M_Y\nabla^M_X\right) \quad \text{and}$$
$$R^{\nabla^E}(X,Y) = \nabla^E_{[X,Y]} - \left(\nabla^E_X\nabla^E_Y - \nabla^E_Y\nabla^E_X\right).$$

The derivative of R^{∇^E} with respect to ∇^M and ∇^E is the tensor field $\nabla^{M,E}_X(R^{\nabla^E})$ given, for any $X,Y,Z \in \Gamma(TM)$, $\alpha \in \Gamma(E)$, by:

$$\nabla^{M,E}_X(R^{\nabla^E})(Y,Z,\alpha) = \nabla^E_X(R^{\nabla^E}(Y,Z)\alpha) - R^{\nabla^E}(\nabla^M_X Y,Z)\alpha - R^{\nabla^E}(Y,\nabla^M_X Z)\alpha$$
$$-R^{\nabla^E}(Y,Z)\nabla^E_X\alpha. \tag{2}$$

Let K^M, ric^M, and s^M denote the sectional curvature, the Ricci curvature, and the scalar curvature of $(M,\langle\ ,\ \rangle_{TM})$, respectively.

We recall the definition of the Sasaki metric g_S on E, we consider its restriction h to the sphere bundles $E^{(r)} = \left\{a \in E, \langle a,a\rangle_E = r^2\right\}$ $(r>0)$, and we give the expressions of the different curvatures of $(E^{(r)},h)$.

For any $a \in E$, there exists an injective linear map $h^a : T_xM \longrightarrow T_aE$ given in a coordinates system (x_i,β_j) on E associated with a coordinate system $(x_i)_{i=1}^n$ on M and a local trivialization $(s_1,\ldots,s_m)$ of E by:

$$h^a(u) = \sum_{i=1}^n u_i\partial_{x_i} - \sum_{k=1}^m\left(\sum_{i=1}^n\sum_{j=1}^m u_i\beta_j\Gamma^k_{ij}\right)\partial_{\beta_k},$$

where

$$u=\sum_{i=1}^{n}u_i\partial_{x_i},\ \nabla^E_{\partial_{x_i}}s_j=\sum_{k=1}^{m}\Gamma_{ij}^k s_k \quad \text{and} \quad a=\sum_{i=1}^{m}\beta_i s_i.$$

Moreover, if $\mathcal{H}_a E$ denotes the image of h^a, then:

$$TE=\mathcal{V}E\oplus\mathcal{H}E,$$

where $\mathcal{V}E=\ker d\pi_E$. For any $\alpha\in\Gamma(E)$ and for any $X\in\Gamma(TM)$, we denote by $\alpha^v\in\Gamma(TE)$ and $X^h\in\Gamma(TE)$ the vertical and horizontal vector field associated with α and X, respectively. The flow of α^v is given by $\Phi^\alpha(t,a)=a+t\alpha(\pi_E(a))$ and X^h is given by $X^h(a)=h^a(X(\pi_E(a)))$.

The Sasaki metric g_s on E is determined by the formulas:

$$g_s(X^h,Y^h)=\langle X,Y\rangle_{TM}\circ\pi_E,\ g_s(\alpha^v,\beta^v)=\langle\alpha,\beta\rangle_E\circ\pi_E \quad \text{and} \quad g_s(X^h,\alpha^v)=0,$$

for all $X,Y\in\Gamma(TM)$ and $\alpha,\beta\in\Gamma(E)$.

For any $X\in\Gamma(TM)$ and $\alpha\in\Gamma(E)$, X^h is tangent to $E^{(r)}$; however, α^v is not tangent to $E^{(r)}$. Therefore, we define the *tangential lift* of α by:

$$\alpha^t(a)=\alpha^v(a)-\langle\alpha,a\rangle_E\frac{U(a)}{r^2},\quad (a)\in E,$$

where U is the vertical vector field on E whose flow is given by $\Phi(t,(a))=e^t a$. We have:

$$T_aE^{(r)}=\left\{X^h+\alpha^t\ /\ X\in T_xM \text{ and } \alpha\in E_x \text{ with } \langle\alpha,a\rangle_E=0\right\}.$$

The restriction h of g_s to $E^{(r)}$ is given by:

$$h(X^h,Y^h)=\langle X,Y\rangle_{TM}\circ\pi_E,\quad h(X^h,\alpha^t)=0,$$
$$h(\alpha^t,\beta^t)(a)=\langle\alpha,\beta\rangle_E-\frac{\langle\alpha,a\rangle_E\langle\beta,a\rangle_E}{r^2}=\langle\bar\alpha,\bar\beta\rangle_E,$$

where $\alpha,\beta\in\Gamma(E),\ X,Y\in\Gamma(TM)$ and $\bar\alpha=\alpha-\frac{\langle\alpha,a\rangle_E}{r^2}a$.

The following proposition can be established in the same way as the classical case where $E=TM$, $\langle\ ,\ \rangle_E=\langle\ ,\ \rangle_{TM}$, and $\nabla^E=\nabla^M$.

Proposition 2.1. *We have:*

$$[\alpha^t,\beta^t]=\frac{\langle\alpha,a\rangle_E}{r^2}\beta^t-\frac{\langle\beta,a\rangle_E}{r^2}\alpha^t,\ [X^h,\alpha^t]=\left(\nabla^E_X\alpha\right)^t \quad \textit{and}$$
$$[X^h,Y^h](a)=[X,Y]^h(a)+(R^{\nabla^E}(X,Y)a)^t,$$

where R^{∇^E} is the curvature of ∇^E given by $R^{\nabla^E}(X,Y)=\nabla^E_{[X,Y]}-\left(\nabla^E_X\nabla^E_Y-\nabla^E_Y\nabla^E_X\right)$.

To compute the Riemannian invariants of $(E^{(r)},h)$ (Levi–Civita connection and the different curvatures), we will use the following facts:

(i) The projection $\pi_E:(E^{(r)},h)\longrightarrow(M,\langle\ ,\ \rangle_{TM})$ is a Riemannian submersion with totally geodesic fibers and, hence, the different Riemannian invariants can be computed using O'Neill formulas (see [1, chap. 9]).

Here, the O'Neill shape tensor, say B, is given by the expression of $[X^h, Y^h]$. Therefore, by virtue of Proposition 2.1, we get:

$$B_{X^h}Y^h((a)) = \frac{1}{2}\mathcal{V}[X^h, Y^h](a) = \frac{1}{2}(R^{\nabla^E}(X,Y)a)^v$$
$$= \frac{1}{2}(R^{\nabla^E}(X,Y)a)^t, \tag{3}$$

$B_{\alpha^t} = 0$ and $h(B_{X^h}\alpha^t, Y^h) = -h(B_{X^h}Y^h, \alpha^t)$ for any $\alpha \in \Gamma(E)$, $X, Y \in \Gamma(TM)$, and $(a) \in E^{(r)}$.

(ii) O'Neill's formulas involve the Riemannian invariants of $(M, \langle\ ,\ \rangle_{TM})$, the tensor B and the Riemannian invariants of the restriction of h to the fibers.

Based on these facts, the Levi–Civita connection $\overline{\nabla}$ of $(E^{(r)}, h)$ is given by:

$$\overline{\nabla}_{X^h}Y^h(a) = (\nabla^M_X Y)^h(a) + \frac{1}{2}(R^{\nabla^E}(X,Y)a)^t,$$
$$\overline{\nabla}_{X^h}\alpha^t = B_{X^h}\alpha^t + (\nabla^E_X\alpha)^t,\ \overline{\nabla}_{\alpha^t}X^h = B_{X^h}\alpha^t,$$
$$(\overline{\nabla}_{\alpha^t}\beta^t)(a) = -\frac{\langle\beta, a\rangle}{r^2}\alpha^t \quad \text{and} \quad h(B_{X^h}\alpha^t, Y^h) = -h(B_{X^h}Y^h, \alpha^t), \tag{4}$$

$X, Y \in \Gamma(TM)$, $\alpha, \beta \in \Gamma(E)$, and $a \in E^{(r)}$. Note that, if $(X_i)_{i=1}^n$ is a local orthonormal frame of TM, $X \in \Gamma(TM)$, and $\alpha \in \Gamma(E)$, then:

$$B_{X^h}\alpha^t = \frac{1}{2}\sum_{i=1}^n \langle R^{\nabla^E}(X, X_i)\alpha, a\rangle_E X_i^h. \tag{5}$$

Remark 1. When $E = TM$, $\langle\ ,\ \rangle_E = \langle\ ,\ \rangle_{TM}$ and $\nabla^E = \nabla^M$, we have a simple expression of $B_{X^h}\alpha^t$ thanks to the symmetries of $R^{\nabla^E} = R^M$, namely:

$$(B_{X^h}Y^t)(a) = \frac{1}{2}R^M(Y(\pi_E(a)), a)X(\pi_E(a)), \qquad X, Y \in \Gamma(TM). \tag{6}$$

The fibers are totally geodesic submanifolds, and a direct computation shows that the curvature, the Ricci curvature, and the scalar curvature of the restriction of the metric to the fibers are given by:

$$R^v(\alpha^t, \beta^t)\gamma^t = \frac{1}{r^2}\left(h(\alpha^t, \gamma^t)\beta^t - h(\beta^t, \gamma^t)\alpha^t\right),$$
$$\mathrm{ric}^v(\alpha^t, \beta^t) = \frac{1}{r^2}(m-2)h(\alpha^t, \beta^t) \quad \text{and} \quad s^v = \frac{1}{r^2}(m-1)(m-2).$$

To compute the different curvatures of $(E^{(r)}, h)$, we need the following formulas.

Proposition 2.2. *For any $X, Y, Z \in \Gamma(TM)$, $\alpha, \beta \in \Gamma(E)$, and $a \in E$, we have:*

$$h((\overline{\nabla}_{X^h}B)_{Y^h}Z^h, \alpha^t)(a) = -\frac{1}{2}\langle\nabla^{M,E}_X(R^{\nabla^E})(Y, Z, \alpha), a\rangle_E.$$

Moreover, if $\langle \alpha(x), a\rangle_E = \langle \beta(x), a\rangle_E = 0$*, then:*

$$h((\overline{\nabla}_{\alpha^t} B)_{X^h} Y^h, \beta^t)(a) = \frac{1}{2}\langle R^{\nabla^E}(X,Y)\alpha, \beta\rangle_E(x) + h(B_{Y^h}\alpha^t, B_{X^h}\beta^t)(a) - h(B_{X^h}\alpha^t, B_{Y^h}\beta^t)(a).$$

Proof. Suppose first that $\langle \alpha(x), a\rangle_E = \langle \beta(x), a\rangle_E = 0$. We have:

$$\begin{aligned} h((\overline{\nabla}_{\alpha^t} B)_{X^h} Y^h, \beta^t) &= h(\overline{\nabla}_{\alpha^t}(B_{X^h}Y^h), \beta^t) - h(B_{\overline{\nabla}_{\alpha^t}X^h}Y^h, \beta^t) \\ &\quad - h(B_{X^h}\overline{\nabla}_{\alpha^t}Y^h, \beta^t) \\ &= \alpha^t.h(B_{X^h}Y^h, \beta^t) - h(B_{X^h}Y^h, \overline{\nabla}_{\alpha^t}\beta^t) \\ &\quad + h(B_{Y^h}\overline{\nabla}_{\alpha^t}X^h, \beta^t) + h(\overline{\nabla}_{\alpha^t}Y^h, B_{X^h}\beta^t) \\ &= \alpha^t.h(B_{X^h}Y^h, \beta^t) - h(B_{X^h}Y^h, \overline{\nabla}_{\alpha^t}\beta^t) \\ &\quad + h(B_{Y^h}\alpha^t, B_{X^h}\beta^t) - h(B_{X^h}\alpha^t, B_{Y^h}\beta^t). \end{aligned}$$

From (4) and the definition of α^t, we get:

$$\overline{\nabla}_{\alpha^t}\beta^t(a) = 0 \quad \text{and} \quad (\alpha^t.h(B_{X^h}Y^h, \beta^t))(a) = (\alpha^v.h(B_{X^h}Y^h, \beta^t))(a).$$

However:

$$\begin{aligned} \alpha^v.h(B_{X^h}Y^h, \beta^t)(a) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} h(B_{X^h}Y^h(a+t\alpha), \beta^t(a+t\alpha)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Big[h(B_{X^h}Y^h(a+t\alpha), \beta^v(a+t\alpha)) \\ &\quad - \frac{1}{r^2}\langle \beta, a+t\alpha\rangle_E h(B_{X^h}Y^h(a+t\alpha), U(a+t\alpha))\Big] \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\langle R^{\nabla^E}(X,Y)(a+t\alpha), \beta\rangle_E(x) \\ &= \frac{1}{2}\langle R^{\nabla^E}(X,Y)\alpha, \beta\rangle_E(x), \end{aligned}$$

which complete to establish the second formula.

On the other hand:

$$\begin{aligned} h((\overline{\nabla}_{X^h} B)_{Y^h} Z^h, \alpha^t)(a) &= h(\overline{\nabla}_{X^h}(B_{Y^h}Z^h), \alpha^t)(a) \\ &\quad - h(B_{\overline{\nabla}_{X^h}Y^h}Z^h, \alpha^t)(a) - h(B_{Y^h}\overline{\nabla}_{X^h}Z^h, \alpha^t)(a) \\ &= X^h.h(B_{Y^h}Z^h, \alpha^t)(a) - \frac{1}{2}\langle R^{\nabla^E}(Y,Z)a, \nabla^E_X\alpha\rangle_E \\ &\quad - \frac{1}{2}\langle R^{\nabla^E}(\nabla^M_X Y, Z)a, \alpha\rangle_E \\ &\quad - \frac{1}{2}\langle R^{\nabla^E}(Y, \nabla^M_X Z)a, \alpha\rangle_E \\ &= \frac{1}{2}\langle R^{\nabla^E}(Y,Z)\nabla^E_X\alpha + R^{\nabla^E}(\nabla^M_X Y, Z)\alpha \\ &\quad + R^{\nabla^E}(Y, \nabla^M_X Z)\alpha, a\rangle_E + X^h.h(B_{Y^h}Z^h, \alpha^t)(a). \end{aligned}$$

The key point is that if $\phi^X_t(x)$ is the integral curve of X passing through x, then the integral curve of X^h at a is the ∇^E-parallel section a^t along $\phi^X_t(x)$

with $a^0 = a$. Therefore:

$$\begin{aligned} X^h.h(B_{Y^h}Z^h, \alpha^t)(a) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} h(B_{Y^h}Z^h, \alpha^t)(a^t) \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\langle R^{\nabla^E}(Y(\phi_t^X(x)), Z(\phi_t^X(x)))\alpha(\phi_t^X(x)), a^t\rangle_E \\ &= -\frac{1}{2}\langle \nabla_X^E(R^{\nabla^E}(Y,Z)\alpha)(x), a\rangle_E. \end{aligned}$$

This completes the proof.

□

Proposition 2.3. *Let $P \subset T_aE^{(r)}$ be a plane. Then:*

1. *If* $\operatorname{rank}(E) = 2$, *then there exists a basis* $\{X^h + \alpha^t, Y^h\}$ *of* P *satisfying:*

$$\alpha \in E_x,\ X, Y \in T_xM,\ |X|^2 + |\alpha|^2 = |Y|^2 = 1,\ \langle X, Y\rangle_{TM} = 0 \quad \textit{and} \quad \langle \alpha, a\rangle_E = 0.$$

The sectional curvature of $(E^{(r)}, h)$ at P is given by:

$$\begin{aligned} K(P) = \langle R^M(X,Y)X,Y\rangle_{TM} - \frac{3}{4}|R^{\nabla^E}(X,Y)a|^2 + \frac{1}{4}\sum_{i=1}^n \langle R^{\nabla^E}(Y,X_i)\alpha, a\rangle_E^2 \\ + \langle \nabla_Y^{M,E}(R^{\nabla^E})(X,Y,\alpha), a\rangle_E. \end{aligned}$$

2. *If* $\operatorname{rank}(E) \geq 3$, *then there exists a basis* $\{X^h + \alpha^t, Y^h + \beta^t\}$ *of* P *satisfying:*

$$\begin{aligned} &\alpha, \beta \in E_x,\ X, Y \in T_xM,\ |X|^2 + |\alpha|^2 = |Y|^2 + |\beta|^2 = 1, \\ &\langle X, Y\rangle_{TM} = \langle \alpha, \beta\rangle_E = 0 \quad \textit{and} \quad \langle \alpha, a\rangle_E = \langle \beta, a\rangle_E = 0. \end{aligned}$$

The sectional curvature of $(E^{(r)}, h)$ at P is given by:

$$\begin{aligned} K(P) = \langle R^M(X,Y)X,Y\rangle_{TM} &+ \frac{1}{r^2}|\alpha|^2|\beta|^2 + 3\langle R^{\nabla^E}(X,Y)\alpha, \beta\rangle_E \\ &- \frac{3}{4}\langle R^{\nabla^E}(X,Y)a, R^{\nabla^E}(X,Y)a\rangle_E \\ &+ \frac{1}{4}\sum_{i=1}^n \left(\langle R^{\nabla^E}(X,X_i)\beta, a\rangle_E + \langle R^{\nabla^E}(Y,X_i)\alpha, a\rangle_E\right)^2 \\ &- \sum_{i=1}^n \langle R^{\nabla^E}(X,X_i)\alpha, a\rangle_E \langle R^{\nabla^E}(Y,X_i)\beta, a\rangle_E \\ &+ \langle \nabla_Y^{M,E}(R^{\nabla^E})(X,Y,\alpha) - \nabla_X^{M,E}(R^{\nabla^E})(X,Y,\beta), a\rangle_E, \end{aligned}$$

where $(X_i)_{i=1}^n$ is any orthonormal basis of T_xM.

Proof. If the rank of E is equal to 2, then $\dim T_aE^{(r)} = n+1$ and $P \cap \{X^h, X \in T_xM\} \neq 0$ and, hence, P contains a unit vector Y^h. We take a unit vector $X^h + \alpha^t$ orthogonal to Y^h to get a basis $(X^h + \alpha^t, Y^h)$ of P.

If $\operatorname{rank}(E) > 2$, we take an orthonormal basis $(X^h + \alpha^t, Y^h + \beta^t)$ of P, that is:

$$|X|^2 + |\alpha|^2 = |Y|^2 + |\beta|^2 = 1, \quad \langle X, Y\rangle_{TM} + \langle \alpha, \beta\rangle_E = 0 \quad \text{and} \quad \langle \alpha, a\rangle_E = \langle \beta, a\rangle_E = 0.$$

We write $(\frac{1}{2}(|X|^2 - |Y|^2), \langle X, Y\rangle_{TM}) = \rho(\cos(\mu), \sin(\mu))$ with $\mu \in [0, \frac{\pi}{2})$ and $\rho > 0$. Then, the vectors

$$U = \cos\left(\frac{\mu}{2}\right)(X^h + \alpha^t) + \sin\left(\frac{\mu}{2}\right)(Y^h + \beta^t) \quad \text{and}$$
$$V = -\sin\left(\frac{\mu}{2}\right)(X^h + \alpha^t) + \cos\left(\frac{\mu}{2}\right)(Y^h + \beta^t)$$

constitute a basis of P satisfying the desired relations.

Let us compute the sectional curvature at P. We denote by R the curvature tensor of $(E^{(r)}, h)$:

$$\begin{aligned}
K(P) &= h(R(X^h + \alpha^t, Y^h + \beta^t)(X^h + \alpha^t), Y^h + \beta^t)\\
&= h(R(X^h + \alpha^t, Y^h + \beta^t)X^h, Y^h) + h(R(X^h + \alpha^t, Y^h + \beta^t)X^h, \beta^t)\\
&\quad + h(R(X^h + \alpha^t, Y^h + \beta^t)\alpha^t, Y^h)\\
&\quad + h(R(X^h + \alpha^t, Y^h + \beta^t)\alpha^t, \beta^t)\\
&= h(R(X^h, Y^h)X^h, Y^h) + h(R(X^h, \beta^t)X^h, Y^h) + h(R(\alpha^t, Y^h)X^h, Y^h)\\
&\quad + h(R(\alpha^t, \beta^t)X^h, Y^h)\\
&\quad + h(R(X^h, Y^h)X^h, \beta^t) + h(R(X^h, \beta^t)X^h, \beta^t) + h(R(\alpha^t, Y^h)X^h, \beta^t)\\
&\quad + h(R(\alpha^t, \beta^t)X^h, \beta^t)\\
&\quad + h(R(X^h, Y^h)\alpha^t, Y^h) + h(R(X^h, \beta^t)\alpha^t, Y^h) + h(R(\alpha^t, Y^h)\alpha^t, Y^h)\\
&\quad + h(R(\alpha^t, \beta^t)\alpha^t, Y^h)\\
&\quad + h(R(X^h, Y^h)\alpha^t, \beta^t) + h(R(X^h, \beta^t)\alpha^t, \beta^t) + h(R(\alpha^t, Y^h)\alpha^t, \beta^t)\\
&\quad + h(R(\alpha^t, \beta^t)\alpha^t, \beta^t)\\
&= h(R(X^h, Y^h)X^h, Y^h) + 2h(R(X^h, Y^h)X^h, \beta^t) + 2h(R(X^h, Y^h)\alpha^t, Y^h)\\
&\quad + 2h(R(X^h, Y^h)\alpha^t, \beta^t)\\
&\quad + h(R(X^h, \beta^t)X^h, \beta^t) + 2h(R(\alpha^t, Y^h)X^h, \beta^t) + 2h(R(\alpha^t, \beta^t)X^h, \beta^t)\\
&\quad + h(R(\alpha^t, Y^h)\alpha^t, Y^h) + 2h(R(\alpha^t, \beta^t)\alpha^t, Y^h) + h(R(\alpha^t, \beta^t)\alpha^t, \beta^t).
\end{aligned}$$

Recall that the projection $\pi_E : (E^{(r)}, h) \longrightarrow (M, \langle\ ,\ \rangle_{TM})$ is a Riemannian submersion with totally geodesic fibers and O'Neill shape tensor B is given by (3). Therefore: we can use O'Neill's formulas for curvature given in [1, chap. 9 p. 241]. From these formulas, we have $h(R(\alpha^t, \beta^t)X^h, \beta^t) = h(R(\alpha^t, \beta^t)\alpha^t, Y^h) = 0$ and hence:

$$\begin{aligned}
K(P) = {} & h(R(X^h, Y^h)X^h, Y^h) + h(R(X^h, \beta^t)X^h, \beta^t)\\
& + h(R(\alpha^t, Y^h)\alpha^t, Y^h) + h(R(\alpha^t, \beta^t)\alpha^t, \beta^t)\\
& + 2h(R(X^h, Y^h)X^h, \beta^t) + 2h(R(X^h, Y^h)\alpha^t, Y^h)\\
& + 2h(R(X^h, Y^h)\alpha^t, \beta^t) + 2h(R(\alpha^t, Y^h)X^h, \beta^t).
\end{aligned}$$

Let us give every term in this expression using O'Neill's formulas and Proposition (2.2):

$$\begin{aligned}
h(R(X^h, Y^h)X^h, Y^h) &= \langle R^M(X, Y)X, Y\rangle_{TM}\\
&\quad - \frac{3}{4}\langle R^{\nabla^E}(X, Y)a, R^{\nabla^E}(X, Y)a\rangle_E,\\
h(R(X^h, \beta^t)X^h, \beta^t) &= h((\overline{\nabla}_{\beta^t}B)_{X^h}X^h, \beta^t)\\
&\quad + h(B_{X^h}\beta^t, B_{X^h}\beta^t) = h(B_{X^h}\beta^t, B_{X^h}\beta^t),
\end{aligned}$$

$$\begin{aligned}
h(R(\alpha^t,Y^h)\alpha^t,Y^h) &= h((\overline{\nabla}_{\alpha^t}B)_{Y^h}Y^h,\alpha^t)\\
&\quad +h(B_{Y^h}\alpha^t,B_{Y^h}\alpha^t) = h(B_{Y^h}\alpha^t,B_{Y^h}\alpha^t),\\
h(R(\alpha^t,\beta^t)\alpha^t,\beta^t) &= \frac{1}{r^2}|\alpha|^2|\beta|^2,\\
2h(R(X^h,Y^h)X^h,\beta^t) &= 2h((\overline{\nabla}_{X^h}B)_{X^h}Y^h),\beta^t)\\
&= -\langle \nabla^{M,E}_X(R^{\nabla^E})(X,Y,\beta),a\rangle_E,\\
2h(R(X^h,Y^h)\alpha^t,Y^h) &= -2h((\overline{\nabla}_{Y^h}B)_{X^h}Y^h),\alpha^t)\\
&= \langle \nabla^{M,E}_Y(R^{\nabla^E})(X,Y,\alpha),a\rangle_E,\\
2h(R(X^h,Y^h)\alpha^t,\beta^t) &= 2h((\overline{\nabla}_{\alpha^t}B)_{X^h}Y^h,\beta^t)\\
&\quad -2h((\overline{\nabla}_{\beta^t}B)_{X^h}Y^h,\alpha^t)+2h(B_{X^h}\alpha^t,B_{Y^h}\beta^t)\\
&\quad -2h(B_{X^h}\beta^t,B_{Y^h}\alpha^t)\\
&= 2\langle R^{\nabla^E}(X,Y)\alpha,\beta\rangle_E - 2h(B_{X^h}\alpha^t,B_{Y^h}\beta^t)\\
&\quad +2h(B_{X^h}\beta^t,B_{Y^h}\alpha^t),\\
2h(R(\alpha^t,Y^h)X^h,\beta^t) &= -2h(R(X^h,\beta^t)Y^h,\alpha^t)\\
&= -2h((\overline{\nabla}_{\beta^t}B)_{X^h}Y^h,\alpha^t) - 2h(B_{Y^h}\beta^t,B_{X^h}\alpha^t)\\
&= \langle R^{\nabla^E}(X,Y)\alpha,\beta\rangle_E - 2h(B_{X^h}\alpha^t,B_{Y^h}\beta^t).
\end{aligned}$$

To complete the proof, we need to compute the quantity:

$$\begin{aligned}
Q = h(B_{X^h}\beta^t,B_{X^h}\beta^t) + h(B_{Y^h}\alpha^t,B_{Y^h}\alpha^t)\\
-4h(B_{X^h}\alpha^t,B_{Y^h}\beta^t) + 2h(B_{Y^h}\alpha^t,B_{X^h}\beta^t).
\end{aligned}$$

When $E=TM$, $\langle\ ,\ \rangle_E = \langle\ ,\ \rangle_{TM}$, and $\nabla^E=\nabla^M$, one can use the formula (6) to recover the expression of the sectional curvature given in [7]. In the general case, we use instead (5) and we get:

$$\begin{aligned}
Q &= \frac{1}{4}\sum_{i=1}^n \langle R^{\nabla^E}(X,X_i)\beta,a\rangle_E^2 + \frac{1}{4}\sum_{i=1}^n \langle R^{\nabla^E}(Y,X_i)\alpha,a\rangle_E^2\\
&\quad -\sum_{i=1}^n \langle R^{\nabla^E}(X,X_i)\alpha,a\rangle_E\langle R^{\nabla^E}(Y,X_i)\beta,a\rangle_E\\
&\quad +\frac{1}{2}\sum_{i=1}^n \langle R^{\nabla^E}(Y,X_i)\alpha,a\rangle_E\langle R^{\nabla^E}(X,X_i)\beta,a\rangle_E.
\end{aligned}$$

This completes the proof. □

Example 1. Let $M=S^2$ with its canonical metric $\langle\ ,\ \rangle_{TM}$, $E=TM$, and $\nabla^E=\nabla^M$. Let us compute the sectional curvature of $(T^{(1)}M,h)$. According to Proposition 2.3, if P is a plane in $T_{(x,u)}T^{(1)}M$, then $P=\mathrm{span}\{X^h+Z^t,Y^h\}$ with $X,Y,Z\in T_xM$, $|X|^2+|Z|^2=|Y|^2=1$ and $\langle Z,u\rangle_{TM}=0$. The curvature R^M is given by $R^M(X,Y)Z=\langle X,Z\rangle_{TM}Y-\langle Y,Z\rangle_{TM}X$. Hence:

$$\begin{aligned}
K(P) &= \langle R^M(X,Y)X,Y\rangle_{TM} - \frac{3}{4}|R^M(X,Y)u|^2 + \frac{1}{4}|R^M(Z,u)Y|^2\\
&= |X|^2 - \frac{3}{4}\left(\langle X,u\rangle_{TM}^2 + \langle Y,u\rangle_{TM}^2|X|^2\right) + \frac{1}{4}\left(\langle Z,Y\rangle_{TM}^2 + \langle u,Y\rangle_{TM}^2|Z|^2\right).
\end{aligned}$$

If $Z=0$ then $K(P)=\frac{1}{4}$. If $Z\neq 0$, then $\{Z,u\}$ becomes an orthogonal basis of T_xM and:

$$1=|Y|^2=\langle Y,u\rangle_{TM}^2+\frac{1}{|Z|^2}\langle Y,Z\rangle_{TM}^2.$$

Thus:

$$K(P)=|X|^2+\frac{1}{4}|Z|^2-\frac{3}{4}\left(\langle X,u\rangle_{TM}^2+\langle Y,u\rangle_{TM}^2|X|^2\right).$$

If $X=0$, then $K(P)=\frac{1}{4}$. If $X\neq 0$, then $\{X,Y\}$ is an orthogonal basis and hence:

$$1=|u|^2=\langle Y,u\rangle_{TM}^2+\frac{1}{|X|^2}\langle X,u\rangle_{TM}^2,$$

and hence, $K(P)=\frac{1}{4}$. Therefore, $(T^{(1)}M,h)$ has constant sectional curvature $\frac{1}{4}$. Actually, from the fact that $(T^{(1)}M,h)$ has constant sectional curvature and from the long exact homotopy sequence associated with the S^1-fibration $T^{(1)}M\longrightarrow S^2$, one can see easily that $T^{(1)}M$ is simply connected and, hence, it is diffeomorphic to the sphere S^3. This has been proved first in [11].

Proposition 2.4. *Let $a\in E^{(r)}$, $X,Y\in T_xM$, $\alpha,\beta\in E_x$, and $(X_i)_{i=1}^n$ be any orthonormal basis of T_xM. Then:*

1. *The Ricci curvature of $(E^{(r)},h)$ is given by:*

$$\begin{aligned}\mathrm{ric}(X^h+\alpha^t,Y^h+\beta^t)=&\frac{(m-2)}{r^2}\langle\overline{\alpha},\overline{\beta}\rangle_E+\mathrm{ric}^M(X,Y)\\&-\frac{1}{2}\sum_{i=1}^n\langle R^{\nabla^E}(X,X_i)a,R^{\nabla^E}(Y,X_i)a\rangle_E\\&-\frac{1}{2}\sum_{i=1}^n\langle\nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i,X,\beta)\\&+\nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i,Y,\alpha),a\rangle_E\\&+\frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n\langle R^{\nabla^E}(X_i,X_j)a,\alpha\rangle_E\\&\langle R^{\nabla^E}(X_i,X_j)a,\beta\rangle_E.\end{aligned}$$

2. *The scalar curvature of $(E^{(r)},h)$ is given by:*

$$\tau^r(a)=s^M(x)+\frac{1}{r^2}(m-1)(m-2)-\frac{1}{4}\xi_x(a,a),$$

where

$$\xi_x(a,b)=\sum_{j=1}^n\sum_{i=1}^n\langle R^{\nabla^E}(X_i,X_j)a,R^{\nabla^E}(X_i,X_j)b\rangle_E,\quad a,b\in E_x.$$

Proof. We will use the O'Neil formulas for the Ricci curvature and scalar curvature given in [1, Proposition 9.36, Corollary 9.37]. From these formulas, Proposition 2.2, and the fact that the fibers are Einstein, we get:

$$\begin{aligned}
\mathrm{ric}(X^h,Y^h) &= \mathrm{ric}^M(X,Y) - 2\sum_{i=1}^n h(B_{X^h}X_i^h, B_{Y^h}X_i^h) = \mathrm{ric}^M(X,Y)\\
&\quad -\frac{1}{2}\sum_{i=1}^n \langle R^{\nabla^E}(X,X_i)a, R^{\nabla^E}(Y,X_i)a\rangle_E,\\
\mathrm{ric}(\alpha^t,\beta^t) &= \frac{(m-2)}{r^2}\langle\overline{\alpha},\overline{\beta}\rangle_E\\
&\quad +\sum_{i=1}^n h(B_{X_i^h}\alpha^t, B_{X_i^h}\beta^t)\\
&= \frac{(m-2)}{r^2}\langle\overline{\alpha},\overline{\beta}\rangle_E\\
&\quad +\frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n \langle R^{\nabla^E}(X_i,X_j)a,\alpha\rangle_E\langle R^{\nabla^E}(X_i,X_j)a,\beta\rangle_E,\\
\mathrm{ric}(X^h,\beta^t) &= -h(\check{\delta}BX^h,\beta^t) = \sum_{i=1}^n h((\overline{\nabla}_{X_i^h}B)_{X_i^h}X,\beta^t)\\
&= -\frac{1}{2}\sum_{i=1}^n \langle\nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i,X,\beta),a\rangle_E.
\end{aligned}$$

This establishes the expression of the Ricci curvature. The scalar curvature is given by $\tau^r = s^M\circ\pi_E + s^v + |B|^2$, which completes the proof. □

3. On the Sign of the Different Curvatures of $(E^{(r)}, h)$

In this section, we study the sign of sectional curvature, Ricci curvature, and scalar curvature of sphere bundles $E^{(r)}$ equipped with the Sasaki metric h.

Throughout this section, $(M,\langle\ ,\ \rangle_{TM})$ is a Riemannian manifold of dimension n and $(E,\langle\ ,\ \rangle_E)$ is a Euclidean vector bundle of rank m with an invariant connection ∇^E.

3.1. The Case $R^{\nabla^E}=0$

Note that $R^{\nabla^E}=0$ if and only if the O'Neill shape tensor of the Riemannian submersion $\pi_E:(E^{(r)},h)\longrightarrow(M,\langle\ ,\ \rangle_{TM})$ vanishes, which is equivalent to $E^{(r)}$ being locally the Riemannian product of M and the fiber. Therefore, we have the following results.

Proposition 3.1. *Suppose that $R^{\nabla^E}=0$ and $m=2$. Then, using the notations in Propositions 2.3 and 2.4:*

$$\begin{aligned}
K(P) = \langle R^M(X,Y)X,Y\rangle_{TM},\quad &\mathrm{ric}(X^h+\alpha^t, Y^h+\beta^t)\\
&= \mathrm{ric}^M(X,Y)\quad \textit{and}\quad \tau^r(a) = s^M(\pi_E(a)).
\end{aligned}$$

Proposition 3.2. *Suppose that $R^{\nabla^E}=0$ and $m\geq 3$. Then:*

1. *$(M,\langle\ ,\ \rangle_{TM})$ has constant scalar curvature if and only if $(E^{(r)},h)$ has constant scalar curvature,*

2. $(M, \langle\ ,\ \rangle_{TM})$ *is locally symmetric if and only if* $(E^{(r)}, h)$ *is locally symmetric,*
3. $(M, \langle\ ,\ \rangle_{TM})$ *is Einstein with Einstein constant* $\frac{m-2}{r^2}$ *if and only if* $(E^{(r)}, h)$ *is Einstein with the same Einstein constant,*
4. $(E^{(r)}, h)$ *can never have a constant sectional curvature.*

Proof. We have noticed that $R^{\nabla^E} = 0$ if and only $E^{(r)}$ is locally the Riemannian product of M and the fiber. However, the fiber is a sphere endowed with its canonical metric and has constant sectional curvature. For instance, a Riemannian product is locally symmetric if and only if its components are locally symmetric which permits to prove 2. The same argument permits to prove the other assertions. □

For the Euclidean vector bundles with large rank compared to the dimension of the base, the following theorem constitutes a converse to the third assertion in Proposition 3.2. Note that the rank of the Atiyah vector bundle $E(M, k)$ is $\frac{n(n+1)}{2}$ and, hence, it satisfies the hypothesis of the next theorem.

Theorem 3.1. *Suppose that* $m - 1 > \frac{n(n-1)}{2}$ *where* m *is the rank of* E *and* $n = \dim M$*. Then:*

1. $(E^{(r)}, h)$ *is Einstein with Einstein constant* λ *if and only if* $R^{\nabla^E} = 0$, $\lambda = \frac{(m-2)}{r^2}$ *and* M *is Einstein with Einstein constant* $\frac{(m-2)}{r^2}$.
2. $(E^{(r)}, h)$ *can never has constant sectional curvature.*

Proof. 1. If $(E^{(r)}, h)$ is Einstein, then, according to Proposition 2.4, we have for any $x \in M$, $X \in T_xM$, $a \in E_x^{(r)}$ and $\alpha \in E_x$ with $\langle \alpha, a\rangle_E = 0$:

$$\lambda|\alpha|^2 = \frac{(m-2)}{r^2}|\alpha|^2 + \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle R^{\nabla^E}(X_i, X_j)a, \alpha\rangle_E^2. \tag{7}$$

Fix $x \in M$, $a \in E_x^{(r)}$, and an orthonormal basis $(X_i)_i$ of T_xM, and choose an orthonormal family $(\alpha_1, \ldots, \alpha_{m-1})$ of elements in the orthogonal of a. For any $k = 1, \ldots, m-1$, define the vector $U_k \in \mathbb{R}^{\frac{n(n-1)}{2}}$ by putting:

$$U_k = \Big(\langle R^{\nabla^E}(X_1, X_2)a, \alpha_k\rangle_E, R^{\nabla^E}(X_1, X_3)a, \alpha_k\rangle_E, \\ \ldots, \langle R^{\nabla^E}(X_{n-1}, X_n)a, \alpha_k\rangle_E\Big).$$

If we take $\alpha = \alpha_k$ in (7), we get that the Euclidean norm of U_k satisfies $|U_k|^2 = 2\left(\lambda - \frac{(m-2)}{r^2}\right)$. Moreover, if we take $\alpha = \alpha_k + \alpha_l$ with $l \neq k$, we get that $\langle U_l, U_k\rangle = 0$. Thus, $(U_1, \ldots, U_{m-1})$ is an orthogonal family of vectors in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Since $m - 1 > \frac{n(n-1)}{2}$, they must be linearly dependent. However, they have the same norm, so they must vanish. This completes the proof of the first assertion.

2. If $(E^{(r)}, h)$ has a constant sectional curvature, then it is Einstein and hence $R^{\nabla^E} = 0$. However, according to the expression of the sectional

curvature given in Proposition 2.3, it cannot be constant. This completes the proof.

□

3.2. The Case $\nabla^{M,E}(R^{\nabla^E})=0$

If $\nabla^{M,E}(R^{\nabla^E})=0$, then R^{∇^E} is invariant under parallel transport of ∇^M and ∇^E and, hence, there exists a constant $\mathbf{K}>0$, such that for any $X,Y\in\Gamma(TM)$, $\alpha\in\Gamma(E)$:

$$|R^{\nabla^E}(X,Y)\alpha|\leq\mathbf{K}|X||Y||\alpha|. \tag{8}$$

The following theorem generalizes a result obtained in [7].

Theorem 3.2. *Suppose that $\nabla^{M,E}(R^{\nabla^E})=0$ and the sectional curvature of M is bounded below by a positive constant C. Then:*

1. *At every point of $E^{(r)}$, there exists a tangent plane with positive sectional curvature.*
2. *If* $\operatorname{rank}(E)=2$*, then the sectional curvature of $(E^{(r)},h)$ is non-negative if $r^2\leq\frac{4C}{3\mathbf{K}}$.*
3. *If* $\operatorname{rank}(E)\geq 3$*, then the sectional curvature of $(E^{(r)},h)$ is non-negative if:*

$$C-\frac{3}{4}r^2\mathbf{K}^2\left(4+3r^2(n-2)\mathbf{K}+\frac{3}{4}r^4(n-2)^2\mathbf{K}^2\right)\geq 0. \tag{9}$$

In particular, for r sufficiently small, the sectional curvature of $(E^{(r)},h)$ is non-negative.

Proof. Let $P\subset T_aE^{(r)}$ be a plane. Then, there exists an orthonormal basis $\{X^h+\alpha^t,Y^h+\beta^t\}$ of P satisfying $|X|^2+|\alpha|^2=|Y|^2+|\beta|^2=1$, $\langle X,Y\rangle_{TM}=\langle\alpha,\beta\rangle_E=0$ and $\langle\alpha,a\rangle_E=\langle\beta,a\rangle_E=0$. Put $X=\cos(t)\widetilde{X}$, $\alpha=\sin(t)\widetilde{\alpha}$, $Y=\cos(s)\widetilde{Y}$, $\beta=\sin(s)\widetilde{\beta}$ and $a=r\widetilde{a}$ with $s,t\in[0,\pi/2]$ and $|\widetilde{X}|=|\widetilde{Y}|=|\widetilde{\alpha}|=|\widetilde{\beta}|=1$. We replace in the expression of $K(P)$ given in Proposition 2.3 and we get:

$$\begin{aligned}K(P)&=A\cos^2(t)\cos^2(s)+\frac{1}{r^2}\sin^2(t)\sin^2(s)\\&\quad+B\cos(t)\cos(s)\sin(t)\sin(s)+D\cos^2(t)\sin^2(s)+E\sin^2(t)\cos^2(s),\end{aligned}$$

where

$$\begin{aligned}A&=K^M(\{\widetilde{X},\widetilde{Y}\})-\frac{3}{4}r^2|R^{\nabla^E}(\widetilde{X},\widetilde{Y})\widetilde{a}|^2,\\B&=3\langle R^{\nabla^E}(\widetilde{X},\widetilde{Y})\widetilde{\alpha},\widetilde{\beta}\rangle_E-r^2\sum_{i=1}^n\langle R^{\nabla^E}(\widetilde{X},X_i)\widetilde{\alpha},\widetilde{a}\rangle_E\langle R^{\nabla^E}(\widetilde{Y},X_i)\widetilde{\beta},\widetilde{a}\rangle_E\\&\quad+\frac{r^2}{2}\sum_{i=1}^n\langle R^{\nabla^E}(\widetilde{X},X_i)\widetilde{\beta},\widetilde{a}\rangle_E\langle R^{\nabla^E}(\widetilde{Y},X_i)\widetilde{\alpha},\widetilde{a}\rangle_E,\\D&=\frac{r^2}{4}\sum_{i=1}^n\langle R^{\nabla^E}(\widetilde{X},X_i)\widetilde{\beta},\widetilde{a}\rangle_E^2,\quad E=\frac{r^2}{4}\sum_{i=1}^n\langle R^{\nabla^E}(\widetilde{Y},X_i)\widetilde{\alpha},\widetilde{a}\rangle_E^2.\end{aligned}$$

1. If $\cos(t) = \cos(s) = 0$, then $K(P) = \frac{1}{r^2} > 0$ and, hence, sectional curvature of $(E^{(r)}, h)$ can never be nonpositive.

   Let us prove now the second assertion and the third assertion. If $X = 0$ or $Y = 0$, then $K(P) \geq 0$. Suppose now that $X \neq 0$ and $Y \neq 0$, so we can choose $X_1 = \widetilde{X}$ and $X_2 = \widetilde{Y}$ and get:
$$A \geq C - \frac{3}{4}r^2\mathbf{K}^2 \quad \text{and} \quad B \geq -\frac{3\mathbf{K}}{2}\left(2 + r^2(n-2)\mathbf{K}\right).$$
2. If $\mathrm{rank}(E) = 2$, we can choose $\beta = 0$ and hence:
$$K(P) \geq (C - \frac{3}{4}r^2\mathbf{K})\cos^2(t)\cos^2(s) + \frac{1}{r^2}\sin^2(t)\sin^2(s).$$
   Thus the sectional curvature is non-negative if: $r^2 \leq \frac{4C}{3\mathbf{K}}$.
3. Suppose that $\mathrm{rank}(E) > 2$. Then, using the estimations of A and B given above, we get:
$$K(P) \geq \left(C - \frac{3}{4}r^2\mathbf{K}^2\right)\cos^2(t)\cos^2(s) + \frac{1}{r^2}\sin^2(t)\sin^2(s)$$
$$-\frac{3\mathbf{K}}{2}\left(2 + r^2(n-2)\mathbf{K}\right)\cos(t)\cos(s)\sin(t)\sin(s).$$
   The right side of this inequality, say Q, can be arranged in the following way:
$$Q = \left[\frac{1}{r}\sin(t)\sin(s) - \frac{3r\mathbf{K}}{4}\left(2 + r^2(n-2)\mathbf{K}\right)\cos(t)\cos(s)\right]^2$$
$$+\left(C - \frac{3}{4}r^2\mathbf{K}^2\left(4 + 3r^2(n-2)\mathbf{K} + \frac{3}{4}r^4(n-2)^2\mathbf{K}^2\right)\right)\cos^2(t)\cos^2(s).$$
   This ends the proof of the last assertion.

□

Remark 2. 1. In the classical case, i.e., $E = TM$, $\langle\ ,\ \rangle_E = \langle\ ,\ \rangle_{TM}$, and $\nabla^E = \nabla^M$, the hypotheses $\nabla^M(R^M) = 0$ and M has positive sectional curvature imply that the sectional curvature of M is bounded below by a positive constant. Thus, in this case, our result is the same as the result obtained in [7].
2. The left side of the inequality (9), say Q, goes to C when r goes to 0 which permitted as to get our result. In some cases, the constant $\mathbf{K}$ can depend on a parameter and by varying this parameter one can make $Q > 0$. This is the case in Theorem 4.3.

Theorem 3.3. *Suppose that $\nabla^{M,E}(R^{\nabla^E}) = 0$, $R^{\nabla^E} \neq 0$ and there exists a positive constant ρ, such that $\mathrm{ric}^M(X, X) \geq \rho|X|^2$, for any $X \in \Gamma(TM)$. Then:*

1. *If $\mathrm{rank}(E) = 2$, then $(E^{(r)}, h)$ has non-negative Ricci curvature for $r^2 \leq \frac{2\rho}{n\mathbf{K}^2}$, where the constant $\mathbf{K}$ is given in (8).*
2. *If $\mathrm{rank}(E) > 2$, then $(E^{(r)}, h)$ has positive Ricci curvature for $r^2 < \frac{2\rho}{n\mathbf{K}^2}$, where the constant $\mathbf{K}$ is given in (8).*

Proof. For any $a \in E^{(r)}$, $X \in T_xM$ and $\alpha \in E_x$, such that $|X|^2 + |\alpha|^2 = 1$ and $\langle \alpha, a\rangle_E = 0$, we have from Proposition 2.4 that:

$$\begin{aligned}\mathrm{ric}(X^h + \alpha^t, X^h + \alpha^t) = &\frac{(m-2)}{r^2}|\alpha|^2 + \mathrm{ric}^M(X,X) - \frac{1}{2}\sum_{i=1}^n |R^{\nabla^E}(X, X_i)a|^2\\ &-\sum_{i=1}^n \langle \nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i, X, \alpha), a\rangle_E\\ &+\frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n \langle R^{\nabla^E}(X_i, X_j)a, \alpha\rangle_E^2.\end{aligned}$$

Let us write $X = \cos(t)\hat{X}$, $\alpha = \sin(t)\hat{\alpha}$ and $\hat{a} = a/r$ where $\hat{X}$ and $\hat{\alpha}$ are unit vectors.

Suppose that $\nabla^{M,E}(R^{\nabla^E}) = 0$. We obtain:

$$\begin{aligned}\mathrm{ric}(X^h + \alpha^t, X^h + \alpha^t) = &\cos^2(t)\left(\mathrm{ric}^M(\hat{X},\hat{X}) - \frac{r^2}{2}\sum_{i=1}^n |R^{\nabla^E}(\hat{X}, X_i)\hat{a}|^2\right)\\ &+\sin^2(t)\left(\frac{(m-2)}{r^2} + \frac{r^2}{4}\sum_{i=1}^n\sum_{j=1}^n \langle R^{\nabla^E}(X_i, X_j)\hat{a}, \hat{\alpha}\rangle_E^2\right).\end{aligned}$$

From the hypothesis on ric^M and (8), we get:

$$\mathrm{ric}(X^h + \alpha^t, X^h + \alpha^t) \geq \left(\rho - \frac{nr^2\mathbf{K}^2}{2}\right)\cos^2(t) + \frac{(m-2)}{r^2}\sin^2(t).$$

This shows the two assertions. □

3.3. Ricci and Scalar Curvatures

The two following theorems are a generalization of [7, Theorem 3, Theorem 1] established in the case when $E = TM$.

Theorem 3.4. *If M is compact with positive Ricci curvature and* $\mathrm{rank}(E) \geq 3$, *then, for r sufficiently small, the Ricci curvature of the sphere bundle $(E^{(r)}, h)$ is positive.*

Proof. Suppose now that M is compact with positive Ricci curvature and put $X = \cos(t)\hat{X}$, $\alpha = \sin(t)\hat{\alpha}$ and $\hat{a} = \frac{a}{r}$ where $\hat{X} \in T_xM$, $\hat{\alpha} \in E_x$, $|\hat{X}| = |\hat{\alpha}| = 1$

and $a \in E^{(r)}$. We have:

$$\begin{aligned}
\mathrm{ric}(X^h+\alpha^t,X^h+\alpha^t) &= \cos^2(t)\,\mathrm{ric}^M(\hat{X},\hat{X}) + \frac{(m-2)}{r^2}\sin^2(t)\\
&\quad -\frac{1}{2}r^2\cos^2(t)\sum_{i=1}^n |R^{\nabla^E}(\hat{X},X_i)\hat{a}|^2\\
&\quad -r\cos(t)\sin(t)\sum_{i=1}^n \langle \nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i,\hat{X})\hat{\alpha},\hat{a}\rangle_E\\
&\quad +\frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n \langle R^{\nabla^E}(X_i,X_j)a,\alpha\rangle_E^2,\\
&\geq \cos^2(t)\mathrm{ric}^M(\hat{X},\hat{X}) + \frac{(m-2)}{r^2}\sin^2(t)\\
&\quad -\frac{1}{2}r^2\cos^2(t)\sum_{i=1}^n |R^{\nabla^E}(\hat{X},X_i)\hat{a}|^2\\
&\quad -r\,\cos(t)\sin(t)\sum_{i=1}^n \langle \nabla^{M,E}_{X_i}(R^{\nabla^E})(X_i,\hat{X})\hat{\alpha},\hat{a}\rangle_E.
\end{aligned}$$

Since M is compact, there exist positive constants L_1 and L_2, such that for any $x \in M$ and for any unit vectors $X, Y, Z \in T_xM$, $\alpha, \beta \in E_x$:

$$|R^{\nabla^E}(X,Y)Z| \leq L_1 \quad \text{and} \quad |\langle \nabla^{M,E}_X(R^{\nabla^E})(Y,Z)\alpha,\beta\rangle_E| \leq L_2.$$

On the other hand, there is a positive number ϵ, such that $\mathrm{ric}^M(X,X) \geq \epsilon$, for any unit vector X. Then, using the above estimations, we get:

$$\begin{aligned}
\mathrm{ric}(X^h+\alpha^t,X^h+\alpha^t) &\geq \cos^2(t)(\epsilon - \frac{1}{2}r^2nL_1^2) + \frac{(m-2)}{r^2}\sin^2(t) - rnL_2\cos(t)\sin(t)\\
&= \left(\sqrt{A}\cos(t) - \frac{B}{2\sqrt{A}}\sin(t)\right)^2 + C\sin^2(t),
\end{aligned}$$

where $A = \epsilon - \frac{1}{2}r^2nL_1^2$, $B = rnL_2$, $C = \left(\frac{m-2}{r^2} - \frac{B^2}{4A}\right)$ and r taken such that $A, C > 0$. Then, the right side of this inequality is positive for every t. □

Theorem 3.5. *Let $(M, \langle\ ,\ \rangle_{TM})$ be a compact Riemannian manifold and $(E, \langle\ ,\ \rangle_E)$ be a Euclidean vector bundle with an invariant connection ∇^E. Then, for r sufficiently small, the scalar curvature of $(E^{(r)}, h)$ is positive.*

Proof. Suppose now that M is compact and put $\hat{a} = \frac{a}{r}$, where $a \in E^{(r)}$. We have:

$$\tau^r(a) = s^M(\pi_E(a)) + \frac{1}{r^2}(m-1)(m-2) - \frac{1}{4}r^2\xi_{\pi_E(a)}(\hat{a},\hat{a}).$$

Since M is compact, there exists positive constants L_1 and L_2, such that for any $x \in M$ and for any unit vectors $X, Y \in T_xM$, $\alpha, \beta \in E_x$:

$$|\langle R^M(X,Y)X,Y\rangle_{TM}| \leq L_1 \quad \text{and} \quad |R^{\nabla^E}(X,Y)\alpha| \leq L_2.$$

Then:

$$\tau^r(a) \geq \frac{1}{r^2}(m-1)(m-2) + \frac{1}{4}n(n-1)(4L_1 - rL_2^2.).$$

This means that τ^r is positive on $E^{(r)}$ when r is sufficiently small. $\square$

Let $E \longrightarrow M$ be a vector bundle or rank m. Recall that its associated sphere bundle is the quotient $S(E) = E\backslash\{0\}/\sim$, where $a \sim b$ if there exists $t > 0$, such that $a = tb$. Let $\langle\ ,\ \rangle_E$ be a Euclidean product on E. The associated $O(m)$-principal bundle has a connection, so there exits a connection ∇^E on E which preserves the metric $\langle\ ,\ \rangle_E$. Since $S(E)$ can be identified to $E^{(r)}$ for any r, using Theorems 3.4 and 3.5, we get the following corollary which has been proved in [12] by a different method.

Corollary 3.1. *Let $E \longrightarrow M$ be a vector bundle over a compact Riemannian manifold and $S(E) \longrightarrow M$ its associated sphere bundle. Then:*

1. *If the Ricci curvature of M is positive, then $S(E)$ admits a complete Riemannian metric of positive curvature.*
2. *$S(E)$ admits a complete Riemannian metric of positive scalar curvature.*

We will end this section with a result which has been proved in [2] when $E = TM$, $\langle\ ,\ \rangle_{TM} = \langle\ ,\ \rangle_E$, and ∇^E is the Levi-Civita connection of $\langle\ ,\ \rangle_{TM}$.

Theorem 3.6. *Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold and $(E, \langle\ ,\ \rangle_E)$ be a Euclidean vector bundle with an invariant connection ∇^E. Then, the sphere bundle $(E^{(r)}, h)$ equipped with the Sasaki metric has constant scalar curvature if and only if:*

$$\xi = \frac{|R^{\nabla^E}|^2}{m}\langle\ ,\ \rangle_E, \tag{10}$$

$$4ms^M - r^2|R^{\nabla^E}|^2 = constant. \tag{11}$$

where $\xi(a,b) = \sum_{j=1}^n \left(\sum_{i=1}^n \langle R^{\nabla^E}(X_i,X_j)a, R^{\nabla^E}(X_i,X_j)b\rangle_E\right)$, *for any* $a, b \in \Gamma(E)$.

Proof. The scalar curvature τ^r is given, for $a \in E^{(r)}$, by:

$$\tau^r(a) = s^M(\pi_E(a)) + \frac{1}{r^2}(m-1)(m-2) - \frac{1}{4}\xi_{\pi_E(a)}(a,a).$$

Suppose that τ is constant along $E^{(r)}$. For fixed $x \in M$, $\tau^r(a)$ does not depend on the choice of the vector $a \in E_x^{(r)}$. This implies that ξ_x is proportional to the metric $\langle\ ,\ \rangle_E$ and the coefficient of proportionality is necessarily equal to $|R^{\nabla^E}|^2/m$. $\square$

4. Sasaki Metric on the Sphere Bundle of the Atiyah Euclidean Vector Bundle Associated with a Riemannian Manifold

We have seen in the last section that many results obtained on the sphere bundles of tangent bundles over Riemannian manifolds can be generalized to any Euclidean vector bundle. In this section, we will express these results in the case of the sphere bundle of the Atiyah Euclidean vector bundle introduced in Sect. 1 to get some new interesting geometric situations and to open new horizons for further explorations.

4.1. The Atiyah Euclidean Vector Bundle and the Supracurvature of a Riemannian Manifold

Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold, $k > 0$, and $(E(M, k), \langle\ ,\ \rangle_k, \nabla^E)$ the associated Atiyah Euclidean vector bundle defined in the Introduction. Let $K^M : \mathrm{so}(TM) \longrightarrow \mathrm{so}(TM)$ be the curvature operator given by $K^M(X \wedge Y) = R^M(X, Y)$ where $X \wedge Y(Z) = \langle Y, Z\rangle_{TM} X - \langle X, Z\rangle_{TM} Y$.

The curvature R^{∇^E} of ∇^E (we refer to as the supracurvature of $(M, \langle\ ,\ \rangle_{TM}, k)$) was computed in [4, Theorem 3.1]. It is given by the following formulas:

$$\begin{aligned} R^{\nabla^E}(X,Y)Z &= \left\{R^M(X,Y)Z + H_Y H_X Z - H_X H_Y Z\right\} \\ &\quad + \left\{-\frac{1}{2}\nabla^M_Z (K^M)(X \wedge Y)\right\}, \\ R^{\nabla^E}(X,Y)F &= \left\{(R^{\nabla^E}(X,Y)F)_{TM}\right\} \\ &\quad + \left\{[R^M(X,Y), F] + H_Y H_X F - H_X H_Y F\right\}, \\ \langle (R^{\nabla^E}(X,Y)F)_{TM}, Z\rangle_k &= -\langle R^{\nabla^E}(X,Y)Z, F\rangle_k, \end{aligned} \tag{12}$$

$X, Y, Z \in \Gamma(TM)$, $F \in \Gamma(\mathrm{so}(TM))$. We denote by $E^{(r)}(M, k)$ the sphere bundle of radius r associated with $E(M, k)$ and h the Sasaki metric on $E^{(r)}(M, k)$.

The supracurvature is deeply related to the geometry of $(M, \langle\ ,\ \rangle_{TM})$. Let us compute it in some particular cases. This computation will be useful in the proof of Theorem 4.1 where we will characterize the Riemannian manifolds with vanishing supracurvature.

Supracurvature of the Riemannian Product of Riemannian Manifolds.

Proposition 4.1. *Let $(M, \langle\ ,\ \rangle_{TM})$ be the Riemannian product of p Riemannian manifolds $(M_1, \langle\ ,\ \rangle_1), \ldots, (M_p, \langle\ ,\ \rangle_p)$. Then, the supracurvature of $(M, \langle\ ,\ \rangle_{TM})$ at a point $x = (x_1, \ldots, x_p)$ is given by:*

$$\begin{cases} R^{\nabla^E}[(X_1,\ldots,X_p),(Y_1,\ldots,Y_p)](Z_1,\ldots,Z_p) = \left(R^{\nabla^{E_1}}(X_1,Y_1)Z_1,\ldots,R^{\nabla^{E_p}}(X_p,Y_p)Z_p\right), \\ \quad R^{\nabla^E}[(X_1,\ldots,X_p),(Y_1,\ldots,Y_p)](F) = \left(R^{\nabla^{E_1}}(X_1,Y_1)F_1,\ldots,R^{\nabla^{E_p}}(X_p,Y_p)F_p\right), \end{cases}$$

where $X_i, Y_i, Z_i \in T_{x_i}M_i$, $F \in \mathrm{so}(T_xM)$, $F_i = \mathrm{pr}_i \circ F_{|TM_i}$, $R^{\nabla^{E_i}}$ is the supracurvature of $(M_i, \langle\ ,\ \rangle_i, k)$ and $i = 1, \ldots, p$.

Proof. It is an immediate consequence of the following formulas:

$$\begin{aligned} R^M[X,Y](Z) &= \left(R^{M_1}(X_1,Y_1)Z_1,\ldots,R^{M_p}(X_p,Y_p)Z_p\right), \\ H_X Y &= \left(H^1_{X_1}Y_1,\ldots,H^p_{X_p}Y_p\right), \\ H_X F &= \left(H^1_{X_1}F_1,\ldots,H^p_{X_p}F_p\right), \\ \nabla^M_X(K^M)(X \wedge Y) &= \left(\nabla_{Z_1}(K^{M_1})(X_1 \wedge Y_1),\ldots,\nabla_{Z_p}(K^{M_p})(X_p \wedge Y_p)\right), \end{aligned}$$

where $X = (X_1, \ldots, X_p)$, $Y = (Y_1, \ldots, Y_p)$, $Z = (Z_1, \ldots, Z_p)$, and $F_i = \mathrm{pr}_i \circ F_{|TM_i}$. □

Supracurvature of Riemannian manifolds with Constant Curvature.

Proposition 4.2. *Suppose that* $(M, \langle\ ,\ \rangle_{TM})$ *has constant sectional curvature* c *and put* $\varpi = \frac{1}{4}c(2-ck)$. *Then, for any* $X, Y \in \Gamma(TM)$ *and* $F \in \Gamma(\mathrm{so}(TM))$:

$$R^{\nabla^E}(X,Y)Z = -2\varpi X \wedge Y(Z) \quad \text{and} \quad R^{\nabla^E}(X,Y)F = -2\varpi[X \wedge Y, F].$$

Proof. The expression of R^{∇^E} is given by (12). We have $H_X Y = -\frac{1}{2}R^M(X,Y) = \frac{1}{2}cX \wedge Y$. Moreover, since the curvature is constant, then $\nabla^M(K^M) = 0$.

Now, if $(X_i)_{i=1}^n$ is a local frame of orthonormal vector fields, then:

$$\begin{aligned}\langle H_X F, Y\rangle_{TM} &= -\frac{1}{2}k\,\mathrm{tr}(F \circ R^M(X,Y)) = -\frac{1}{2}ck\sum_{i=1}^n \langle F(X_i), X \wedge Y(X_i)\rangle_{TM}\\ &= -\frac{1}{2}ck\sum_{i=1}^n (\langle Y, X_i\rangle_{TM}\langle F(X_i), X\rangle_{TM} - \langle X, X_i\rangle_{TM}\langle F(X_i), Y\rangle_{TM})\\ &= -ck\langle F(Y), X\rangle_{TM}.\end{aligned}$$

Thus, $H_X F = ckF(X)$. Therefore:

$$\begin{aligned}[H_Y, H_X]Z &= \frac{1}{2}(H_Y R^M(Z,X) + H_X R^M(Y,Z))\\ &= \frac{1}{2}ck(R^M(Z,X)Y + R^M(Y,Z)X)\\ &= -\frac{1}{2}ckR^M(X,Y)Z.\end{aligned}$$

Thus:

$$R^{\nabla^A}(X,Y)Z = \frac{1}{2}(2-ck)R^M(X,Y)Z = -\frac{1}{2}c(2-ck)X \wedge Y(Z).$$

On the other hand:

$$\begin{aligned}[H_Y, H_X]F &= ck(H_Y F(X) - H_X F(Y))\\ &= -\frac{1}{2}ck(R^M(Y, F(X)) + R^M(F(Y), X)),\\ &= -\frac{1}{2}c^2k([F, X \wedge Y]).\end{aligned}$$

This completes the proof. □

Supracurvature of some locally symmetric spaces. Let G be a compact connected Lie group with $\mathfrak{g}$ its Lie algebra and K be a closed subgroup of G with $\mathfrak{k}$ its Lie algebra. Denote by $\pi : G \longrightarrow G/K$ the canonical projection. Suppose that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is Ad_K-invariant, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and the restriction of the Killing form B of $\mathfrak{g}$ to p is negative definite. The scalar product $\langle\ ,\ \rangle_{\mathfrak{p}} = \lambda B_{|\mathfrak{p}\times\mathfrak{p}}$ with $\lambda < 0$ defines a G-invariant Riemannian metric $\langle\ ,\ \rangle_{G/K}$ on G/K which is locally symmetric. For any $X \in \mathfrak{k}$, we denote by Φ_X the restriction of ad_X to $\mathfrak{p}$, then:

$$\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}}) = \Phi_{\mathfrak{k}} \oplus (\Phi_{\mathfrak{k}})^{\perp}, \tag{13}$$

where $(\Phi_{\mathfrak{k}})^{\perp}$ is the orthogonal with respect to the invariant scalar product on $\mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}})$, $(A, B) \mapsto -\mathrm{tr}(AB)$.

Proposition 4.3. *The supracurvature of* $(G/K, \langle\ ,\ \rangle_{G/K}, k)$ *at* $\pi(e)$ *is given by:*

$$R^{\nabla^E}(X,Y)Z = [[X,Y],Z] + \frac{k}{4}\left([Y, U(\Phi_{[X,Z]})] - [X, U(\Phi_{[Y,Z]})]\right),$$
$$R^{\nabla^E}(X,Y)F = [\Phi_{[X,Y]}, \Phi_{X^F - \frac{k}{4}U(F)}] + [\Phi_{[X,Y]}, F^\perp],$$

where $X, Y, Z \in T_{\pi(e)}G/K = \mathfrak{p}$, $F = \Phi_{X^F} + F^\perp \in \mathrm{so}(\mathfrak{p}, \langle\ ,\ \rangle_{\mathfrak{p}}) = \Phi_{\mathfrak{k}} \oplus (\Phi_{\mathfrak{k}})^\perp$ *and* $U(F)$ *is the element of* $\mathfrak{k}$ *given by:*

$$U(F) = \sum_{i=1}^{n} [X_i, F(X_i)],$$

where $(X_1, \ldots, X_n)$ *is an orthonormal basis of* $\mathfrak{p}$.

Proof. The expression of R^{∇^E} is given by (12). The curvature of G/K at $\pi(e)$ is given by (see [1, Proposition 7.72]):

$$R^{G/K}(X,Y)Z = [[X,Y],Z], \quad X, Y, Z \in \mathfrak{p},$$

$\nabla^{G/K}(K^{G/K}) = 0$ and $H_X Y = -\frac{1}{2}R^{G/K}(X,Y) = -\frac{1}{2}\Phi_{[X,Y]}$. Choose $(X_i)_{i=1}^n$ an orthonormal basis of $\mathfrak{p}$. We have:

$$\begin{aligned}
\langle H_X F, Y\rangle_k &\overset{(1)}{=} -\frac{1}{2}k\,\mathrm{tr}(F \circ R^{G/K}(X,Y)) \\
&= \frac{k}{2}\sum_i \langle F(X_i), [[X,Y],X_i]\rangle_{\mathfrak{p}} \\
&= \frac{\lambda k}{2}\sum_i B(F(X_i), [[X,Y],X_i]) \\
&= -\frac{k}{2}\sum_i \langle [X,[X_i,F(X_i)]], Y\rangle_{\mathfrak{p}}.
\end{aligned}$$

Thus, $H_X F = -\frac{k}{2}[X, U(F)]$. We deduce that:

$$\begin{aligned}
H_Y H_X Z - H_X H_Y Z &= -\frac{1}{2}H_Y(\Phi_{[X,Z]}) + \frac{1}{2}H_X(\Phi_{[Y,Z]}) \\
&= \frac{k}{4}[Y, U(\Phi_{[X,Z]})] - \frac{k}{4}[X, U(\Phi_{[Y,Z]})], \\
H_Y H_X F - H_X H_Y F &= \frac{k}{4}\Phi_{[Y,[X,U(F)]]} - \frac{k}{4}\Phi_{[X,[Y,U(F)]]} \\
&= -\frac{k}{4}[\Phi_{[X,Y]}, \Phi_{U(F)}].
\end{aligned}$$

This gives the first formula. For the second formula, we have:

$$\begin{aligned}
R^{\nabla^E}(X,Y)F &= [R^M(X,Y), F] + H_Y H_X F - H_X H_Y F \\
&= [\Phi_{[X,Y]}, \Phi_{X^F} + F^\perp] - \frac{k}{4}[\Phi_{[X,Y]}, \Phi_{U(F)}] \\
&= [\Phi_{[X,Y]}, \Phi_{X^F - \frac{k}{4}U(F)}] + [\Phi_{[X,Y]}, F^\perp].
\end{aligned}$$

This completes the proof. □

Supracurvature of complex projective spaces. Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \longrightarrow P^n(\mathbb{C})$ be the natural projection and $\pi_s : S^{2n+1} \longrightarrow P^n(\mathbb{C})$ its restriction to $S^{2n+1} \subset \mathbb{C}^{n+1}\backslash\{0\}$. For any $m \in S^{2n+1}$, put $F_m = \ker((\pi_s)_*)_m$ and let $F_m^\perp$ be the orthogonal complementary subspace to F_m in $T_m(S^{2n+1})$:

$$T_m(S^{2n+1}) = F_m \oplus F_m^\perp.$$

We introduce the Riemannian metric $\langle\ ,\ \rangle_{P^n(\mathbb{C})}$ on $P^n(\mathbb{C})$, so that the restriction of $(\pi_s)_*$ to $F_m^\perp$ is an isometry onto $T_{\pi(m)}(P^n(\mathbb{C}))$. Let J_0 be the canonical complex structures on $\mathbb{C}^{n+1}$ and the standard complex structures J on $P^n(\mathbb{C})$ is given by:

$$J(\pi_s)_*v = (\pi_s)_*J_0v, \quad v \in F_m^\perp.$$

Proposition 4.4. *The curvature and the supracurvature of $(P^n(\mathbb{C}), g, k)$ are given by:*

$$\begin{aligned}
R^{P^n(\mathbb{C})}(X,Y)Z &= \langle X,Z\rangle_{P^n(\mathbb{C})}Y - \langle Y,Z\rangle_{P^n(\mathbb{C})}X \\
&\quad -2\langle JY,X\rangle_{P^n(\mathbb{C})}JZ + \langle JZ,Y\rangle_{P^n(\mathbb{C})}JX - \langle JZ,X\rangle_{P^n(\mathbb{C})}JY, \\
R^{\nabla^E}(X,Y)Z &= (k-1)\big(\langle Y,Z\rangle_{P^n(\mathbb{C})}X \\
&\quad -\langle X,Z\rangle_{P^n(\mathbb{C})}Y + 2\langle JY,X\rangle_{P^n(\mathbb{C})}JZ\big) \\
&\quad +((2n+3)k-1)\big(\langle JZ,X\rangle_{P^n(\mathbb{C})}JY \\
&\quad -\langle JZ,Y\rangle_{P^n(\mathbb{C})}JX\big), \\
R^{\nabla^E}(X,Y)F &= \left(\frac{k}{2}-1\right)[F, X\wedge Y + JX\wedge JY] \\
&\quad +2\langle JY,X\rangle_{P^n(\mathbb{C})}[F,J] \\
&\quad +\frac{k}{2}\left([J\circ F\circ J, X\wedge Y] - J\circ F(X)\wedge JY\right. \\
&\quad \left.-JX\wedge J\circ F(Y)\right),
\end{aligned}$$

where $X,Y,Z \in \Gamma(TP^n(\mathbb{C}))$ and $F \in \Gamma(\mathrm{so}(TP^n(\mathbb{C})))$.

Proof. The projection $\pi_s : S^{2n+1} \longrightarrow P^n(\mathbb{C})$ is a Riemannian submersion with totally geodesic fiber and its O'Neill shape tensor is given by $A_{X^h}Y^h = -\langle J_0X^h, Y^h\rangle_{\mathbb{C}^{n+1}}J_0N$ where N is the radial vector field and X^h, Y^h are the horizontal lift of $X,Y \in \Gamma(TP^n(\mathbb{C}))$. The expression of $R^{P^n(\mathbb{C})}$ follows from the formulas:

$$\begin{aligned}
&\langle R^{S^{2n+1}}(X^h,Y^h)Z^h, T^h\rangle_{S^{2n+1}} \\
&\quad = \langle R^{P^n(\mathbb{C})}(X,Y)Z,T\rangle_{P^n(\mathbb{C})}\circ\pi_s - 2\langle A_{X^h}Y^h, A_{Z^h}T^h\rangle_{S^{2n+1}} \\
&\qquad +\langle A_{Y^h}Z^h, A_{X^h}T^h\rangle_{S^{2n+1}} - \langle A_{X^h}Z^h, A_{Y^h}T^h\rangle_{S^{2n+1}}, \\
&R^{S^{2n+1}}(X^h,Y^h)Z^h = -(X^h\wedge Y^h)Z^h.
\end{aligned}$$

To compute the supracurvature, we use (12). We choose an orthonormal frame $(X_i)_{i=1}^{2n}$ of $\Gamma(TP^n(\mathbb{C}))$. We have:

$$\begin{aligned}\langle H_X F, Y\rangle_{P^n(\mathbb{C})} &= \frac{k}{2}\sum_{i=1}^{2n}\langle R^{P^n(\mathbb{C})}(X,Y)X_i, F(X_i)\rangle_{P^n(\mathbb{C})}\\ &= \frac{k}{2}\sum_{i=1}^{2n}\big[\langle X, X_i\rangle_{P^n(\mathbb{C})}\langle Y, F(X_i)\rangle_{P^n(\mathbb{C})}\\ &\quad -\langle Y, X_i\rangle_{P^n(\mathbb{C})}\langle X, F(X_i)\rangle_{P^n(\mathbb{C})} - 2\langle JY, X\rangle_{P^n(\mathbb{C})}\langle JX_i, F(X_i)\rangle_{P^n(\mathbb{C})}\\ &\quad +\langle JX_i, Y\rangle_{P^n(\mathbb{C})}\langle JX, F(X_i)\rangle_{P^n(\mathbb{C})}\\ &\quad -\langle JX_i, X\rangle_{P^n(\mathbb{C})}\langle JY, F(X_i)\rangle_{P^n(\mathbb{C})}\big]\\ &= \frac{k}{2}\left(2\langle F(X), Y\rangle_{P^n(\mathbb{C})} - 2\mathrm{tr}(F\circ J)\langle JX, Y\rangle_{P^n(\mathbb{C})}\right.\\ &\quad \left. -\langle JX, F(JY)\rangle_{P^n(\mathbb{C})} + \langle JY, F(JX)\rangle_{P^n(\mathbb{C})}\right).\end{aligned}$$

Thus:

$$H_X F = k(F(X) - \mathrm{tr}(F\circ J)JX - J\circ F\circ J(X)).$$

However, from (1), $H_X Z = -\frac{1}{2}R^{P^n(\mathbb{C})}(X,Z)$, and hence:

$$H_Y H_X Z = -\frac{k}{2}(R^{P^n(\mathbb{C})}(X,Z)Y - \mathrm{tr}(R^{P^n(\mathbb{C})}(X,Z)\circ J)JY - J\circ R^{P^n(\mathbb{C})}(X,Z)\circ J(Y)).$$

However, $R^{P^n(\mathbb{C})}(X,Z)\circ J = J\circ R^{P^n(\mathbb{C})}(X,Z)$ and a direct computation gives that $\mathrm{tr}(J\circ R^{P^n(\mathbb{C})}(X,Y)) = 4(n+1)\langle JY, X\rangle_{P^n(\mathbb{C})}$.

Therefore:

$$\begin{aligned}H_Y H_X Z &= k\left(2(n+1)\langle JZ, X\rangle_{P^n(\mathbb{C})}JY - R^{P^n(\mathbb{C})}(X,Z)Y\right)\\ &= k\left(\langle Y, Z\rangle_{P^n(\mathbb{C})}X - \langle X, Y\rangle_{P^n(\mathbb{C})}\right.\\ &\quad \left. Z - \langle JY, Z\rangle_{P^n(\mathbb{C})}JX + \langle JY, X\rangle_{P^n(\mathbb{C})}JZ + 2(n+1)\langle JZ, X\rangle_{P^n(\mathbb{C})}JY\right).\end{aligned}$$

Thus:

$$\begin{aligned}H_Y H_X Z - H_X H_Y Z = k(&\langle Y, Z\rangle_{P^n(\mathbb{C})}X - \langle X, Z\rangle_{P^n(\mathbb{C})}Y + 2\langle JY, X\rangle_{P^n(\mathbb{C})}JZ\\ &+(2n+3)\left(\langle JZ, X\rangle_{P^n(\mathbb{C})}JY - \langle JZ, Y\rangle_{P^n(\mathbb{C})}JX\right)).\end{aligned}$$

Then:

$$\begin{aligned}R^{\nabla^E}(X,Y)Z = (k-1)&\left(\langle Y, Z\rangle_{P^n(\mathbb{C})}X - \langle X, Z\rangle_{P^n(\mathbb{C})}Y + 2\langle JY, X\rangle_{P^n(\mathbb{C})}JZ\right)\\ &+((2n+3)k-1)\left(\langle JZ, X\rangle_{P^n(\mathbb{C})}JY - \langle JZ, Y\rangle_{P^n(\mathbb{C})}JX\right).\end{aligned}$$

On the other hand:

$$\begin{aligned}H_Y H_X F &= k\left(H_Y F(X) - \mathrm{tr}(F\circ J)H_Y JX - H_Y J\circ F\circ J(X)\right)\\ &= \frac{k}{2}(Y\wedge F(X) + JY\wedge F\circ J(X)\\ &\quad +JY\wedge J\circ F(X) - Y\wedge J\circ F\circ J(X)\\ &\quad +2\langle J\circ F(X) - F\circ J(X), Y\rangle_{P^n(\mathbb{C})}J\\ &\quad -\mathrm{tr}(F\circ J)\left(Y\wedge JX - JY\wedge X + 2\langle X, Y\rangle_{P^n(\mathbb{C})}J\right)).\end{aligned}$$

Therefore, since $F(X)\wedge Y+X\wedge F(Y)=[F,X\wedge Y]$:

$$H_YH_XF-H_XH_YF=\frac{k}{2}\left([X\wedge Y+JX\wedge JY,F]\right.$$
$$\left.+[J\circ F\circ J,X\wedge Y]-J\circ F(X)\wedge JY-JX\wedge J\circ F(Y)\right),$$

and

$$[R^{P^n(\mathbb{C})}(X,Y),F]=-[X\wedge Y+JX\wedge JY$$
$$+2\langle JY,XJ\rangle_{P^n(\mathbb{C})},F]=-[X\wedge Y+JX\wedge JY,F]-2\langle JY,X\rangle_{P^n(\mathbb{C})}[J,F].$$

Thus:

$$\begin{aligned}R^{\nabla^E}(X,Y)F&=[R^{P^n(\mathbb{C})}(X,Y),F]+H_YH_XF-H_XH_YF\\&=(\frac{k}{2}-1)[F,X\wedge Y+JX\wedge JY]\\&\quad+2\langle JY,X\rangle_{P^n(\mathbb{C})}[F,J]+\frac{k}{2}[J\circ F\circ J,X\wedge Y]\\&\quad-\frac{k}{2}(J\circ F(X)\wedge JY+JX\wedge J\circ F(Y)).\end{aligned}$$

□

It is obvious that if $(M,\langle\ ,\ \rangle_{TM})$ is flat, then, for any $k>0$, the supracurvature of $(M,\langle\ ,\ \rangle_{TM},k)$ vanishes. Furthermore, according to Propositions 4.1 and 4.2, if $(M,\langle\ ,\ \rangle_{TM})$ is the Riemannian product of p Riemannian manifolds all having constant sectional curvature $\frac{2}{k}$, then the supracurvature of $(M,\langle\ ,\ \rangle_{TM},k)$ vanishes. Actually, such are the only cases where the supracurvature vanishes.

Theorem 4.1. *Let $(M,\langle\ ,\ \rangle_{TM})$ be a connected Riemannian manifold. Then, the supracurvature of $(M,\langle\ ,\ \rangle_{TM},k)$ vanishes if and only if the Riemannian universal cover of $(M,\langle\ ,\ \rangle_{TM})$ is isometric to:* $(\mathbb{R}^n,\langle\ ,\ \rangle_0)\times\mathbb{S}^{n_1}\left(\sqrt{\frac{k}{2}}\right)\times\cdots\times\mathbb{S}^{n_p}\left(\sqrt{\frac{k}{2}}\right)$ *where $\mathbb{S}^{n_i}\left(\sqrt{\frac{k}{2}}\right)$ is the Riemannian sphere of dimension n_i, of radius $\sqrt{\frac{k}{2}}$ and constant curvature $\frac{2}{k}$.*

Proof. Suppose that the supracurvature of $(M,\langle\ ,\ \rangle_{TM},k)$ vanishes and consider the Riemannian covering $(N,\langle\ ,\ \rangle_{TN})$ of $(M,\langle\ ,\ \rangle_{TM})$. Since $(M,\langle\ ,\ \rangle_{TM})$ and $(N,\langle\ ,\ \rangle_{TN})$ are locally isometric, then the supracurvature of $(N,\langle\ ,\ \rangle_{TN},k)$ vanishes. This implies by virtue of (12) that $(N,\langle\ ,\ \rangle_{TN})$ is locally symmetric and for any $X,Y\in\Gamma(TN)$:

$$\langle R^N(X,Y)X,Y\rangle_{TN}=\langle H_XY,H_XY\rangle_k\geq 0.$$

Thus, $(N,\langle\ ,\ \rangle_{TN})$ has non-negative sectional curvature. Since N is simply connected, then $(N,\langle\ ,\ \rangle_{TN})$ is a symmetric space. However, a simply connected symmetric space is the Riemannian product of a Euclidean space and a finite family of irreducible symmetric spaces (see [1, Theorem 7.76]). Thus, $(N,\langle\ ,\ \rangle_{TN})=(E,\langle\ ,\ \rangle_0)\times(N_1,\langle\ ,\ \rangle_1)\times\ldots\times(N_p,\langle\ ,\ \rangle_p)$ where $(E,\langle\ ,\ \rangle_0)$ is flat and the $(N_i,\langle\ ,\ \rangle_i)$ are irreducible symmetric spaces with non-negative sectional curvature. This implies that the N_i are compact and Einstein with positive scalar curvature. According to Proposition 4.1, the vanishing of the supracurvature of $(N, \langle\ ,\ \rangle_{TN}, k)$ implies the vanishing of the supracurvature of $(N_i, \langle\ ,\ \rangle_i, k)$ for $i = 1, \dots, p$.

Let $i \in \{1, \dots, p\}$ and denote by n_i the dimension of N_i. The symmetric space N_i can be identified to G/K, where G is the component of the identity of the group of isometries of $(N_i, \langle\ ,\ \rangle_i)$ and K is the isotropy at some point. Moreover, the Lie algebra $\mathfrak{g}$ of G has a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Since N_i is Einstein and irreducible, the metric in restriction to $\mathfrak{p}$ is proportional to the restriction of the Killing form (see [1]).

The vanishing of the supracurvature of $(N_i, \langle\ ,\ \rangle_i, k)$ implies, by virtue of the second formula in Proposition 4.3, $[\Phi_{[\mathfrak{p},\mathfrak{p}]}, \Phi_{\mathfrak{k}}^{\perp}] = 0$. This relation and the fact that $[\mathfrak{p}, \mathfrak{p}]$ is an ideal of $\mathfrak{k}$ imply that $\Phi_{[\mathfrak{p},\mathfrak{p}]}$ is an ideal of $\mathrm{so}(\mathfrak{p})$. However, if $\dim \mathfrak{p} \neq 4$, then the real Lie algebra $\mathrm{so}(\mathfrak{p})$ is simple (see [8, Theorem 6.105]), and in this case, $\Phi_{[\mathfrak{p},\mathfrak{p}]} = 0$ or $\Phi_{[\mathfrak{p},\mathfrak{p}]} = \mathrm{so}(\mathfrak{p})$. If $\Phi_{[\mathfrak{p},\mathfrak{p}]} = 0$, then $R^{N_i} = 0$, and we get the result. Otherwise, $\dim \mathfrak{k} \geq \dim \Phi_{\mathfrak{k}} \geq \dim \mathrm{so}(\mathfrak{p}) = \frac{n_i(n_i-1)}{2}$. Therefore:

$$\dim G = \dim \mathfrak{k} + n_i \geq \frac{n_i(n_i+1)}{2}.$$

However, the dimension of the group of isometries is always less or equal to $\frac{n_i(n_i+1)}{2}$ with equality when the manifold has constant curvature. Thus, $\dim G = \frac{n(n+1)}{2}$, and hence, N_i has constant curvature. If $\dim \mathfrak{p} = 4$, $(N_i, \langle\ ,\ \rangle_i)$ is a Einstein four dimensional homogeneous space with positive scalar curvature and according to the main result in [5], $(N_i, \langle\ ,\ \rangle_i)$ is isometric to $\mathbb{S}^4(r)$, $\mathbb{S}^2(r) \times \mathbb{S}^2(r)$ or $P^2(\mathbb{C})$. However, Proposition 4.4 shows that the supracurvature of $P^2(\mathbb{C})$ does not vanish and Proposition 4.2 shows that $\mathbb{S}^n(r)$ has vanishing supracurvature if and only if $r = \sqrt{\frac{k}{2}}$. This completes the proof. □

4.2. Geometry of $(E^{(r)}(M, k), h)$ When M is Locally Symmetric

The following proposition is a key step to apply Theorems 3.2 and 3.3 to $E(M, k)$.

Proposition 4.5. *If M is locally symmetric, then $\nabla^{M,E}(R^{\nabla^E}) = 0$.*

Proof. Assume that M is locally symmetric which is equivalent to $\nabla^M(K^M) = 0$. Note first that $\nabla^{M,E}(R^{\nabla^E}) = 0$ if and only if, for any curve $\gamma : [a, b] \longrightarrow M$, $V_1, V_2, V_3 : [a, b] \longrightarrow TM$ parallel vector fields along c and $F : [a, b] \longrightarrow \mathrm{so}(TM)$ parallel section along c, then $R^{\nabla^E}(V_1, V_2)V_3$ and $R^{\nabla^E}(V_1, V_2)F$ are parallel along c. However, $R^M(V_1, V_2)V_3$ is parallel, $H_{V_1}V_2$ and $H_{V_1}F$ are also parallel, and using (12), we can conclude. □

The following theorem is an immediate consequence of Theorems 3.2, 3.3, and Proposition 4.5.

Theorem 4.2. *1. If $(M, \langle\ ,\ \rangle_{TM})$ is locally symmetric and its sectional curvature is positive, then, for r sufficiently small, $(E^{(r)}(M, k), h)$ has nonnegative sectional curvature.*

2. *If M is compact with positive Ricci curvature or locally symmetric with positive Ricci curvature, then, for r sufficiently small, the Ricci curvature of $(E^{(r)}(M,k),h)$ is positive.*

When M has positive constant sectional curvature, one can apply Theorem 4.2, but in this case, we can apply Remark 2 to get a better result.

Theorem 4.3. *Let $(M, \langle\ ,\ \rangle_{TM})$ be a Riemannian manifold with positive constant sectional curvature c. Then, for k close to $\frac{2}{c}$, $(E^{(r)}(M,k),h)$ has non-negative sectional curvature.*

Proof. Let us find in this case a $\mathbf{K}$ as in (8). For any $X,Y,Z \in \Gamma(TM)$ and $F \in \Gamma(\mathrm{so}(TM))$, we have:

$$|R^{\nabla^E}(X,Y)(Z+F)| \leq |R^{\nabla^E}(X,Y)Z| + |R^{\nabla^E}(X,Y)F|.$$

From Proposition 4.2, we get that:

$$|R^{\nabla^E}(X,Y)Z| \leq 4|\varpi||X||Y||Z| \quad \text{and} \quad R^{\nabla^E}(X,Y)F = 2\varpi\left(F(X)\wedge Y + X \wedge F(Y)\right),$$

where $\varpi = \frac{1}{4}c(2-ck)$. Let us compute $|F(X)\wedge Y|$. Let $(X_i)_{i=1}^n$ be a local orthonormal frame of TM. Then:

$$\begin{aligned}|F(X)\wedge Y|^2 &= -k\mathrm{tr}((F(X)\wedge Y)^2)\\ &= k\sum_{i=1}^n \langle F(X)\wedge Y(X_i), F(X)\wedge Y(X_i)\rangle_{TM}\\ &= k\sum_{i=1}^n \langle \langle Y, X_i\rangle_{TM} F(X) - \langle F(X), X_i\rangle_{TM} Y,\\ &\qquad \langle \langle Y, X_i\rangle_{TM} F(X) - \langle F(X), X_i\rangle_{TM} Y\rangle_{TM}\\ &= 2k|F(X)|^2|Y|^2 + 2k\langle F(X), Y\rangle_{TM}^2 \leq 4k|F|^2|X|^2|Y|^2.\end{aligned}$$

Finally:

$$|R^{\nabla^E}(X,Y)(Z+F)| \leq 8k|\varpi||X||Y|(|Z|+|F|).$$

Therefore, we can take $\mathbf{K} = 8k|\varpi|$ which goes to zero when k goes to $\frac{2}{c}$. Thus, when k is close to $\frac{2}{c}$, the inequality (9) holds and we get the desired result. □

4.3. Riemannian Manifolds Whose $(E^{(r)}(M,k),h)$ is Einstein

It has been proved in [2] that $(T^{(r)}M,h)$ is Einstein if and only if $\dim M = 2$ and either M is flat or has constant curvature $\frac{1}{r^2}$. We have a more rich situation in the case of $(E^{(r)}(M,k),h)$.

Theorem 4.4. *Let $(M, \langle\ ,\ \rangle_{TM})$ be a connected Riemannian manifold. Then:*

1. *$(E^{(r)}(M,k),h)$ is Einstein with Einstein constant λ if and only if the Riemannian covering of $(M, \langle\ ,\ \rangle_{TM})$ is locally isometric to the Riemannian product $\mathbb{S}^p\left(\sqrt{\frac{k}{2}}\right)\times\ldots\times\mathbb{S}^p\left(\sqrt{\frac{k}{2}}\right)$ of q spheres of dimension p and radius $\sqrt{\frac{k}{2}}$ with:*

$$\lambda = \frac{2(p-1)}{k} = \frac{qp(qp+1)-4}{2r^2}.$$

2. $(E^{(r)}(M,k),h)$ *can never have constant sectional curvature.*

Proof. This is an immediate consequence of Theorems 3.1 and 4.1. □

4.4. Scalar Curvature of $(E^{(r)}(M,k),h)$

As an application of Theorem 3.6, we have the following result:

Theorem 4.5. *Suppose that* $(M,\langle\ ,\ \rangle_{TM})$ *has constant sectional curvature* c. *Then,* $(E^{(r)}(M,k),h)$ *has constant scalar curvature if and only if either* (*i*) $n=3$, (*ii*) $c=0$, *or* (*iii*) $c=\frac{2}{k}$.

Proof. The scalar curvature τ is given by, for $(x,Z+F)\in E^{(r)}(M,k)$:

$$\tau(x,Z+F)=n(n-1)c+\frac{1}{r^2}(m-1)(m-2)-\frac{1}{4}\xi_x(Z+F,Z+F),$$

where

$$\xi_x(Z+F,Z+F)=2\varpi^2(n-1)|Z+F|^2+2\varpi^2(n-3)|F|^2,\quad \varpi=\frac{1}{4}c(2-ck).$$

According to Theorem 3.6, τ is constant if and only if $n=3$ or $\varpi=0$ and we get the desired result. □

We end this subsection by giving all two-dimensional Riemannian manifolds $(M,\langle\ ,\ \rangle_{TM})$ for which $(E^{(r)}(M,k),h)$ has constant scalar curvature.

Proposition 4.6. *Let* $(M,\langle\ ,\ \rangle_{TM})$ *be a two-dimensional Riemannian manifold with curvature* $R^M(X,Y)=-CX\wedge Y$ *with* $C\in C^\infty(M)$. *Then, for any* $X,Y\in\Gamma(TM)$ *and* $F\in\Gamma(\mathrm{so}(TM))$:

$$R^{\nabla^E}(X,Y)Z=-\varpi X\wedge Y(Z)+\frac{1}{2}Z(C)X\wedge Y \quad \textit{and}$$
$$R^{\nabla^E}(X,Y)F=-\varpi[X\wedge Y,F]+k\langle F(X),Y\rangle_{TM}\mathrm{grad}(C),$$

where $\varpi=\frac{1}{2}C(2-kC)$ *and* $X\wedge Y$ *is the skew-symmetric endomorphism of* TM *given by:*

$$X\wedge Y(Z)=\langle Y,Z\rangle_{TM}X-\langle X,Z\rangle_{TM}Y.$$

Proof. According to (12):

$$R^{\nabla^E}(X,Y,Z)=R^M(X,Y,Z)+H_YH_XZ-H_XH_YZ-\frac{1}{2}\nabla^M_Z(K^M)(X\wedge Y),$$

where $H_XY=-\frac{1}{2}R^M(X,Y)=\frac{1}{2}CX\wedge Y$ and:

$$\begin{aligned}\langle H_XF,Y\rangle_{TM}&=-\frac{1}{2}k\ \mathrm{tr}(F\circ R^M(X,Y))\\ &=-\frac{1}{2}Ck\sum_{i=1}^n\langle F(X_i),X\wedge Y(X_i)\rangle_{TM}=-Ck\langle F(Y),X\rangle_{TM}.\end{aligned}$$

Thus $H_XF=CkF(X)$ and:

$$H_YH_XZ-H_XH_YZ=\frac{1}{2}C^2k(X\wedge Z(Y)-Y\wedge Z(X))=\frac{1}{2}C^2kX\wedge Y(Z).$$

Moreover:

$$\begin{aligned}\nabla^M_Z(K^M)(X\wedge Y) &= \nabla^M_Z(K^M(X\wedge Y)) - K^M(\nabla^M_Z X\wedge Y) - K^M(X\wedge\nabla^M_Z Y)\\ &= -\nabla^M_Z(CX\wedge Y) + C\nabla^M_Z X\wedge Y + CX\wedge\nabla^M_Z Y\\ &= -Z(C)X\wedge Y.\end{aligned}$$

By adding the expressions above, we get the first formula.

On the other hand:

$$R^{\nabla^E}(X,Y)F = \left\{(R^{\nabla^E}(X,Y)F)_{TM}\right\} + \left\{[R^M(X,Y),F] + H_Y H_X F - H_X H_Y F\right\},$$

where

$$\begin{aligned}\langle (R^{\nabla^E}(X,Y)F)_{TM}, Z\rangle_k &= -\langle R^{\nabla^E}(X,Y)Z, F\rangle_k\\ &= -\frac{1}{2}Z(C)\langle X\wedge Y, F\rangle_k\\ &= -\frac{k}{2}\langle \mathrm{grad}(C), Z\rangle_{TM}\sum_{i=1}^n \langle X\wedge Y(X_i), F(X_i)\rangle_{TM}\\ &= k\langle F(X), Y\rangle_{TM}\langle \mathrm{grad}(C), Z\rangle_{TM}.\end{aligned}$$

Thus, $(R^{\nabla^E}(X,Y)F)_{TM} = k\langle F(X),Y\rangle_{TM}\mathrm{grad}(C)$. Furthermore:

$$\begin{aligned}[H_Y, H_X]F &= Ck(H_Y F(X) - H_X F(Y))\\ &= -\frac{1}{2}Ck(R^M(Y,F(X)) + R^M(F(Y),X)) = -\frac{1}{2}C^2k([F, X\wedge Y]).\end{aligned}$$

This completes the proof. □

Theorem 4.6. *Let $(M,\langle\ ,\ \rangle_{TM})$ be a two-dimensional Riemannian manifold. Then, $(E^{(r)}(M,k),h)$ has constant scalar curvature if and only if $(M,\langle\ ,\ \rangle_{TM})$ has constant curvature $C=0$ or $C=\frac{2}{k}$.*

Proof. We choose an orthonormal basis (X_1,X_2), such that $Ric^M(X_i) = \rho_i X_i$, and we put $F_{12} = \frac{1}{\sqrt{2k}}X_1\wedge X_2$. The family (X_1,X_2,F_{12}) is a local orthonormal frame of $E(M,k)$. We have for any vector field Z:

$$R^{\nabla^E}(X_1,X_2)Z = -\frac{1}{2}C(2-kC)X_1\wedge X_2(Z) \quad \text{and}$$

$$R^{\nabla^E}(X_1,X_2)F_{12} = -\sqrt{\frac{k}{2}}\mathrm{grad}(C).$$

Then:

$$\begin{aligned}\xi(X_i,X_i) &= 2\varpi^2 + k(X_i(C))^2,\ i=1,2\\ \xi(F_{12},F_{12}) &= k|\mathrm{grad}(C)|^2,\\ \xi(X_1,X_2) &= kX_1(C)X_2(C),\\ \xi(X_i,F_{12}) &= \varpi^2\sqrt{2k}\langle \mathrm{grad}(C), X_1\wedge X_2(X_i)\rangle,\ i=1,2.\end{aligned}$$

On the other hand:

$$|R^{\nabla^E}|^2 = 4\varpi^2 + 2k|\mathrm{grad}(C)|^2.$$

Suppose that $(E^{(r)}(M,k),h)$ has constant scalar curvature. The equation (10) gives for F_{12}:

$$4\varpi^2 - k|\mathrm{grad}(C)|^2 = 0.$$

We eliminate $|\mathrm{grad}(C)|^2$ in Eq. (11), to find:

$$24C - 3C^2(2-kC)^2 = \text{constant}.$$

Therefore, C must be constant and $C=0$ or $C=\frac{2}{k}$. □

5. The Sasaki Metric with Positive Scalar Curvature on the Unit Bundle of Three-Dimensional Unimodular Lie Groups

The purpose of this section is to prove the following result.

Theorem 5.1. *Let G be a three-dimensional connected unimodular Lie group. Then, there exists a left invariant Riemannian metric on G, such that $(T^{(1)}G, h)$ has positive scalar curvature.*

Proof. Let G be a connected three-dimensional unimodular Lie group with left invariant metric. Using an argument developed in [9], there exists an orthonormal basis (X_1, X_2, X_3) of left invariant vector fields, such that:

$$[X_1, X_2] = mX_3, \quad [X_1, X_3] = nX_2 \quad \text{and} \quad [X_2, X_3] = pX_1.$$

By straightforward computation using the Koszul formula, we get that the Levi–Civita connexion in this case is given by:

$$\nabla_{X_1} = \frac{1}{2}(-m+n+p)X_2 \wedge X_3, \nabla_{X_2} = \frac{1}{2}(m+n+p)X_1 \wedge X_3, \quad \text{and}$$
$$\nabla_{X_3} = \frac{1}{2}(m+n-p)X_1 \wedge X_2.$$

From this formula, we deduce that:

$$\begin{aligned} R^G(X_1, X_2) &= \nabla_{[X_1,X_2]} - [\nabla_{X_1}, \nabla_{X_2}] \\ &= \frac{1}{2}m(m+n-p)X_1 \wedge X_2 \\ &\quad - \frac{1}{4}(-m+n+p)(m+n+p)[X_2 \wedge X_3, X_1 \wedge X_3] \\ &= \frac{1}{2}\left[m(m+n-p) - \frac{1}{2}((n+p)^2 - m^2)\right] X_1 \wedge X_2 \\ &= \left(\frac{3}{4}m^2 + \frac{1}{2}m(n-p) - \frac{1}{4}(n+p)^2\right) X_1 \wedge X_2. \end{aligned}$$

In a similar way, we get:

$$R^G(X_1, X_2) = \left(\frac{3}{4}m^2 + \frac{1}{2}m(n-p) - \frac{1}{4}(n+p)^2\right) X_1 \wedge X_2 = \mu_{12} X_1 \wedge X_2,$$
$$R^G(X_1, X_3) = \left(\frac{3}{4}n^2 + \frac{1}{2}n(p+m) - \frac{1}{4}(m-p)^2\right) X_1 \wedge X_3 = \mu_{13} X_1 \wedge X_3,$$
$$R^G(X_2, X_3) = \left(\frac{3}{4}p^2 + \frac{1}{2}p(n-m) - \frac{1}{4}(m+n)^2\right) X_2 \wedge X_3 = \mu_{23} X_2 \wedge X_3.$$

By virtue of Proposition 2.4, the scalar curvature of the unit tangent sphere bundle $(T^{(1)}G, h)$ of G equipped with the Sasaki metric is given by, for any $a \in T^{(1)}G$:

$$\tau(a) = 1 - \mu_{12} - \mu_{13} - \mu_{23} - \frac{1}{4}\xi(a,a),$$

where $\xi(a,a) = \sum_{i,j=1}^{3} |R^G(X_i, X_j)a|^2$. We have:

$$\xi(X_1, X_1) = 2(\mu_{12}^2 + \mu_{13}^2), \quad \xi(X_2, X_2) = 2(\mu_{12}^2 + \mu_{23}^2) \quad \text{and} \quad \xi(X_3, X_3) = 2(\mu_{13}^2 + \mu_{23}^2).$$

Put

$$\begin{aligned}
\lambda_1 &= 2\tau(X_1) = \mu_{12}^2 + \mu_{13}^2 + 2(\mu_{12} + \mu_{13} + \mu_{23} - 1),\\
\lambda_2 &= 2\tau(X_2) = \mu_{12}^2 + \mu_{23}^2 + 2(\mu_{12} + \mu_{13} + \mu_{23} - 1),\\
\lambda_3 &= 2\tau(X_3) = \mu_{13}^2 + \mu_{23}^2 + 2(\mu_{12} + \mu_{13} + \mu_{23} - 1).
\end{aligned}$$

Then, the scalar curvature τ of $(T^{(1)}G, h)$ is positive if and only if $\lambda_i < 0$ for all $i \in \{1, 2, 3\}$. There are values for parameters m, n and p for which λ_i is negative for all $i \in \{1, 2, 3\}$.

1. For $m = \frac{1}{2}$, $n = \frac{1}{3}$ and $p = \frac{1}{4}$: In this case, the Lie group G is isomorphic to the group $SO(3)$, or $SU(2)$:
$$\lambda_1 = -\frac{267223}{165888}, \quad \lambda_2 = -\frac{269771}{165888} \quad \text{and} \quad \lambda_3 = -\frac{266131}{165888}.$$
2. For $m = \frac{1}{2}$, $n = \frac{1}{3}$, and $p = -\frac{1}{4}$: $G \cong SL(2,\mathbb{R})$ or $O(1, 2)$:
$$\lambda_1 = -\frac{243799}{165888}, \quad \lambda_2 = -\frac{241979}{165888} \quad \text{and} \quad \lambda_3 = -\frac{260179}{165888}.$$
3. For $m = \frac{1}{2}$, $n = \frac{1}{3}$, and $p = 0$: $G \cong E(2)$:
$$\lambda_1 = -\frac{16411}{10368}, \quad \lambda_2 = -\frac{16211}{10368} \quad \text{and} \quad \lambda_3 = -\frac{16711}{10368}.$$
4. For $m = \frac{1}{2}$, $n = \frac{1}{3}$, and $p = 0$: $G \cong E(1, 1)$:
$$\lambda_1 = -\frac{20491}{10368}, \quad \lambda_2 = -\frac{20531}{10368} \quad \text{and} \quad \lambda_3 = -\frac{20551}{10368}.$$
5. For $m = \frac{1}{2}$, $n = -\frac{1}{3}$, and $p = 0$: $G \cong H(3,\mathbb{R})$:
$$\lambda_1 = -\frac{7}{8}, \quad \lambda_2 = -\frac{7}{8} \quad \text{and} \quad \lambda_3 = -\frac{11}{8}.$$

□

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