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MATHEMATISCHE NACHRICHTEN

On para-Kähler Lie algebroids and contravariant pseudo-Hessian structures

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Abstract

In this paper, we generalize all the results obtained on para-Kähler Lie algebras in [3] to para-Kähler Lie algebroids. In particular, we study exact para-Kähler Lie algebroids as a generalization of exact para-Kähler Lie algebras. This study leads to a natural generalization of pseudo-Hessian manifolds, we call them contravariant pseudo-Hessian Manifolds. Contravariant pseudo-Hessian manifolds have many similarities with Poisson manifolds. We explore these similarities which, among others, leads to a powerful machinery to build examples of non trivial pseudo-Hessian structures. Namely, we will show that given a finite dimensional commutative and associative algebra $(\mathcal{A}, .)$, the orbits of the action Φ of $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(a, \mu) = \exp\left(L_a^*\right)(\mu)$ are pseudo-Hessian manifolds, where $L_a(b) = a.b$. We illustrate this result by considering many examples of associative commutative algebras and show that the resulting pseudo-Hessian manifolds are very interesting.

KEYWORDS

associative commutative algebras, left symmetric algebroids, para-Kähler Lie algebroids, pseudo-Hessian manifolds, symplectic Lie algebroids

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1 | INTRODUCTION

Recall that a Lie algebroid is a vector bundle $A \longrightarrow M$ together with an anchor map $\rho : A \longrightarrow TM$ and a Lie bracket $[\ ,\ ]_A$ on $\Gamma(A)$ such that, for any $a, b \in \Gamma(A)$, $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b.$$

Lie algebroids are now a central notion in differential geometry and constitute an active domain of research. They have many applications in various part of mathematics and physics (see for instance [6–8,18]). It is a well-established fact that many classical geometrical structures involving the tangent bundle of a manifold (which has a natural structure of Lie algebroid) can be generalized to the context of Lie algebroids. Thus the notions of connections on Lie algebroids, symplectic Lie algebroids, pseudo-Riemannian Lie algebroids and so on are now usual notions in differential geometry with many applications in physics (see for instance [4,10]). On the other hand, it is important to point out that Lie algebroids generalize also Lie algebras and, for instance, if one obtains a result on the curvature of pseudo-Riemannian Lie algebroids this result holds for the curvature of pseudo-Euclidean Lie algebras and hence for the curvature of left invariant pseudo-Riemannian metrics on Lie groups.

In this paper, we study para-Kähler Lie algebroids as a generalization of both para-Kähler manifolds and left invariant para-Kähler structures on Lie groups. Para-Kahler Lie algebroids were introduced by Leichtnam, Tang and Weinstein in [14] and have been studied in [16,17]. In our study, we adopt a different point of view from [16,17], we recover some of their results and give new ones.

A *para-Kähler* structure on a manifold M is a pair (g, K) where g is a pseudo-Riemannian metric and K is a parallel (with respect to the Levi–Civita connection of g) skew-symmetric endomorphism field satisfying $K^2 = Id_{TM}$. The paper [9] contains a survey on para-Kähler geometry and contains many references. When the manifold is a Lie group G, the metric and the para-complex structure are considered left-invariant, they are both determined by their restrictions to the Lie algebra $\mathfrak{g}$ of G. In a such situation, $(\mathfrak{g}, g_e, K_e)$ is called *para-Kähler Lie algebra*.

A para-Kähler Lie algebroid is a Lie algebroid (A, M, ρ) together with a pseudo-Euclidean product $\langle\ ,\ \rangle$ on A and a bundle isomorphism $K : A \longrightarrow A$ such that $K^2 = \mathrm{Id}_A$, K is skew-symmetric with respect to $\langle\ ,\ \rangle$ and $\nabla K = 0$ where ∇ is the Levi–Civita connection associated to $\langle\ ,\ \rangle$.

The authors realized a complete study of para-Kähler Lie algebras in [3] and, our first motivation was to generalize the results obtained in this study to the context of para-Kähler Lie algebroids. This has been done successfully and constitutes the first part of this paper. The generalization was not straightforward and many new phenomenas appeared due to the anchor. Moreover, as it happens often in mathematics, during our investigations when studying a special class of para-Kähler Lie algebroids, we came across a new structure which turned out to be a natural generalization of the notion of pseudo-Hessian manifolds. We will call it contravariant pseudo-Hessian manifold and some of its remarkable properties constitute the second part of this paper. Let us give briefly the definition of this structure and some of its properties.

Recall that a pseudo-Hessian manifold is a locally affine manifold (M, D) endowed with a pseudo-Riemannian metric such that g is locally given by $Dd\phi$ where ϕ is a function. This is equivalent to $S = Dg$ is totally symmetric. Pseudo-Hessian geometry is an active domain of research which has many applications in economic theory, in system modeling and optimization as well as in statistical theory. One can consult [1,22] to find out more about this geometry and its origins. A contravariant pseudo-Hessian manifold is triple (M, D, h) where (M, D) is a locally affine manifold and h is a symmetric bivector field such that the tensor $T \in \Gamma\left(\otimes^3 TM\right)$ given by $T(\alpha, \beta, \gamma) = D_{h_\#(\alpha)} h(\beta, \gamma)$ is totally symmetric, where $h_\# : T^*M \longrightarrow TM$ is given by $\beta\left(h_\#(\alpha)\right) = h(\alpha, \beta)$. When h is invertible, $\left(M, D, h^{-1}\right)$ is a pseudo-Hessian manifold. There are many similarities between Poisson manifolds as a generalization of symplectic manifolds and contravariant pseudo-Hessian manifolds as a generalization of pseudo-Hessian manifolds. Indeed, if (M, D, h) is a contravariant pseudo-Hessian manifold then $\mathrm{Im}\, h_\#$ is an integrable distribution and defines a singular foliation whose leaves are pseudo-Hessian manifolds. There is an analogue of Darboux–Weinstein theorem near a regular point (see Theorem 6.11) and $\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ and the bracket $[\ ,\ ]_h$ on $\Omega^1(M)$ given by

$$\prec \mathcal{D}_\alpha \beta, X \succ = \nabla_X h(\alpha, \beta) + \prec \nabla^*_{h_\#(\alpha)} \beta, X \succ \quad \text{and} \quad [\alpha, \beta]_{\mathcal{D}} = \nabla^*_{h_\#(\alpha)} \beta - \nabla^*_{h_\#(\beta)} \alpha,$$

satisfy $\left(T^*M, M, h_\#, [\ ,\ ]_h\right)$ is a Lie algebroid and $\mathcal{D}$ is a torsionless flat connection for this Lie algebroid. Moreover, for any $x \in M$, $\mathcal{A}_x = \ker h_\#(x)$ carries a natural structure of commutative associative algebra. On the other hand, let $(\mathcal{A}, .)$ be a commutative associative algebra $(\mathcal{A}, .)$. Denote by D the canonical affine connection on $\mathcal{A}^*$ and define the symmetric bivector field h on $\mathcal{A}^*$ by

$$h(\alpha, \beta)(\mu) = \prec \mu, \alpha(\mu).\beta(\mu) \succ, \quad \alpha, \beta \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*, \mathcal{A}),\ \mu \in \mathcal{A}^*.$$

Then $(\mathcal{A}^*, D, h)$ is a contravariant pseudo-Hessian manifolds and the leaves of the foliation associated to $\mathrm{Im}\, h_\#$ are the orbits of the action Φ of $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(a, \mu) = \exp(L_a^*)(\mu)$ where $L_a(b) = a.b$ (see Theorem 7.3). Thus the orbits of Φ are pseudo-Hessian manifolds. This gives powerful machinery to build examples of pseudo-Hessian structures. We will show that the pseudo-Hessian structure of these orbits is not trivial since their Hessian curvature is not zero. We illustrate this result by considering many examples of associative commutative algebras and we will show the pseudo-Hessian structures of these orbits are not trivial since their Hessian curvatures are not zero.

We give now the organization of this paper. In Section 2, we recall some basic facts about Lie algebroids and connections on Lie algebroids. A Lie algebroid with a torsionless and flat connection was called Koszul–Vinberg algebroid in [20] and left symmetric algebroid in [15]. These algebroids play a central role in the study of para-kähler Lie algebroids. We adopt the terminology of left symmetric algebroids as in [15] and we give some of their geometrical properties we will use later. In Section 3, we start the study of para-Kähler Lie algebroids and the main result here is Theorem 3.7 which states that a para-Kähler Lie algebroid is obtained from two left symmetric algebroids on two dual vector bundles compatible in some sense. In

Sections 4 and 5, we study exact para-Kähler Lie algebroids and the related notions of $\mathbb{S}$-matrices and quasi-S-matrices on a left symmetric algebroid. If (M, D) is an affine manifold then (TM, M, D) becomes a left symmetric algebroid and a symmetric $\mathbb{S}$-matrix on (TM, M, D) defines a contravariant pseudo-Hessian structure on (M, D). Section 6 is devoted to the study of this new structure. In Section 7, we study linear contravariant pseudo-Hessian manifolds and we give many examples.

Notations: Let $A \longrightarrow M$ be a vector bundle and let $F : A \longrightarrow A$ be a bundle endomorphism. We denote by $\Gamma(A)$ the space of its sections and by $F^* : A^* \longrightarrow A^*$ the dual endomorphism. For any $X \in A_x$ and $\alpha \in A_x^*$, we denote $\alpha(X)$ by $\prec \alpha, X \succ$. We consider the vector bundle $\Phi(A) := A \oplus A^*$ endowed with the two nondegenerate bilinear forms $\langle\ ,\ \rangle_0$, Ω_0 and $K_0 : \Phi(A) \longrightarrow \Phi(A)$ given by

$$\langle u+\alpha, v+\beta\rangle_0 = \prec \alpha, v \succ + \prec \beta, u \succ,$$

$$\Omega_0(u+\alpha, v+\beta) = \prec \beta, u \succ - \prec \alpha, v \succ \quad \text{and} \quad K_0(u+\alpha) = u-\alpha. \tag{1.1}$$

Let $\omega \in \Gamma\left(\wedge^2 A^*\right)$ which is nondegenerate. We denote by $\flat : A \longrightarrow A^*$ the bundle isomorphism given by $\flat(v) = \omega(v, .)$.

2 | LIE ALGEBROIDS, CONNECTIONS, LEVI–CIVITA CONNECTIONS, LEFT SYMMETRIC ALGEBROIDS AND SYMPLECTIC LIE ALGEBROIDS

Through this paper, we will use some well-known basic notions, namely, anchored bundles, Lie algebroids, connections on Lie algebroids and symplectic Lie algebroids. In this section, we recall the definitions of these notions, we give some of their properties and some basic examples. For more details one can consult [6–8,11,18]. We recall the definition of left symmetric algebroids known also as Koszul–Vinberg algebroids. They are Lie algebroids which will play a central role in the study of para-Kähler Lie algebroids. We give some of their properties we will use later.

Lie algebroids and their immediate properties An *anchored vector bundle* is a triple (A, M, ρ) where $p : A \longrightarrow M$ is a vector bundle and $\rho : A \longrightarrow TM$ is a bundle homomorphism called anchor. An homomorphism between two anchored vector bundles (A, M, ρ) and (A', M, ρ') is a bundle homomorphism $\phi : A \longrightarrow A'$ such that $\rho = \rho' \circ \phi$. The sum of two anchored bundles (A, M, ρ) and (B, M, ρ') is the anchored bundle $(A \oplus B, M, \rho \oplus \rho')$.

Let (A, M, ρ) be an anchored vector bundle. A bracket on $\Gamma(A)$ is skew-symmetric $\mathbb{R}$-bilinear map

$$[\ ,\ ]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A).$$

It is called *anchored* if for any $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$,

$$[a, fb]_A = f[a, b]_A + \rho(a)(f)b. \tag{2.1}$$

By using a classical argument, we can deduce from this relation that $[\ ,\ ]_A$ is local in the sense that if a section a vanishes on an open set U then for any $b \in \Gamma(A)$, $[a, b]_A$ vanishes on U. The *torsion* of $[\ ,\ ]_A$ is the map $\tau_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \longrightarrow \mathcal{X}(M)$ given by

$$\tau_{[\ ,\ ]_A}(a, b) = \rho([a, b]_A) - [\rho(a), \rho(b)]. \tag{2.2}$$

$\tau_{[\ ,\ ]_A}$ is $\mathbb{R}$-bilinear, skew-symmetric and, for any $f \in C^\infty(M)$,

$$\tau_{[\ ,\ ]_A}(fa, b) = \tau_{[\ ,\ ]_A}(a, fb) = f\tau_{[\ ,\ ]_A}(a, b).$$

So $\tau_{[\ ,\ ]_A} \in \Gamma\left(\wedge^2 A^* \otimes TM\right)$. In order to study under which conditions $[\ ,\ ]_A$ is a Lie bracket, we introduce the Jacobiator of $[\ ,\ ]_A$ as $J_{[\ ,\ ]_A} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$J_{[\ ,\ ]_A}(a, b, c) = \left[[a, b]_A, c\right]_A + \left[[b, c]_A, a\right]_A + \left[[c, a]_A, b\right]_A.$$

J is $\mathbb{R}$-trilinear and skew-symmetric. Thus $[\ ,\ ]_A$ is a Lie bracket if and only if $J_{[\ ,\ ]_A} = 0$. However, this equation is not tensorial and may be very difficult to check in concrete situations. Nevertheless, for any $a, b, c \in \Gamma(A)$ and any $f \in C^\infty(M)$,

$$J_{[\ ,\ ]_A}(a, b, fc) = fJ_{[\ ,\ ]_A}(a, b, c) + \tau_{[\ ,\ ]_A}(a, b)(f)c. \tag{2.3}$$

This relation shows that $J_{[\,,\,]_A}$ is local and if $\tau_{[\,,\,]_A}$ vanishes then $J_{[\,,\,]_A}$ becomes a tensor, namely, $J_{[\,,\,]_A} \in \Gamma\left(\wedge^3 A^* \otimes TM\right)$. This shows also that if $J_{[\,,\,]_A}$ vanishes then $\tau_{[\,,\,]_A}$ does. The following proposition is an immediate consequence of (2.3) and gives us an useful way of checking if an anchored bracket is actually a Lie bracket.

Proposition 2.1. *Let (A, M, ρ) be an anchored bundle and let $[\,,\,]_A$ be an anchored bracket on $\Gamma(A)$. Then the following assertions are equivalent:*

(i) *$\left(\Gamma(A), [\,,\,]_A\right)$ is a Lie algebra, i.e., J_A vanishes identically.*

(ii) *For any $x \in M$ there exists an open set U of M containing x and a basis of sections $(a_1, \dots, a_r)$ over U such that*

$$J_{[\,,\,]_A}\left(a_i, a_j, a_k\right) = 0 \quad \textit{and} \quad \tau\left(a_i, a_j\right) = 0, \quad 1 \le i < j < k \le r.$$

Definition 2.2. A *Lie algebroid* is an anchored vector bundle (A, M, ρ) together with an anchored bracket $[\,,\,]_A$ satisfying (i) or (ii) of Proposition 2.1. A Lie algebroid is called transitive if its anchor map is onto on every point and it is called regular if the rank of ρ is constant on M.

There are some well-known properties of a Lie algebroid $\left(A, M, \rho, [\,,\,]_A\right)$.

(a) The induced map $\rho : \Gamma(A) \longrightarrow \mathcal{X}(M)$ is a Lie algebras homomorphism.

(b) The smooth distribution $\mathrm{Im}\,\rho$ is integrable in the sense of Sussmann [24] and, for any leaf L of $\mathrm{Im}\,\rho$, $\left(A_{|L}, L, \rho, [\,,\,]_A\right)$ is a transitive Lie algebroid.

(c) For any $x \in M$, there is an induced Lie bracket say $[\,,\,]_x$ on $\mathfrak{g}_x = \ker(\rho_x) \subset A_x$ which makes it into a finite dimensional Lie algebra.

(d) The map $d_A : \Gamma(\wedge A^*) \longrightarrow \Gamma(\wedge A^*)$ by

$$d_A Q\left(a_1, \dots, a_p\right) = \sum_{i=1}^{p} (-1)^{i+1} \rho(a_i).Q\left(a_1, \dots, \hat{a}_i, \dots, a_p\right) - \sum_{1 \le i < j \le p} (-1)^{i+j+1} Q\left([a_i, a_j]_A, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_p\right),$$

is a differential, i.e., $d_A^2 = 0$. In particular, for any $a, b \in \Gamma(A)$, $f \in C^\infty(M)$ and $Q \in \Gamma(\wedge A^*)$,

$$d_A f(a) = \rho(a)(f) \quad \text{and} \quad d_A Q(a, b) = \rho(a).Q(b) - \rho(b).Q(a) - Q([a, b]_A).$$

These two relations show that there is a correspondence between Lie algebroids structure on (A, M) and differentials on $\Gamma(\wedge A^*)$.

(e) The bracket $[\,,\,]_A$ extends to a new bracket denoted in the same way on the sections of $\wedge A = A \oplus \ldots \oplus \wedge^{\mathrm{rank} A} A$ called *generalized Schouten–Nijenhuis bracket*. Its properties are the same as those of the usual Schouten–Nijenhuis bracket and the anchor extends also to give a map $\rho : \wedge A \longrightarrow \wedge TM$ which preserves Schouten–Nijenhuis brackets. It is important here to point out that if $\Pi \in \Gamma\left(\wedge^2 A\right)$ satisfies $[\Pi, \Pi]_A = 0$ then $\pi = \rho(\Pi)$ is a Poisson tensor on M.

Some examples of Lie algebroids

1. The basic example of a Lie algebroid over M is the tangent bundle itself, with the identity mapping as anchor.

2. Every finite dimensional Lie algebra is a Lie algebroid over a one point space.

3. Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket.

4. Let (M, π) be a Poisson manifold. The bivector field π defines a bundle homomorphism $\pi_\# : T^*M \longrightarrow TM$ and a bracket on $\Omega^1(M)$ by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\#(\alpha)}\beta - \mathcal{L}_{\pi_\#(\beta)}\alpha - d\pi(\alpha, \beta)$$

such that $\left(T^*M, M, \pi_\#, [\,,\,]_\pi\right)$ is a Lie algebroid.

5. Let $\mathfrak{g} \xrightarrow{\tau} \mathcal{X}(M)$ be an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M, i.e., a morphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Consider $\left(A, M, \rho, [\ ,\ ]_A\right)$, where $A = M \times \mathfrak{g}$ as a trivial bundle and

$$\rho((m,\xi)) = \tau(\xi)(m) \quad \text{and} \quad [\xi,\eta]_A = \mathcal{L}_{\rho(\xi)}\eta - \mathcal{L}_{\rho(\eta)}\xi + [\xi,\eta]_{\mathfrak{g}}, \quad \eta,\xi \in \Gamma(A) = C^\infty(M,\mathfrak{g}).$$

By using (ii) of Proposition 2.1, it is easy to check that $\left(A, M, \rho, [\ ,\ ]_A\right)$ is a Lie algebroid.

Connections on Lie algebroids Given a Lie algebroid $\left(A, M, \rho, [\ ,\ ]_A\right)$, an A-connection on a vector bundle $E \longrightarrow M$ is a $\mathbb{R}$-bilinear operator $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying:

$$\nabla_{fa} s = f\,\nabla_a s \quad \text{and} \quad \nabla_a(fs) = f\nabla_a s + \rho(a)(f)s,$$

for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$. We shall call A-connections on the vector bundle $A \longrightarrow M$ *linear A-connections*. The curvature of an A-connection ∇ on E is formally identical to the usual definition

$$R(a,b)s = \nabla_a\nabla_b s - \nabla_b\nabla_a s - \nabla_{[a,b]_A} s,$$

where $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$. The connection ∇ is called flat if R vanishes identically. The dual of ∇ is the A-connection ∇^* on E^* given by

$$\prec \nabla^*_a\alpha, s \succ = \rho(a).\prec \alpha, s\succ - \prec \alpha, \nabla_a s \succ, \tag{2.4}$$

for any $a \in \Gamma(A), s \in \Gamma(E), \alpha \in \Gamma(E^*)$.

There is a notion of parallel transport associated to an A-connection ∇ on $q : E \longrightarrow M$. An A-path is a curve $\alpha : [a,b] \longrightarrow A$ such that

$$\forall t \in [a,b], \quad \rho(\alpha(t)) = c'(t),$$

where $c = p \circ \alpha : [a,b] \longrightarrow M$. Given an A-path $\alpha : [a,b] \longrightarrow A$, we denote by $\Gamma_\alpha(E)$ the vector space of curves $s : [a,b] \longrightarrow E$ such that $q \circ s = p \circ \alpha$. The connection ∇ defines a unique derivative $\nabla^\alpha : \Gamma_\alpha(E) \longrightarrow \Gamma_\alpha(E)$ and a parallel transport $\tau_\alpha : E_{c(a)} \longrightarrow E_{c(b)}$ given by $\tau_\alpha(s_1) = s(a)$ where $s \in \Gamma_\alpha(E)$ is uniquely determined by $\nabla^\alpha s = 0$ and $s(a) = s_1$.

The canonical connection on the adjoint bundle Let $\left(A, M, \rho, [\ ,\ ]_A\right)$ be a Lie algebroid such that ρ has a constant rank over M. Then *the adjoint bundle* $\mathfrak{g} = \ker\rho \longrightarrow M$ is a vector bundle of Lie algebras. Define $\nabla^{\mathfrak{g}} : \Gamma(A) \times \Gamma(\mathfrak{g}) \longrightarrow \Gamma(\mathfrak{g})$ by

$$\nabla^{\mathfrak{g}}_a s = [a,s]_A. \tag{2.5}$$

This defines a A-connection on $\mathfrak{g}$ satisfying, for any $a, b \in \Gamma(A)$, $s_1, s_2 \in \Gamma(\mathfrak{g})$,

$$R^{\nabla^{\mathfrak{g}}}(a,b) = 0 \quad \text{and} \quad \nabla^{\mathfrak{g}}_a[s_1,s_2]_A = \left[\nabla^{\mathfrak{g}}_a s_1, s_2\right]_A + \left[s_1, \nabla^{\mathfrak{g}}_a s_2\right]_A, \tag{2.6}$$

for any $a \in \Gamma(A)$ and $s_1, s_2 \in \Gamma(\mathfrak{g})$. For any $x, y \in M$ lying in the same leaf of the characteristic foliation, there exists an A-path $\alpha : [a,b] \longrightarrow A$ such that $p \circ \alpha(a) = x$ and $p \circ \alpha(b) = y$. The second relation in (2.6) shows that the parallel transport $\tau_\alpha : \mathfrak{g}_x \longrightarrow \mathfrak{g}_y$ is an isomorphism of Lie algebras.

The Levi–Civita connection of a pseudo-Riemannian Lie algebroid A pseudo-Riemannian metric of signature (p,q) on a Lie algebroid $\left(A, M, \rho, [\ ,\ ]_A\right)$ is the data, for any $x \in M$, of a nondegenerate product $\langle\ ,\ \rangle_x$ of signature (p,q) on the fiber A_x such that, for any local sections a, b of A, the function $\langle a, b\rangle$ is smooth. A Lie algebroid with a pseudo-Riemannian metric is called *pseudo-Riemannian Lie algebroid.*

The most interesting fact about pseudo-Riemannian Lie algebroids is the existence on the analogous of the Levi–Civita connection. Indeed, if $\langle\ ,\ \rangle$ is a pseudo-Riemannian metric on a Lie algebroid $\left(A, M, \rho, [\ ,\ ]_A\right)$ then the formula

$$2\langle\nabla_a b, c\rangle = \rho(a).\langle b,c\rangle + \rho(b).\langle a,c\rangle - \rho(c).\langle a,b\rangle + \langle [c,a]_A, b\rangle + \langle [c,b]_A, a\rangle + \langle [a,b]_A, c\rangle$$

defines a linear A-connection which is characterized by the two following properties:

(i) ∇ is metric, i.e., $\rho(a).\langle b,c\rangle = \langle \nabla_a b, c\rangle + \langle b, \nabla_a c\rangle$,

(ii) ∇ is torsion free, i.e., $\nabla_a b - \nabla_b a = [a,b]_A$.

We call ∇ the *Levi–Civita A-connection* associated to $\langle\ ,\ \rangle$. Moreover, $\langle\ ,\ \rangle$ and ρ define a symmetric bivector field $h \in \Gamma(TM \otimes TM)$ by

$$h(\alpha,\beta) = \langle \# \circ \rho^*(\alpha), \# \circ \rho^*(\beta)\rangle = \prec \beta, \rho \circ \# \circ \rho^*(\alpha) \succ = \prec \alpha, \rho \circ \# \circ \rho^*(\beta) \succ, \tag{2.7}$$

where $\# : A^* \longrightarrow A$ is the isomorphism associated to $\langle\ ,\ \rangle$. The following relations are easy to check:

$$\mathfrak{g}_x^\perp = \# \circ \rho^*\left(T_x^*M\right),\ \mathfrak{g}_x \cap \mathfrak{g}_x^\perp = \# \circ \rho^*\left(\ker h_x\right) \quad \text{and} \quad \ker \rho_x^* \subset \ker h_x,\ x \in M,\ \mathfrak{g}_x = \ker \rho_x.$$

These relations show that h_x is nondegenerate if and only if $\mathfrak{g}_x$ is $\langle\ ,\ \rangle$-nondegenerate and ρ_x is onto.

Symplectic Lie algebroids A *symplectic* form on a Lie algebroid $\left(A, M, \rho, [\ ,\ ]_A\right)$ is a bilinear skew-symmetric nondegenerate form $\omega \in \Gamma\left(\wedge^2 A^*\right)$ such that, for any $a,b,c \in \Gamma(A)$,

$$d_A\omega(a,b,c) := \rho(a).\omega(b,c) + \rho(b).\omega(c,a) + \rho(c).\omega(a,b) - \omega([a,b]_A, c) - \omega([b,c]_A, a) - \omega([c,a]_A, b) = 0.$$

We call $\left(A, M, \rho, [\ ,\ ]_A, \omega\right)$ a *symplectic Lie algebroid*. There is a natural Poisson structure on M associated to $\left(A, M, \rho, [\ ,\ ]_A, \omega\right)$.

Proposition 2.3. *Let $\left(A, M, \rho, [\ ,\ ]_A, \omega\right)$ be a symplectic Lie algebroid and let $\flat : A \longrightarrow A^*$, $a \mapsto \omega(a, .)$. Then the bivector field π given, for any $\alpha, \beta \in \Omega^1(M)$, by*

$$\pi(\alpha,\beta) = \omega\left(\flat^{-1} \circ \rho^*(\alpha), \flat^{-1} \circ \rho^*(\beta)\right) = \prec \beta, \rho \circ \flat^{-1} \circ \rho^*(\alpha) \succ, \tag{2.8}$$

is a Poisson tensor. Moreover, $\flat^{-1} \circ \rho^* : T^*M \longrightarrow A$ is a Lie algebroid homomorphism, where T^*M is endowed with the Lie algebroid structure associated to π.

Proof. It is standard and one can see for instance [6]. □

Let $\left(A, M, \rho, [\ ,\ ]_A, \omega\right)$ be a symplectic Lie algebroid, as in the case of a pseudo-Riemannian Lie algebroid, we have

$$\mathfrak{g}_x^\omega = \flat^{-1} \circ \rho^*\left(T_x^*M\right),\ \mathfrak{g}_x \cap \mathfrak{g}_x^\omega = \flat^{-1} \circ \rho^*\left(\ker \pi_x\right) \quad \text{and} \quad \ker \rho_x^* \subset \ker \pi_x,\ x \in M,\ \mathfrak{g}_x = \ker \rho_x, \tag{2.9}$$

where $\mathfrak{g}_x^\omega$ is the orthogonal of $\mathfrak{g}_x$ with respect to ω. These relations show that π_x is invertible if and only if $\mathfrak{g}_x$ is ω-nondegenerate and ρ_x is onto.

Left symmetric algebroids Left symmetric algebroids appeared first in [20] as Koszul–Vinberg algebroids were studied in more details in [15].

Let (A, M, ρ) be an anchored vector bundle. A *right-anchored product* on (A, M, ρ) is $\mathbb{R}$-bilinear map $T : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ such that, for any $a, b \in \Gamma(A)$ and any $f \in C^\infty(M)$,

$$T_{fa}b = f\,T_a b \quad \text{and} \quad T_a(fb) = f\,T_a b + \rho(a)(f)b. \tag{2.10}$$

To T we associate the anchored bracket $[a,b]_T = T_a b - T_b a$. The curvature of T is the $\mathbb{R}$-trilinear map

$$R^T : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A) \quad \text{given by} \quad R^T(a,b)c = [T_a, T_b]c - T_{[a,b]_T}c.$$

The curvature R^T satisfies, for any $a, b, c \in \Gamma(A)$,

$$\begin{cases} R^T(a,b)c = -R^T(b,a)c, \\ R^T(fa,b)c = R^T(a,fb)c = f\,R^T(a,b)c, \\ R^T(a,b)fc = f\,R^T(a,b)c + \tau_{[\ ,\ ]_T}(a,b)(f)c, \end{cases} \tag{2.11}$$

and the Bianchi's identity

$$R^T(a,b)c + R^T(b,c)a + R^T(c,a)b = J_{[\ ,\ ]_T}(a,b,c). \tag{2.12}$$

From theses relations we deduce that R^T is local and T is called Lie-admissible if $[\ ,\ ]_T$ induces a Lie algebroid structure on (A, M, ρ).

Remark 2.4. We preferred to distinguish between A-connections which are defined on Lie algebroids and right-anchored products which are defined over anchored bundles not yet a Lie algebroid.

The following proposition follows easily from Proposition 2.1 and (2.12).

Proposition 2.5. *Let (A, M, ρ) be an anchored bundle and let T be a right-anchored product on $\Gamma(A)$. Then T is Lie-admissible if and only for any $x \in M$ there exists an open set U of M containing x and a basis of sections $(a_1, \dots, a_r)$ over U such that*

$$R^T\left(a_i, a_j\right)a_k + R^T\left(a_j, a_k\right)a_i + R^T(a_k, a_i)a_j = 0 \quad \text{and} \quad \tau_{[\ ,\ ]_T}\left(a_i, a_j\right) = 0, \quad 1 \le i < j < k \le r.$$

The following proposition is a consequence of (2.11).

Proposition 2.6. *Let (A, M, ρ) be an anchored bundle and let T be a right-anchored product on $\Gamma(A)$. Then the following assertions are equivalent.*

(i) *The curvature R^T vanishes identically.*

(ii) *For any $x \in M$ there exists an open set U of M containing x and a basis of sections $(a_1, \dots, a_r)$ over U such that*

$$R^T\left(a_i, a_j\right)a_k = 0 \quad \text{and} \quad \tau_{[\ ,\ ]_T}\left(a_i, a_j\right) = 0, \quad 1 \le i < j \le r,\ 1 \le k \le r.$$

Definition 2.7. A *left symmetric algebroid* is an anchored bundle (A, M, ρ) together with a right-anchored product T satisfying (i) or (ii) of Proposition 2.6.

It is obvious that if (A, M, ρ, T) is a left symmetric algebroid then $\left(A, M, \rho, [\ ,\ ]_T\right)$ is a Lie algebroid.

Remark 2.8.

1. A right-anchored product is Lie-admissible if the Jacobiator of the associated anchored bracket vanishes. Proposition 2.5 gives a subtle way of checking the Lie-admissibility. We will use it in a crucial way in the study of para-Kähler Lie algebroids.
2. A left symmetric algebroid is an anchored bundle with a right-anchored product whose curvature vanishes. However, it is important to keep in mind that the vanishing of the curvature is not a tensorial equation and (ii) of Proposition 2.6 could be very useful in concrete situations.

Some examples of left symmetric algebroids

1. Any left symmetric algebra is obviously a left symmetric algebroid.
2. Let M be a smooth manifold. A right-anchored product on $(TM, M, \mathrm{id}_{TM})$ is just a linear connection, a Lie-admissible right-anchored product on $(TM, M, \mathrm{id}_{TM})$ is just a torsion-free linear connection and a left symmetric product on $(TM, M, \mathrm{id}_{TM})$ is just a flat linear connection.
3. Let $(S, .)$ be a left symmetric algebra, i.e, for any $a, b, c \in S$,

$$\mathrm{assoc}(a,b,c) = \mathrm{assoc}(b,a,c),$$

where $\mathrm{assoc}(a,b,c) = (a.b).c - a.(b.c)$. It is know that $[a,b] = a.b - b.a$ is a Lie bracket on A. Let $\tau : S \longrightarrow \mathcal{X}(M)$ an action of $(S, [\ ,\ ])$ on a smooth manifold M. Consider the anchored bundle (A, M, ρ) where A is the trivial bundle $M \times S$ and $\rho(m,a) = \tau(a)(m)$. Define on $\Gamma(A) = C^\infty(M, S)$ the product T by

$$T_{s_1}s_2 = \mathcal{L}_{\rho(s_1)}s_2 + s_1.s_2.$$

By using (ii) of Proposition 2.6, it is easy to check that (A, M, ρ, T) is a left symmetric algebroid. This example has been given in [15].

4. Let $\mathfrak{g} \xrightarrow{\Gamma} \mathcal{X}(M)$ be an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M, i.e., a morphism of Lie algebras from $\mathfrak{g}$ to the Lie algebra of vector fields on M. Let $r \in \wedge^2\mathfrak{g}$ be a solution of the classical Yang–Baxter equation, i.e.,

$$[r, r] = 0,$$

where $[r, r] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is defined by

$$[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)]),$$

and $r : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ denotes also the linear map given by $\alpha(r(\beta)) = r(\alpha, \beta)$. We denote by π^r the Poisson tensor on M image of r by Γ. Write

$$r = \sum_{i,j} a_{ij} u_i \wedge u_j$$

and put, for $\alpha, \beta \in \Omega^1(M)$,

$$\mathcal{D}^r_\alpha \beta := \sum_{i,j} a_{ij} \alpha(U_i) \mathcal{L}_{U_j} \beta,$$

where $U_i = \Gamma(u_i)$. We get a map $\mathcal{D}^r : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ which is a a right-anchored product on $\left(T^*M, M, \pi^r_\#\right)$. It was proved in [5] that $\mathcal{D}^r$ is Lie-admissible and left symmetric.

We finish this section by an useful lemma.

Lemma 2.9. *Let (A, M, ρ, T) be a left symmetric algebroid such that $\rho(a) = 0$ implies $T_a = 0$. Then, for any regular point $m \in M$ and $a \in A_m$, there exist an open set U around m and $s \in \Gamma(U)$ such that, for any $s' \in \Gamma(U)$, $T_{s'} s = 0$ and $s(m) = a$.*

Proof. Denote by q the rank of ρ at m. According to the local splitting theorem near a regular point (see [11] Theorem 1.1), there exists a coordinates system $\left(x_1, \ldots, x_q, y_1, \ldots, y_{n-q}\right)$ around m and a basis of sections $\left(a_1, \ldots, a_r\right)$ of A such that

$$\begin{cases} \rho(a_i) = \partial_{x_i}, \ i = 1, \ldots, q, \\ \rho(a_i) = 0, \ i = q+1, \ldots, r, \\ \left[a_i, a_j\right]_A = \sum\limits_{u=q+1}^{r} C^{ij}_u a_u. \end{cases}$$

Put $T_{a_i} a_j = \sum_{u=1}^r \Gamma^u_{ij} a_u$. We look for $s = \sum_{j=1}^r f_j a_j$ satisfying $Ts = 0$ and $s(m) = \sum_{i=1}^r a^0_i a_i$. Since by assumption $T_{a_i} = 0$ for $i = q+1, \ldots, r$, this is equivalent to

$$\frac{\partial f_j}{\partial x_i} = -\sum_{u=1}^{r} f_u \Gamma^j_{iu} \quad \text{and} \quad f_j(m) = a^0_j, \quad i = 1, \ldots, q, \ j = 1, \ldots, r. \tag{2.13}$$

We think of the y_i as parameters and we consider $\alpha : \mathbb{R}^q \longrightarrow \mathbb{R}$ and, for $i = 1, \ldots, q$, $F_i : \mathbb{R}^q \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$, given by

$$\alpha\left(x_1, \ldots, x_q\right) = \left(f_1(x, y), \ldots, f_r(x, y)\right) \quad \text{and} \quad F_i(x, z) = -\left(\sum_{u=1}^{r} z_u \Gamma^j_{iu}(x, y)\right)^r_{j=1}.$$

Thus (2.13) is equivalent to

$$\frac{\partial \alpha}{\partial x_i} = F_i(x, \alpha(x)).$$

According to a well-known theorem (see [23] pp.187) these system of differential equations has solutions if

$$\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j} + \sum_{u=1}^{r} \frac{\partial F_j}{\partial z_u} F^u_i - \sum_{u=1}^{r} \frac{\partial F_i}{\partial z_u} F^u_j = 0, \ \ i, j = 1, \ldots, q.$$

Or, one can check easily by using that $T_{[a_i, a_j]} = 0$ that this condition is equivalent to the vanishing of the curvature. □

3 | PARA-KÄHLER LIE ALGEBROIDS

In this section, we give the definition of a para-Kähler Lie algebroid, its basic properties, a characterization of a such structure and some examples.

Definition of para-Kähler Lie algebroid and its immediate consequences Let $(A, M, \rho, [\ ,\ ]_A)$ be a Lie algebroid. The Nijenhuis torsion of a bundle homomorphism $H : A \longrightarrow A$ is given by

$$N_H(a,b) := [Ha, Hb]_A - H[Ha, b]_A - H[a, Hb]_A + H^2[a,b]_A, \tag{3.1}$$

for any $a, b \in \Gamma(A)$.

The following proposition is a generalization of a well-known fact in differential geometry (see Proposition 4.2 p. 148 [13]). It can be used to give quickly the two equivalent definitions of para-Kähler Lie algebroids established in [17].

Proposition 3.1. *Let $(A, M, \rho, \langle\ ,\ \rangle)$ be a pseudo-Riemannian Lie algebroid and let $K : A \longrightarrow A$ be a skew-symmetric bundle endomorphism such that $K^2 = \mathrm{id}_A$. Define Ω by $\Omega(a,b) = \langle Ka, b\rangle$. Then the following assertions are equivalent:*

- **(i)** $d_A\Omega = 0$ *and* $N_K = 0$.
- **(ii)** $\nabla K = 0$, *where ∇ is the Levi–Civita A-connection associated to $\langle\ ,\ \rangle$.*

Proof. We have, for any $a, b, c \in \Gamma(A)$,

$$\begin{aligned}
d_A\Omega(a,b,c) &= \langle \nabla_a(K)b, c\rangle + \langle \nabla_b(K)c, a\rangle + \langle \nabla_c(K)a, b\rangle,\\
d_A\Omega(a, Kb, Kc) &= \langle \nabla_a(K)Kb, Kc\rangle + \langle \nabla_{Kb}(K)Kc, a\rangle + \langle \nabla_{Kc}(K)a, Kb\rangle,\\
\langle N_K(c, Kb), a\rangle &= \langle \nabla_{Kc}(K)(Kb), a\rangle - \langle \nabla_b(K)(c), a\rangle + \langle \nabla_c(K)(b), a\rangle - \langle \nabla_{Kb}(K)(Kc), a\rangle,
\end{aligned}$$

where $\nabla_a(K)b = \nabla_a(Kb) - K(\nabla_a b)$. These relations show that (ii) implies (i). Moreover, K is skew-symmetric and hence

$$\langle \nabla_a(K)b, c\rangle + \langle \nabla_a(K)c, b\rangle = 0.$$

Thus

$$d_A\Omega(a,b,c) + d_A\Omega(a, Kb, Kc) + \langle N_K(c, Kb), a\rangle = \langle \nabla_a(K)b, c\rangle + \langle \nabla_a(K)Kb, Kc\rangle.$$

Now

$$\langle \nabla_a(K)Kb, Kc\rangle = \langle \nabla_a b, Kc\rangle - \langle K\nabla_a Kb, Kc,\rangle = \langle \nabla_a Kb, c,\rangle - \langle K\nabla_a b, c\rangle = \langle \nabla_a(K)b, c\rangle.$$

Finally,

$$2\langle \nabla_a(K)b, c\rangle = d_A\Omega(a,b,c) + d_A\Omega(a, Kb, Kc) + \langle N_K(c, Kb), a\rangle$$

and the proposition follows. □

Definition 3.2. A *para-Kähler Lie algebroid* is a pseudo-Riemannian Lie algebroid $(A, M, \rho, \langle\ ,\ \rangle)$ endowed with a bundle isomorphism $K : A \longrightarrow A$ satisfying $K^2 = \mathrm{Id}_A$, K is skew-symmetric with respect to $\langle\ ,\ \rangle$ and $\nabla K = 0$, where ∇ is the Levi–Civita A-connection of $\langle\ ,\ \rangle$.

A para-Kähler Lie algebroid $(A, M, \rho, \langle\ ,\ \rangle, K)$ carries a natural bilinear skew-symmetric nondegenerate form Ω_K defined by $\Omega_K(a,b) = \langle Ka, b\rangle$. The following proposition is an immediate consequence of Definition 3.2 and Proposition 3.1.

Proposition 3.3. *Let $(A, M, \rho, \langle\ ,\ \rangle, K)$ be para-Kähler Lie algebroid and let ∇ be its Levi–Civita A-connection. Then:*

- **(i)** *(A, M, ρ, K) is a para-complex Lie algebroid, i.e., $K^2 = \mathrm{Id}_A$, $N_K = 0$ and, for any $x \in M$,* $\dim\ker(K + \mathrm{Id}_A)(x) = \dim\ker(K - \mathrm{Id}_A)(x)$.
- **(ii)** *(A, M, ρ, Ω_K) is a symplectic Lie algebroid and hence $\pi = \rho(\Pi)$ is a Poisson tensor on M, where Π is the inverse of Ω_K.*
- **(iii)** *$A = A^+ \oplus A^-$ where $A^+ = \ker(K - \mathrm{Id}_A)$ and $A^- = \ker(K + \mathrm{Id}_A)$.*

(iv) *A^+ and A^- are isotropic with respect to $\langle\ ,\ \rangle$ and Lagrangian with respect to Ω_K.*

(v) *for any $a \in \Gamma(A)$, $\nabla_a(\Gamma(A^+)) \subset \Gamma(A^+)$ and $\nabla_a(\Gamma(A^-)) \subset \Gamma(A^-)$.*

The following proposition is the first important property of para-Kähler Lie algebroids.

Proposition 3.4. *Let $\left(A, M, \rho, \langle\ ,\ \rangle, K\right)$ be a para-Kähler Lie algebroid then, for any $a^+, b^+ \in \Gamma(A^+)$, $a^-, b^- \in \Gamma(A^-)$,*

$$R(a^+, b^+) = R(a^-, b^-) = 0 \quad \textit{and} \quad R(a^+, a^-)b^+ - R(b^+, a^-)a^+ = R(a^-, a^+)b^- - R(b^-, a^+)a^- = 0,$$

where R is the curvature of the Levi–Civita connection ∇. In particular, for $\epsilon = \pm$, the restriction of ∇ to A^ϵ induces on $\left(A^\epsilon, M, \rho_{|A^\epsilon}\right)$ a left symmetric algebroid structure.

Proof. According to Bianchi's identity (2.12)

$$R(a^+, b^+)a^- + R(b^+, a^-)a^+ + R(a^-, a^+)b^+ = 0.$$

Since $\nabla\Gamma(A^\epsilon) \subset \Gamma(A^\epsilon)$ we get $R(a^+, b^+)a^- \in \Gamma(A^-)$ and $R(b^+, a^-)a^+ + R(a^-, a^+)b^+ \in \Gamma(A^+)$ and hence

$$R(a^+, b^+)a^- = R(b^+, a^-)a^+ + R(a^-, a^+)b^+ = 0.$$

Now, for any $c^+ \in \Gamma(A^+)$,

$$\langle R(a^+, b^+)a^-, c^+\rangle = -\langle a^-, R(a^+, b^+)c^+\rangle = 0.$$

So $R(a^+, b^+) = 0$. In the same way we can show that $R(a^-, b^-) = 0$ and $R(b^-, a^+)a^- + R(a^+, a^-)b^- = 0$. This completes the proof. ☐

Description of para-Kähler Lie algebroids Let $(A, M, \rho, \langle\ ,\ \rangle, K)$ be a para-Kähler Lie algebroid. As an anchored bundle, $(A, M, \rho) = (A^+ \oplus A^-, M, \rho^+ \oplus \rho^-)$ where $\rho^\epsilon = \rho_{|A^\epsilon}$. The map $\flat : A^- \longrightarrow (A^+)^*$, $a \mapsto a^*$, where a^* is given by $\prec a^*, b \succ = \langle a, b\rangle$, realizes a bundle isomorphism between A^- and $(A^+)^*$. According to Proposition 3.4, the restriction of the Levi–Civita connection ∇ to A^+ say S induces on (A^+, M, ρ^+) a structure of left symmetric algebroid. The same thing happens for A^- and by the identification above we get a left symmetric right-anchored product T on $((A^+)^*, M, \rho_1)$ where $\rho_1 = \rho \circ \flat^{-1}$. Thus we can identify $\left(A, M, \rho, \langle\ ,\ \rangle, K\right)$ with $\left(\Phi(A^+), \rho^+ \oplus \rho_1, \langle\ ,\ \rangle_0, , K_0\right)$ (see (1.1)) and, under this identification, the Levi–Civita connection ∇^0 is entirely determined by S and T. Namely, for any $a, b \in \Gamma(A^+)$ and $u, v \in \Gamma((A^+)^*)$, we have

$$\nabla^0_a b = S_a b, \ \nabla^0_u v = T_u v, \ \nabla^0_a u = S^*_a u \quad \text{and} \quad \nabla^0_u a = T^*_u a,$$

where S^* and T^* are the dual of S and T given by (2.4). Moreover, since $\rho^+ \oplus \rho_1 : \Gamma(\Phi(A^+)) \longrightarrow TM$ is a Lie algebra homomorphism, we have

$$[\rho^+(a), \rho^+(b)] = \rho^+\left(S_a b - S_b a\right), \ [\rho_1(u), \rho_1(v)] = \rho_1\left(T_u v - T_v u\right)$$

$$\text{and} \quad [\rho^+(a), \rho_1(u)] = \rho_1\left(S^*_a u\right) - \rho^+\left(T^*_u a\right), \quad \forall a, b \in \Gamma(A^+), \ \forall u, v \in \Gamma((A^+)^*.$$

How to build a para-Kähler Lie algebroid from two left symmetric algebroid structures on two dual vector bundles Let (B, M, ρ_0, S) and (B^*, M, ρ_1, T) be two left symmetric algebroids. We extend the right-anchored products on (B, M, ρ_0) and (B^*, M, ρ_1) to $(\Phi(B), M, \rho_0 \oplus \rho_1)$ by putting, for any $X, Y \in \Gamma(B)$ and for any $\alpha, \beta \in \Gamma(B^*)$,

$$\nabla_{X+\alpha}(Y + \beta) = S_X Y + T^*_\alpha Y + S^*_X \beta + T_\alpha \beta. \tag{3.2}$$

The anchored bracket associated to ∇ is given by

$$[X + \alpha, Y + \beta]_\phi = [X, Y]_B + [\alpha, \beta]_{B^*} + T^*_\alpha Y - T^*_\beta X + S^*_X \beta - S^*_Y \alpha. \tag{3.3}$$

We endow also $\Phi(B)$ with K_0 and $\langle\ ,\ \rangle_0$ given by (1.1).

The question now is under which conditions that ∇ is Lie-admissible or equivalently $[\ ,\ ]_\phi$ is a Lie bracket. It is a crucial step in our study and we will use Proposition 2.5 to get an answer which will turn out to be very useful, particularly, in the next section.

Proposition 3.5. *With the hypothesis above, the following assertions are equivalent:*

(i) *The right-anchored product on* $(\Phi(B), M, \rho_0 \oplus \rho_1)$ *given by* (3.2) *is Lie-admissible.*

(ii) *For any* $X, Y \in \Gamma(B)$ *and* $\alpha, \beta \in \Gamma(B^*)$,

$$R^\nabla(X, \alpha)Y = R^\nabla(Y, \alpha)X \quad \textit{and} \quad R^\nabla(\alpha, X)\beta = R^\nabla(\beta, X)\alpha. \tag{3.4}$$

(iii) *For any* $x \in M$ *there exists an open set* U *containing* x *and a basis of sections* $(a_1, \ldots, a_n)$ *of* B *over* U *such that, for any* $1 \le i, j, k \le n$,

$$R^\nabla(a_i, \alpha_k)a_j = R^\nabla(a_j, \alpha_k)a_i,$$

$$R^\nabla(\alpha_i, a_k)\alpha_j = R^\nabla(\alpha_j, a_k)\alpha_i \quad \textit{and} \quad [\rho_0(a_i), \rho_1(\alpha_j)] = \rho_1\left(S^*_{a_i}\alpha_j\right) - \rho_0\left(T^*_{\alpha_j}a_i\right), \tag{3.5}$$

where $(\alpha_1, \ldots, \alpha_n)$ *is the dual basis of* $(a_1, \ldots, a_n)$ *and* R^∇ *is the curvature of* ∇.

Proof. It is a consequence of Proposition 2.5 and the facts that

$$R^\nabla(X, Y) = 0,\ R^\nabla(\alpha, \beta) = 0,\ \tau_{[\ ,\ ]_\phi}(X, Y) = \tau_{[\ ,\ ]_\phi}(\alpha, \beta) = 0$$

and

$$\tau_{[\ ,\ ]_\phi}(X, \alpha) = \rho_1\left(S^*_X\alpha\right) - \rho_0\left(T^*_\alpha X\right) - [\rho_0(X), \rho_1(\alpha)]$$

for any $X, Y \in \Gamma(B)$ and $\alpha, \beta \in \Gamma(B^*)$. □

The equations (3.4) are the same equations defining left symmetric bialgebroids in [16].

Definition 3.6. Two left symmetric algebroids (B, M, ρ_0, S) and (B^*, M, ρ_1, T) satisfying (3.4) or (3.5) will be called *Lie-extendible* or compatible.

Thus we get the following result.

Theorem 3.7. *Let* (B, M, S, ρ_0) *and* (B^*, M, T, ρ_1) *be two Lie-extendible left symmetric algebroids.*
Then $\left(\Phi(B), M, \rho_0 \oplus \rho_1, \langle\ ,\ \rangle_0, K_0\right)$ *endowed with the Lie algebroid bracket given by* (3.3) *is a para-Kähler Lie algebroid. Moreover, all para-Kähler Lie algebroids are obtained in this way.*

A subtlety of compatible left symmetric algebroids The compatibility between two left symmetric algebroids has a subtle property we will point out now.

Let (B, M, ρ_0, S) and (B^*, M, ρ_1, T) be two left symmetric algebroids. They are compatible if (3.4) holds. Note first that, for any $X, Y \in \Gamma(B)$ and $\alpha, \beta \in \Gamma(B^*)$,

$$R^\nabla(X, \alpha)Y = \left[S_X, T^*_\alpha\right]Y + S_{T^*_\alpha X}Y - T^*_{S^*_X\alpha}Y$$

$$\text{and} \quad R^\nabla(\alpha, X)\beta = \left[T_\alpha, S^*_X\right]\beta + T_{S^*_X\alpha}\beta - S^*_{T^*_\alpha X}\beta. \tag{3.6}$$

On the other hand, Equation (3.4) is not tensorial and it is equivalent to the vanishing of the Jacobiator of the bracket $[\ ,\ ]_\phi$ given by (3.3). We have seen that the vanishing of the Jacobiator implies the vanishing of the torsion. Let compute the torsion of $[\ ,\ ]_\phi$. Since the torsions of $[\ ,\ ]_S$ and $[\ ,\ ]_T$ vanish, we have for any $X, Y \in \Gamma(B)$, $\alpha, \beta \in \Gamma(B^*)$,

$$\tau_{[\ ,\ ]_\phi}(X, Y) = 0 \quad \text{and} \quad \tau_{[\ ,\ ]_\phi}(\alpha, \beta) = 0.$$

Moreover,

$$\tau_{S,T}(X, \alpha) := \tau_{[\ ,\ ]_\phi}(X, \alpha) = [\rho_0(X), \rho_1(\alpha)] - \rho_1\left(S^*_X\alpha\right) + \rho_0\left(T^*_\alpha X\right). \tag{3.7}$$

Then $\tau_{[\ ,\ ]_\phi} = 0$ if and only if the tensor field $\tau_{S,T} \in \Gamma(B^* \otimes B \otimes TM)$ vanishes. By using Bianchi's identity, we get for any $X, Y \in \Gamma(B)$ and $\alpha, \beta \in \Gamma(B^*)$,

$$R^\nabla(X,\alpha)Y - R^\nabla(Y,\alpha)X = J_{[\ ,\ ]_\phi}(X,\alpha,Y) \quad \text{and} \quad R^\nabla(\alpha,X)\beta - R^\nabla(\beta,X)\alpha = J_{[\ ,\ ]_\phi}(\alpha,X,\beta).$$

Thus if $\tau_{S,T}$ vanishes then $\rho_0 \oplus \rho_1$ is a Lie algebras homomorphism and hence $\rho_0 \oplus \rho_1(J_{[\ ,\ ]_\phi}) = 0$. So we get from Bianchi's identity that

$$\rho_0\big(R^\nabla(X,\alpha)Y\big) = \rho_0\big(R^\nabla(Y,\alpha)X\big) \quad \text{and} \quad \rho_1\big(R^\nabla(\alpha,X)\beta\big) = \rho_1\big(R^\nabla(\beta,X)\alpha\big). \tag{3.8}$$

So we get the following proposition.

Proposition 3.8. *Let (B, M, ρ_0, S) and (B^*, M, ρ_1, T) be two left symmetric algebroid structures. Then the following assertions hold:*

- **(i)** *If ρ_0 and ρ_1 are injective then (B, M, S, ρ_0) and (B^*, M, T, ρ_1) are Lie-extendible if and only if $\tau_{S,T} = 0$.*
- **(ii)** *If ρ_0 is injective then the two left symmetric structures are Lie-extendible if and only if, for any $X \in \Gamma(B)$ and $\alpha, \beta \in \Gamma(B^*)$,*

$$R^\nabla(\alpha,X)\beta = R^\nabla(\beta,X)\alpha \quad \textit{and} \quad \tau_{S,T} = 0.$$

- **(iii)** *If ρ_0 is an isomorphism and the two left symmetric structures are Lie-extendible then, for any $X \in \Gamma(B)$ and $\alpha \in \Gamma(B^*)$,*

$$T^*_\alpha X = \mathrm{s}\big(S^*_X\alpha\big) - [X, \mathrm{s}(\alpha)]_B,$$

where $\mathrm{s} = \rho_0^{-1} \circ \rho_1 : B^* \longrightarrow B$.

The general case of (iii) in the proposition above will be studied in the next section devoted to the notion of exact para-Kähler Lie algebroids which generalizes the notion of exact para-Kähler Lie algebras introduced in [2] and studied in more details in [3].

It is important to point out that two compatible left symmetric algebroids give rise to a symmetric bivector field and a Poisson structure on the underlying manifold. Indeed, if (B, M, S, ρ_0) and (B^*, M, T, ρ_1) are two Lie-extendible left symmetric algebroids then $(\Phi(B), M, \rho_0 \oplus \rho_1, \Omega_0)$ is a para-Kähler Lie algebroid and hence there exists a symmetric bivector field h and a Poisson tensor π on M given by (2.7) and (2.8), respectively. One can see easily that $h_\#, \pi_\# : T^*M \longrightarrow TM$ associated to h and π are given by

$$h_\# = \rho_1 \circ \rho_0^* + \rho_0 \circ \rho_1^* \quad \text{and} \quad \pi_\# = \rho_1 \circ \rho_0^* - \rho_0 \circ \rho_1^*. \tag{3.9}$$

Example 3.9. Let (A, M, S, ρ) be a left symmetric algebroid. Then the left symmetric product on A and the trivial left symmetric product on A^* together with the trivial anchor are Lie-extendible so $\big(\Phi(A), M, \rho \oplus 0, \langle\ ,\ \rangle_0, K_0\big)$ endowed with the Lie algebra bracket associated to the left symmetric product

$$\nabla^0_{X+\alpha}(Y + \beta) = S_X Y + S^*_X \beta \tag{3.10}$$

is a para-Kähler Lie algebra. We denote by $[\ ,\ ]^\triangleright$ the Lie bracket associated to ∇^0. We have

$$[X + \alpha, Y + \beta]^\triangleright = [X, Y] + S^*_X\beta - S^*_Y\alpha.$$

Moreover, it is easy to check that $\big(\Phi(A), [\ ,\ ]^\triangleright, \langle\ ,\ \rangle_0\big)$ is a flat pseudo-Riemannian Lie algebroid and $\big(\Phi(A), M, [\ ,\ ]^\triangleright, \Omega_0\big)$ is a symplectic Lie algebroid and Ω_0 is parallel with respect to ∇^0.

4 | EXACT PARA-KÄHLER LIE ALGEBROIDS

In this section, we introduce the notion of exact para-Kähler Lie algebroids which generalizes exact para-Kähler Lie algebras introduced in [2] and studied in more details in [3].

Let (A, M, ρ, S) be a left symmetric algebroid, $\mathrm{r} \in \Gamma(A \otimes A)$ and $\mathrm{r} = \mathfrak{a} + \mathfrak{s}$ the decomposition of r into skew-symmetric and symmetric part. We denote by $\mathrm{r}_\# : A^* \longrightarrow A$ the bundle homomorphism given by $\beta(\mathrm{r}_\#(\alpha)) = \mathrm{r}(\alpha, \beta)$. Put $\rho_\mathrm{r} = \rho \circ \mathrm{r}_\#$ and, for any $\alpha, \beta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,

$$\prec T_\alpha \beta, X \succ := \rho(X).\mathrm{r}(\alpha, \beta) - \mathrm{r}\left(S^*_X \alpha, \beta\right) - \mathrm{r}\left(\alpha, S^*_X \beta\right) + \prec S^*_{\mathrm{r}_\#(\alpha)} \beta, X \succ = S_X \mathrm{r}(\alpha, \beta) + \prec S^*_{\mathrm{r}_\#(\alpha)} \beta, X \succ . \tag{4.1}$$

It is clear that T is a right-anchored product on $\left(A^*, M, \rho_\mathrm{r}\right)$. Let ∇ be the extension of S and T on $(\Phi(A), M, \rho \oplus \rho_\mathrm{r})$ given by (3.2).

Problem 4.1. Under which conditions on (A, M, ρ, r), is $(A^*, M, \rho_\mathrm{r}, T)$ a left symmetric algebroid with (A, M, ρ, S) and $(A^*, M, \rho_\mathrm{r}, T)$ being compatible?

Recall that $(A^*, M, \rho_\mathrm{r}, T)$ is a left symmetric algebroid with (A, M, ρ, S) and $(A^*, M, \rho_\mathrm{r}, T)$ being compatible if and only if of the curvature of T vanishes and, for any $\alpha, \beta \in \Gamma(A^*)$, $X, Y \in \Gamma(A)$,

$$R^\nabla(X, \alpha)Y = R^\nabla(Y, \alpha)X \quad \text{and} \quad R^\nabla(\alpha, X)\beta = R^\nabla(\beta, X)\alpha.$$

Remark first that if the curvature of T vanishes then T is Lie-admissible and hence $\rho_\mathrm{r} : \Gamma(A^*) \longrightarrow \mathcal{X}(M)$ is a Lie algebra homomorphism, i.e., $\rho_\mathrm{r}([\alpha, \beta]_T) = [\rho_\mathrm{r}(\alpha), \rho_\mathrm{r}(\beta)]$. This can be written

$$\rho(\Delta(r)(\alpha, \beta)) = 0, \tag{4.2}$$

where

$$\Delta(r)(\alpha, \beta) = \mathrm{r}_\#([\alpha, \beta]_T) - [\mathrm{r}_\#(\alpha), \mathrm{r}_\#(\beta)]_S. \tag{4.3}$$

We have $\Delta(r) \in \Gamma(A \otimes A \otimes A)$ and Equation (4.2) is tensorial. The following theorem gives an answer to Problem 4.1.

Theorem 4.2. *Let (A, M, ρ, S) be a left symmetric algebroid and let $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$. Then $(A^*, M, \rho_\mathrm{r}, T)$ is a left symmetric algebroid with (A, M, ρ, S) and $(A^*, M, \rho_\mathrm{r}, T)$ being compatible if and only if, for any $\alpha, \beta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,*

$$\rho \circ \Delta(r) = 0, \; S^2\mathfrak{a} = 0$$

$$\text{and} \quad Q_X \Delta(r)(\alpha, \beta) := [X, \Delta(r)(\alpha, \beta)]_S - \Delta(r)\left(S^*_X \alpha, \beta\right) - \Delta(r)\left(\alpha, S^*_X \beta\right) = 0. \tag{4.4}$$

Before proving this theorem, let us give some clarifications on (4.4). First, note that $S\mathfrak{a}$ and $S^2\mathfrak{a}$ are given by

$$S_X \mathfrak{a}(\alpha, \beta) = \rho(X).\mathfrak{a}(\alpha, \beta) - \mathfrak{a}\left(S^*_X \alpha, \beta\right) - \mathfrak{a}\left(\alpha, S^*_X \beta\right) \quad \text{and} \quad S^2_{X,Y}\mathfrak{a} = S_X S_Y \mathfrak{a} - S_{S_X Y}\mathfrak{a}.$$

The second point is that the quantity $Q_X \Delta(r)(\alpha, \beta)$ is tensorial with respect to α and β, it is not tensorial with respect to X. However, when $\rho \circ \Delta(r) = 0$ then it becomes tensorial with respect to X and hence the last equation in (4.4) is tensorial. The proof of Theorem 4.2 is a consequence of the following lemma.

Lemma 4.3. *Let (A, M, ρ, S) be a left symmetric algebroid and let $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$. With the notations above, we have, for any $X, Y \in A$ and for any $\alpha, \beta, \gamma \in A^*$*

$$R^\nabla(X, \alpha)Y = R^\nabla(Y, \alpha)X, \; \prec R^\nabla(\alpha, X)\beta - R^\nabla(\beta, X)\alpha, Y \succ = -2S^2_{X,Y}\mathfrak{a}(\alpha, \beta) \quad \text{and}$$

$$\begin{aligned}\prec R^T(\alpha, \beta)\gamma, X \succ = -\rho(\Delta(r)(\alpha, \beta)). \prec \gamma, X \succ - \prec \gamma, [X, \Delta(r)(\alpha, \beta)]_S \succ + \prec \gamma, \Delta(r)\left(S^*_X \alpha, \beta\right) \succ \\ + \prec \gamma, \Delta(r)\left(\alpha, S^*_X \beta\right) \succ + 2S^2_{X, \mathfrak{s}_\#(\gamma)}\mathfrak{a}(\alpha, \beta) - 2S^2_{X, \mathfrak{a}_\#(\gamma)}\mathfrak{a}(\alpha, \beta).\end{aligned}$$

Proof. Note first that a direct computation using (4.1) gives, for any $X \in \Gamma(A), \alpha \in \Gamma(A^*)$,

$$T^*_\alpha X = \mathrm{r}_\#\left(S^*_X \alpha\right) + \left[\mathrm{r}_\#(\alpha), X\right]_S. \tag{4.5}$$

On the other hand, recall from (3.6) that $R^\nabla(X,\alpha) = \left[S_X, T^*_\alpha\right] + S_{T^*_\alpha X} - T^*_{S^*_X\alpha}$. So by using (4.5) we get

$$\begin{aligned}
R^\nabla(X,\alpha)Y &= S_X \mathrm{r}_\#\left(S^*_Y\alpha\right) + S_X\left[\mathrm{r}_\#(\alpha), Y\right]_S - \mathrm{r}_\#\left(S^*_{S_XY}\alpha\right) - \left[\mathrm{r}_\#(\alpha), S_XY\right]_S \\
&\quad + S_{\mathrm{r}_\#(S^*_X\alpha)}Y + S_{[\mathrm{r}_\#(\alpha),X]_S}Y - \mathrm{r}_\#\left(S^*_Y S^*_X\alpha\right) - \left[\mathrm{r}_\#(S^*_X\alpha), Y\right]_S \\
&= \left[X, \mathrm{r}_\#\left(S^*_Y\alpha\right)\right]_S + S_{\mathrm{r}_\#(S^*_Y\alpha)}X + S_X\left[\mathrm{r}_\#(\alpha), Y\right]_S - \mathrm{r}_\#\left(S^*_{S_XY}\alpha\right) - \left[\mathrm{r}_\#(\alpha), S_XY\right]_S \\
&\quad + S_{\mathrm{r}_\#(S^*_X\alpha)}Y + S_{[\mathrm{r}_\#(\alpha),X]_S}Y - \mathrm{r}_\#\left(S^*_Y S^*_X\alpha\right) - \left[\mathrm{r}_\#\left(S^*_X\alpha\right), Y\right]_S \\
&= \left[X, \mathrm{r}_\#\left(S^*_Y\alpha\right)\right]_S + \left[Y, \mathrm{r}_\#\left(S^*_X\alpha\right)\right]_S + S_{\mathrm{r}_\#(S^*_X\alpha)}Y + S_{\mathrm{r}_\#(S^*_Y\alpha)}X + \left[X, \left[\mathrm{r}_\#(\alpha), Y\right]_S\right]_S \\
&\quad - \mathrm{r}_\#\left(S^*_{S_XY}\alpha + S^*_Y S^*_X\alpha\right) - \left[\mathrm{r}_\#(\alpha), S_XY\right]_S + S_{[\mathrm{r}_\#(\alpha),X]_S}Y + S_{[\mathrm{r}_\#(\alpha),Y]_S}X.
\end{aligned}$$

So

$$R^\nabla(X,\alpha)Y - R^\nabla(Y,\alpha)X = \mathrm{r}_\#\left(\left(R^S(Y,X)\right)^*\alpha\right) + \left[\mathrm{r}_\#(\alpha), [Y,X]_S\right]_S + \left[X, \left[\mathrm{r}_\#(\alpha), Y\right]_S\right]_S + \left[Y, \left[X, \mathrm{r}_\#(\alpha)\right]_S\right]_S = 0.$$

This shows the first relation.

We have from (3.6) that $R^\nabla(\alpha, X) = \left[T_\alpha, S^*_X\right] + T_{S^*_X\alpha} - S^*_{T^*_\alpha X}$. By using (4.1) and (4.5), we get

$$\begin{aligned}
\prec R^\nabla(\alpha,X)\beta, Y \succ &= \prec T_\alpha S^*_X\beta, Y \succ - \prec S^*_X T_\alpha\beta, Y \succ + \prec T_{S^*_X\alpha}\beta, Y \succ - \prec S^*_{T^*_\alpha X}\beta, Y \succ \\
&= S_Y \mathrm{r}\left(\alpha, S^*_X\beta\right) + \prec S^*_{\mathrm{r}_\#(\alpha)} S^*_X\beta, Y \succ - \rho(X). \prec T_\alpha\beta, Y \succ + \prec T_\alpha\beta, S_XY \succ \\
&\quad + S_Y \mathrm{r}\left(S^*_X\alpha, \beta\right) + \prec S^*_{\mathrm{r}_\#(S^*_X\alpha)}\beta, Y \succ - \prec S^*_{\mathrm{r}_\#(S^*_X\alpha)}\beta, Y \succ - \prec S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta, Y \succ \\
&= S_Y \mathrm{r}\left(\alpha, S^*_X\beta\right) + S_Y \mathrm{r}\left(S^*_X\alpha, \beta\right) + \prec S^*_{\mathrm{r}_\#(\alpha)} S^*_X\beta, Y \succ - \rho(X).S_Y\mathrm{r}(\alpha,\beta) \\
&\quad - \rho(X). \prec S^*_{\mathrm{r}_\#(\alpha)}\beta, Y \succ + S_{S_XY}\mathrm{r}(\alpha,\beta) + \prec S^*_{\mathrm{r}_\#(\alpha)}\beta, S_XY \succ - \prec S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta, Y \succ \\
&= -\rho(X).S_Y\mathrm{r}(\alpha,\beta) + S_Y\mathrm{r}\left(\alpha, S^*_X\beta\right) + S_Y\mathrm{r}\left(S^*_X\alpha, \beta\right) + S_{S_XY}\mathrm{r}(\alpha,\beta) \\
&= -S_X S_Y \mathrm{r}(\alpha,\beta) + S_{S_XY}\mathrm{r}(\alpha,\beta),
\end{aligned}$$

and the second relation follows from $\mathrm{r} = \mathfrak{s} + \mathfrak{a}$.

Let us compute the curvature of T. Remark first that from (4.1) we can derive easily that

$$\prec [\alpha,\beta]_T, X \succ := \prec T_\alpha\beta - T_\beta\alpha, X \succ = \prec S^*_{\mathrm{r}_\#(\alpha)}\beta - S^*_{\mathrm{r}_\#(\beta)}\alpha, X \succ + 2S_X\mathfrak{a}(\alpha,\beta). \tag{4.6}$$

By using (4.5) once more, we get

$$\begin{aligned}
\prec T_\alpha T_\beta\gamma, X \succ &= \rho_\mathrm{r}(\alpha). \prec T_\beta\gamma, X \succ - \prec T_\beta\gamma, T^*_\alpha X \succ \\
&= \rho_\mathrm{r}(\alpha)\circ\rho_\mathrm{r}(\beta). \prec \gamma, X \succ - \rho_\mathrm{r}(\alpha). \prec \gamma, T^*_\beta X \succ - \rho_\mathrm{r}(\beta). \prec \gamma, T^*_\alpha X \succ + \prec \gamma, T^*_\beta T^*_\alpha X \succ \\
&= \rho_\mathrm{r}(\alpha)\circ\rho_\mathrm{r}(\beta). \prec \gamma, X \succ - \rho_\mathrm{r}(\alpha). \prec \gamma, T^*_\beta X \succ - \rho_\mathrm{r}(\beta). \prec \gamma, T^*_\alpha X \succ + \prec \gamma, \mathrm{r}_\#\left(S^*_{\mathrm{r}_\#(S^*_X\alpha)}\beta\right) \succ \\
&\quad + \prec \gamma, \left[\mathrm{r}_\#(\beta), \mathrm{r}_\#\left(S^*_X\alpha\right)\right] \succ + \prec \gamma, \mathrm{r}_\#\left(S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta\right) \succ + \prec \gamma, [\mathrm{r}_\#(\beta), [\mathrm{r}_\#(\alpha), X]_S]_S \succ \\
&= \rho_\mathrm{r}(\alpha)\circ\rho_\mathrm{r}(\beta). \prec \gamma, X \succ - \rho_\mathrm{r}(\alpha). \prec \gamma, T^*_\beta X \succ - \rho_\mathrm{r}(\beta). \prec \gamma, T^*_\alpha X \succ \\
&\quad + \prec \gamma, \mathrm{r}_\#\left(S^*_{\mathrm{r}_\#(S^*_X\alpha)}\beta\right) \succ + \prec \gamma, \Delta(\mathrm{r})\left(S^*_X\alpha, \beta\right) \succ + \prec \gamma, \mathrm{r}_\#\left(\left[\beta, S^*_X\alpha\right]_T\right) \succ \\
&\quad + \prec \gamma, \mathrm{r}_\#\left(S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta\right) \succ + \prec \gamma, [\mathrm{r}_\#(\beta), [\mathrm{r}_\#(\alpha), X]_S]_S \succ,
\end{aligned}$$

$$\begin{aligned}\prec T_{[\alpha,\beta]_T}\gamma, X \succ &= \rho_{\mathrm{r}}([\alpha,\beta]_T). \prec \gamma, X \succ - \prec \gamma, T^*_{[\alpha,\beta]_T} X \succ \\ &= \rho_{\mathrm{r}}([\alpha,\beta]_T). \prec \gamma, X \succ - \prec \gamma, \mathrm{r}_\#\big(S^*_X[\alpha,\beta]_T\big) \succ - \prec \gamma, [\mathrm{r}_\#([\alpha,\beta]_T), X]_S \succ .\end{aligned}$$

By using the Jacobi identity for $X, \mathrm{r}_\#(\alpha), \mathrm{r}_\#(\beta)$, we get

$$\begin{aligned}\prec R^T(\alpha,\beta)\gamma, X \succ = &-\rho(\Delta(r)(\alpha,\beta)). \prec \gamma, X \succ - \prec \gamma, [X, \Delta(r)(\alpha,\beta)]_S \succ + \prec \gamma, \Delta(r)\big(S^*_X\alpha, \beta\big) \succ \\ &+ \prec \gamma, \Delta(r)\big(\alpha, S^*_X\beta\big) \succ + \prec Q, \mathfrak{s}(\gamma) \succ - \prec Q, \mathfrak{a}(\gamma) \succ,\end{aligned}$$

where

$$Q = S^*_{\mathrm{r}_\#(S^*_X\alpha)}\beta - S^*_{\mathrm{r}_\#(S^*_X\beta)}\alpha + \big[\beta, S^*_X\alpha\big]_T - \big[\alpha, S^*_X\beta\big]_T + S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta - S^*_{[\mathrm{r}_\#(\beta),X]_S}\alpha + S^*_X[\alpha,\beta]_T.$$

Now, by using (4.6) and the fact that the curvature of S vanishes, we get

$$\begin{aligned}\prec Q, Y \succ = &\prec S^*_{\mathrm{r}_\#(\beta)}S^*_X\alpha, Y \succ - \prec S^*_{\mathrm{r}_\#(\alpha)}S^*_X\beta, Y \succ + 2S_Y\mathfrak{a}\big(\beta, S^*_X\alpha\big) - 2S_Y\mathfrak{a}\big(\alpha, S^*_X\beta\big) + \prec S^*_{[\mathrm{r}_\#(\alpha),X]_S}\beta, Y \succ \\ &- \prec S^*_{[\mathrm{r}_\#(\beta),X]_S}\alpha, Y \succ + \rho(X). \prec [\alpha,\beta]_T, Y \succ - \prec [\alpha,\beta]_T, S_XY \succ \\ = &-2S_Y\mathfrak{a}\big(S^*_X\alpha, \beta\big) - 2S_Y\mathfrak{a}\big(\alpha, S^*_X\beta\big) + \prec S^*_XS^*_{\mathrm{r}_\#(\beta)}\alpha, Y \succ - \prec S^*_XS^*_{\mathrm{r}_\#(\alpha)}\beta, Y \succ + \rho(X). \prec S^*_{\mathrm{r}_\#(\alpha)}\beta, Y \succ \\ &-\rho(X). \prec S^*_{\mathrm{r}_\#(\beta)}\alpha, Y \succ + 2\rho(X).S_Y\mathfrak{a}(\alpha,\beta) - \prec S^*_{\mathrm{r}_\#(\alpha)}\beta, S_XY \succ + \prec S^*_{\mathrm{r}_\#(\beta)}\alpha, S_XY \succ - 2S_{S_XY}\mathfrak{a}(\alpha,\beta) \\ = &\ 2S^2_{X,Y}\mathfrak{a}(\alpha,\beta).\end{aligned}$$

So we get the lemma. □

Corollary 4.4. *Let (A, M, ρ, S) be a left symmetric algebroid such that ρ is into and let $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$. Then $\big(A^*, M, \rho_{\mathrm{r}}, T\big)$ is a left symmetric algebroid with (A, M, ρ, S) and $\big(A^*, M, \rho_{\mathrm{r}}, T\big)$ being compatible if and only if*

$$\Delta(r) = 0 \quad \textit{and} \quad S^2\mathfrak{a} = 0.$$

Let (A, M, ρ, S) be a left symmetric algebroid and $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$ satisfying (4.4). Then $\big(A^*, M, \rho_{\mathrm{r}}, T\big)$ is a left symmetric algebroid compatible with (A, M, ρ, S). According to Theorem 3.7, $\big(\Phi(A), M, \rho \oplus \rho_{\mathrm{r}}\big)$ carries a para-Kähler Lie algebroid structure which will be called exact. This induces on M a symmetric bivector and a Poisson tensor which, by virtue of (3.9), are given by

$$h(\alpha,\beta) = 2\mathfrak{s}(\rho^*(\alpha), \rho^*(\beta)) \quad \text{and} \quad \pi(\alpha,\beta) = 2\mathfrak{a}(\rho^*(\alpha), \rho^*(\beta)). \tag{4.7}$$

Example 4.5. Let (A, M, ρ, S) be a left symmetric algebroid and $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$ which is S-parallel, i.e., $S\mathrm{r} = 0$. Then $S\mathfrak{a} = 0$ and it is easy to check that $\Delta(\mathrm{r}) = 0$. Thus r satisfies (4.4).

5 | PARA-KÄHLER LIE ALGEBROIDS ASSOCIATED TO QUASI $\mathbb{S}$-MATRICES

In this section, we study a class of exact para-Kähler Lie algebroids associated to a kind of solutions of (4.4) we will call quasi S-matrices using the same terminology used in the context of para-Kähler Lie algebras in [3].

Definition 5.1. A quasi $\mathbb{S}$-matrix of a left symmetric algebroid (A, M, ρ, S) is a $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in \Gamma(A \otimes A)$ such that, for any $\alpha, \beta \in \Gamma(A^*)$ and $X \in \Gamma(A)$,

$$\rho \circ \Delta(r) = 0, \;\; S\mathfrak{a} = 0 \quad \text{and} \quad Q_X\Delta(r)(\alpha,\beta) := [X, \Delta(r)(\alpha,\beta)] - \Delta(r)\big(S^*_X\alpha, \beta\big) - \Delta(r)\big(\alpha, S^*_X\beta\big) = 0.$$

The particular case when r is symmetric and $\Delta(r) = 0$ has been considered in [16].

In what follows, we focus our attention on the para-Kähler Lie algebroid structure on $\Phi(A)$ associated to a quasi $\mathbb{S}$-matrix. We show that the Lie algebroid structure can be described in a precise and simple way. Indeed, let r be a quasi S-matrix. Then, according to Theorem 4.2, the right-anchored product T on A^* given by (4.1) is left symmetric and $\left(\Phi(A), [\ ,\ ]^r, \rho+\rho_{\mathrm{r}}, \langle\ ,\ \rangle_0, K_0\right)$ is a para-Kähler Lie algebroid, where

$$[X+\alpha, Y+\beta]^r = [X,Y]_S + S^*_X\beta + T^*_\alpha Y - S^*_Y\alpha - T^*_\beta X + [\alpha,\beta]_T.$$

We have shown in Example 3.9 that $\Phi(A)$ carries a left symmetric product ∇^0 and its associated Lie bracket $[\ ,\ ]^{\triangleright}$ induces on $\Phi(A)$ a para-Kähler Lie algebroid structure. We define a new bracket on $\Phi(A)$ by putting

$$[X+\alpha, Y+\beta]^{\triangleright,r} = [X+\alpha, Y+\beta]^{\triangleright} + \Delta(\mathrm{r})(\alpha,\beta). \tag{5.1}$$

Proposition 5.2. $\left(\Phi(A), [\ ,\ ]^{\triangleright,r}, \rho+0\right)$ *is a Lie algebroid and the linear map*

$$\xi : \left(\Phi(A), [\ ,\ ]^{\triangleright,r}, \rho+0\right) \longrightarrow \left(\Phi(A), [\ ,\ ]^r, \rho+\rho_{\mathrm{r}}\right), X+\alpha \mapsto X - \mathrm{r}_\#(\alpha)+\alpha$$

is an isomorphism of Lie algebroids.

Proof. Clearly ξ is bijective. Let us show that ξ preserves the Lie brackets. It is clear that, for any $X, Y \in \Gamma(A)$, $\xi\left([X,Y]^{\triangleright,r}\right) = [\xi(X),\xi(Y)]^r$. Now, for any $X\in\Gamma(A)$, $\alpha\in\Gamma(A^*)$,

$$\begin{aligned}
\xi\left([X,\alpha]^{\triangleright,r}\right) &= \xi\left(S^*_X\alpha\right)\\
&= -\mathrm{r}_\#\left(S^*_X\alpha\right) + S^*_X\alpha\\
&\overset{(4.5)}{=} -T^*_\alpha X - \left[X, \mathrm{r}_\#(\alpha)\right]_S + S^*_X\alpha\\
&= \left[X, -\mathrm{r}_\#(\alpha)+\alpha\right]^r\\
&= [\xi(X),\xi(\alpha)]^r.
\end{aligned}$$

On the other hand, for any $\alpha,\beta\in\Gamma(A^*)$,

$$\begin{aligned}
\xi\left([\alpha,\beta]^{\triangleright,r}\right) &= \xi(\Delta(\mathrm{r})(\alpha,\beta)) = \Delta(\mathrm{r})(\alpha,\beta),\\
[\xi(\alpha),\xi(\beta)]^r &= \left[-\mathrm{r}_\#(\alpha)+\alpha, -\mathrm{r}_\#(\beta)+\beta\right]^r\\
&= \left[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)\right]_S + [\alpha,\beta]_T - S^*_{\mathrm{r}_\#(\alpha)}\beta + S^*_{\mathrm{r}_\#(\beta)}\alpha - T^*_\alpha\mathrm{r}_\#(\beta) + T^*_\beta\mathrm{r}_\#(\alpha)\\
&\overset{(4.5)}{=} \left[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)\right]_S - \mathrm{r}_\#\left(S^*_{\mathrm{r}_\#(\beta)}\alpha\right) + \mathrm{r}_\#\left(S^*_{\mathrm{r}_\#(\alpha)}\beta\right) + \left[\mathrm{r}_\#(\beta),\mathrm{r}_\#(\alpha)\right]_S - \left[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)\right]_S\\
&= \mathrm{r}_\#([\alpha,\beta]_T) - \left[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)\right]_S\\
&= \Delta(\mathrm{r})(\alpha,\beta).
\end{aligned}$$

□

We can now transport the para-Kähler structure associated to r from $\left(\Phi(A), [\ ,\ ]^r, \langle\ ,\ \rangle_0, K_0\right)$ to $\Phi(A)$ via ξ and we get the following proposition.

Proposition 5.3. *Let* (A, M, ρ, S) *be a left symmetric algebroid and let* $\mathrm{r} = \mathfrak{a}+\mathfrak{s}\in\Gamma(A\otimes A)$ *be a quasi* $\mathbb{S}$*-matrix. Then* $\left(\Phi(A), [\ ,\ ]^{\triangleright,r}, \rho+0, \langle\ ,\ \rangle_r, K_r\right)$ *is a para-Kähler Lie algebroid, where*

$$\langle X+\alpha, Y+\beta\rangle_r = <\alpha, Y> + <\beta, X> -2\mathfrak{s}(\alpha,\beta) \quad \textit{and} \quad K_r(X+\alpha) = X-\alpha-2\mathrm{r}_\#(\alpha).$$

6 | SYMMETRIC QUASI $\mathbb{S}$-MATRICES ON AFFINE MANIFOLDS AND CONTRAVARIANT PSEUDO-HESSIAN STRUCTURES

In this section, we study symmetric quasi $\mathbb{S}$-matrices on the left symmetric algebroid $\left(TM, M, \mathrm{Id}_{TM}, \nabla\right)$ associated to an affine manifold (M,∇). This leads naturally to a new structure we call contravariant pseudo-Hessian structure. There are many similarities between Poisson manifolds as a generalization of symplectic manifolds and contravariant pseudo-Hessian manifolds as a generalization of pseudo-Hessian manifolds and we show some of these similarities.

Symmetric quasi $\mathbb{S}$-matrices on affine manifolds

Let (M,∇) be an affine manifold, i.e., a manifold endowed with a torsionless flat connection. Then $\left(TM,M,Id_{TM},\nabla\right)$ is a left symmetric algebroid and according to Definition 5.1, a symmetric quasi $\mathbb{S}$-matrix on $\left(TM,M,Id_{TM},\nabla\right)$ is a symmetric bivector field h on M such that $\Delta(h)=0$. Let's study this equation more carefully. We denote by $\mathcal{D}$ the right-anchored product on $\left(T^*M,M,h_\#\right)$ associated to h. According to (4.1), we have for any $\alpha,\beta\in\Omega^1(M)$ and $X\in \mathrm{X}(M)$,

$$\prec \mathcal{D}_\alpha\beta, X\succ=\nabla_X h(\alpha,\beta)+\prec\nabla^*_{h_\#(\alpha)}\beta,X\succ \quad\text{and}\quad [\alpha,\beta]_{\mathcal{D}}=\nabla^*_{h_\#(\alpha)}\beta-\nabla^*_{h_\#(\beta)}\alpha. \tag{6.1}$$

Proposition 6.1. *We have, for any $\alpha,\beta,\gamma\in\Omega^1(M)$,*

$$\prec\gamma,\Delta(h)(\alpha,\beta)\succ=\nabla_{h_\#(\beta)}h(\alpha,\gamma)-\nabla_{h_\#(\alpha)}h(\beta,\gamma), \tag{6.2}$$

and

$$\prec\gamma,h_\#\left(\mathcal{D}_\alpha\beta\right)\succ=\prec\gamma,\nabla_{h_\#(\alpha)}h_\#(\beta)\succ+\prec\beta,\Delta(h)(\alpha,\gamma)\succ. \tag{6.3}$$

Proof. Let's compute

$$\begin{aligned}
\prec\gamma,\Delta(h)(\alpha,\beta)\succ &=\prec\gamma,h_\#([\alpha,\beta]_{\mathcal{D}})\succ-\prec\gamma,\left[h_\#(\alpha),h_\#(\beta)\right]\succ\\
&=h\left(\gamma,\nabla^*_{h_\#(\alpha)}\beta\right)-h\left(\gamma,\nabla^*_{h_\#(\beta)}\alpha\right)-\prec\gamma,\left[h_\#(\alpha),h_\#(\beta)\right]\succ\\
&=\nabla_{h_\#(\beta)}h(\alpha,\gamma)-\nabla_{h_\#(\alpha)}h(\beta,\gamma)+h_\#(\alpha).h(\beta,\gamma)\\
&\quad -h_\#(\beta).h(\alpha,\gamma)+h\left(\alpha,\nabla^*_{h_\#(\beta)}\gamma\right)-h\left(\beta,\nabla^*_{h_\#(\alpha)}\gamma\right)-\prec\gamma,\left[h_\#(\alpha),h_\#(\beta)\right]\succ\\
&=\nabla_{h_\#(\beta)}h(\alpha,\gamma)-\nabla_{h_\#(\alpha)}h(\beta,\gamma)+h_\#(\alpha).h(\beta,\gamma)-h_\#(\beta).h(\alpha,\gamma)+\prec\nabla^*_{h_\#(\beta)}\gamma,h_\#(\alpha)\succ\\
&\quad -\prec\nabla^*_{h_\#(\alpha)}\gamma,h_\#(\beta)\succ-\prec\gamma,\nabla_{h_\#(\alpha)}h_\#(\beta)\succ+\prec\gamma,\nabla_{h_\#(\beta)}h_\#(\alpha)\succ\\
&=\nabla_{h_\#(\beta)}h(\alpha,\gamma)-\nabla_{h_\#(\alpha)}h(\beta,\gamma).
\end{aligned}$$

Let's pursue

$$\begin{aligned}
\prec\mathcal{D}_\alpha\beta,h_\#(\gamma)\succ &= \nabla_{h_\#(\gamma)}h(\alpha,\beta)+h\left(\nabla^*_{h_\#(\alpha)}\beta,\gamma\right)\\
&\overset{(6.2)}{=} \nabla_{h_\#(\alpha)}h(\gamma,\beta)+h\left(\nabla^*_{h_\#(\alpha)}\beta,\gamma\right)+\prec\beta,\Delta(h)(\alpha,\gamma)\succ\\
&= h_\#(\alpha).h(\beta,\gamma)-h\left(\nabla^*_{h_\#(\alpha)}\gamma,\beta\right)+\prec\beta,\Delta(h)(\alpha,\gamma)\succ\\
&= \prec\gamma,\nabla_{h_\#(\alpha)}h_\#(\beta)\succ+\prec\beta,\Delta(h)(\alpha,\gamma)\succ.
\end{aligned}$$

□

By using Theorem 4.2, Equation (6.2) and Proposition 5.3 we get the following theorem.

Theorem 6.2. *Let (M,∇) be an affine manifold and let $\left(TM,M,Id_{TM},\nabla\right)$ be its associated left symmetric algebroid. Let h be a symmetric bivector field on M and consider $\mathcal{D}$ the right anchored product given by (6.1). Then $\left(T^*M,M,h_\#,\mathcal{D}\right)$ is a left symmetric algebroid compatible with $\left(TM,M,Id_{TM},\nabla\right)$ if and only if, for any $\alpha,\beta,\gamma\in\Omega^1(M)$,*

$$\nabla_{h_\#(\alpha)}h(\beta,\gamma)-\nabla_{h_\#(\beta)}h(\alpha,\gamma)=0. \tag{6.4}$$

In this case, $\left(TM \oplus T^*M, M, [\ ,\ ]^{\triangleright}, Id_{TM} + 0, \langle\ ,\ \rangle_h, K_h\right)$ *is a para-Kähler Lie algebroid, where*

$$[X + \alpha, Y + \beta]^{\triangleright} = [X, Y] + \nabla^*_X \beta - \nabla^*_Y \alpha,$$

$$\langle X + \alpha, Y + \beta\rangle_h = \prec \alpha, Y \succ + \prec \beta, X \succ - 2h(\alpha, \beta) \quad \textit{and} \quad K_h(X + \alpha) = X - 2h_\#(\alpha) - \alpha.$$

Moreover, the associated symplectic form Ω_0 *is given by*

$$\Omega_0(X + \alpha, Y + \beta) = \prec \beta, X \succ - \prec \alpha, Y \succ .$$

Remark 6.3. This theorem deserves some comments. Indeed, the theorem asserts that, given an affine manifold (M, ∇) and a symmetric bivector field h satisfying (6.4), we have:

1. $\left(T^*M, M, h_\#, \mathcal{D}\right)$ is a left symmetric algebroid,
2. $\left(TM \oplus T^*M, M, [\ ,\ ]^{\triangleright}, Id_{TM} + 0, \langle\ ,\ \rangle_h, \Omega_0, K_h\right)$ is a para-Kähler Lie algebroid.

These two results are a consequence of a long path based on all the results and the constructions performed before. The first assertion introduces a class of left symmetric algebroids and hence a class of Lie algebroids which, to our knowledge, has not been considered before. We will devote the reminder of this section to the study of this class. Also, to our knowledge, the class of para-Kähler Lie algebroids introduced in 2. has not been considered before.

Now once the results are available, one can prove the assertions 1. and 2. directly. For the first assertion, one can compute (a huge computation identical to the one in Lemma 4.3) the curvature of $\mathcal{D}$ and show that it vanishes. For the second assertion, we can use Proposition 3.1 and show either that the Nijenhuis torsion of K_h vanishes and Ω_0 is closed with respect to $[\ ,\ ]^{\triangleright}$ or show that K_h is parallel with respect to the Levi–Civita connection. We have seen in Example 3.9 that Ω_0 is parallel with respect to the Lie-admissible connection $\nabla^0_{X+\alpha}(Y + \beta) = \nabla_X Y + \nabla^*_X \beta$ and hence it is closed. To show that the Nijenhuis torsion with respect to $[\ ,\ ]^{\triangleright}$ vanishes is an easy computation using that $\Delta(h) = 0$. However, one must point out that ∇^0 is not the Levi–Civita connection $\overline{\nabla}$ of $\left(TM \oplus T^*M, M, [\ ,\ ]^{\triangleright}, Id_{TM} + 0, \langle\ ,\ \rangle_h\right)$ and a straightforward computation gives

$$\overline{\nabla}_X Y = \nabla_X Y,$$

$$\overline{\nabla}_X \alpha = \nabla^*_X \alpha - (\nabla_X h)_\#(\alpha),\ \overline{\nabla}_\alpha X = -(\nabla_X h)_\#(\alpha) \quad \text{and} \quad \overline{\nabla}_\alpha \beta = -2(\nabla_{h_\#(\alpha)} h)_\#(\beta) + \nabla^*_{h_\#(\alpha)} \beta - \mathcal{D}_\alpha \beta.$$

With this formula one can check that $\overline{\nabla} K_h = 0$.

We will give now other characterizations of bivector fields satisfying (6.4).

Proposition 6.4. *Let* (M, ∇) *a manifold endowed with a torsionless connection (we don't need to suppose that* ∇ *is flat). Le* h *be a symmetric bivector field on* M*. For any* $f \in C^\infty(M)$*, put* $X_f = h_\#(df)$*. Then the following assertions are equivalent.*

(i) h *satisfies* (6.4).
(ii) *For any* $f, g \in C^\infty(M)$ *and any* $\alpha \in \Omega^1(M)$*,* $d\alpha\left(X_f, X_g\right) = \nabla_{X_f} h_\#(\alpha)(g) - \nabla_{X_g} h_\#(\alpha)(f)$*.*
(iii) *For any* $f, g, \mu \in C^\infty(M)$*,* $\nabla_{X_f} X_\mu(g) = \nabla_{X_g} X_\mu(f)$*.*
(iv) *For any* $x \in M$*, there exists a coordinates system* $(x_1, \dots, x_n)$ *around* x *such that for any* $1 \le k \le n$ *and* $1 \le i < j \le n$*,* $\nabla_{X_{x_i}} X_{x_k}\left(x_j\right) = \nabla_{X_{x_j}} X_{x_k}\left(x_i\right)$*.*

Proof. We have

$$\begin{aligned}
\nabla_{X_f} h(dg, \alpha) &= X_f.h(dg, \alpha) - \prec \nabla^*_{X_f} dg, h_\#(\alpha) \succ - \prec \nabla^*_{X_f} \alpha, X_g \succ \\
&= X_f.h(dg, \alpha) - X_f.h(dg, \alpha) + \nabla_{X_f} h_\#(\alpha)(g) - X_f.h(dg, \alpha) + \alpha\left(\nabla_{X_f} X_g\right) \\
&= -X_f.\alpha\left(X_g\right) + \nabla_{X_f} h_\#(\alpha)(g) + \alpha\left(\nabla_{X_f} X_g\right).
\end{aligned}$$

Thus

$$\nabla_{X_f} h(dg, \alpha) - \nabla_{X_g} h(df, \alpha) = -d\alpha\left(X_f, X_g\right) + \nabla_{X_f} h_\#(\alpha)(g) - \nabla_{X_g} h_\#(\alpha)(f).$$

This relation and the fact that (6.4) is tensorial permit to prove the proposition. □

Remark 6.5. Let (M,∇) as in Proposition 6.4 and h satisfying (6.4). By using (iii) of Proposition 6.4, for any $f,g,\mu\in C^\infty(M)$, we get

$$\begin{aligned}[X_f,X_g](\mu) &= \nabla_{X_f}X_g(\mu)-\nabla_{X_g}X_f(\mu)\\ &= \nabla_{X_\mu}X_g(f)-\nabla_{X_\mu}X_f(g)\\ &= [X_\mu,X_g](f)+[X_f,X_\mu](g)+\nabla_{X_g}X_\mu(f)-\nabla_{X_f}X_\mu(g)\\ &= [X_\mu,X_g](f)+[X_f,X_\mu](g).\end{aligned}$$

Thus

$$[X_f,X_g](\mu)+[X_g,X_\mu](f)+[X_\mu,X_f](g)=0. \tag{6.5}$$

So the triple product $\{.,.,.\}\,:\,C^\infty(M)\times C^\infty(M)\times C^\infty(M)\longrightarrow C^\infty(M)$ given by $\{f,g,\mu\}=[X_f,X_g](\mu)$ satisfies:

1. $\{f,g,\mu\}=-\{g,f,\mu\}$,

2. $\{f,g,\mu\}+\{g,\mu,f\}+\{\mu,f,g\}=0$,

3. $\{f,g,\mu_1\mu_2\}=\{f,g,\mu_1\}\mu_2+\{f,g,\mu_2\}\mu_1$.

Note that we have a similar situation on a Poisson manifold. Indeed, if M is a manifold endowed with a Poisson bracket $\{\ ,\ \}$ then the triple product $\langle\ ,\ ,\ \rangle$ given by $\langle f,g,\mu\rangle=\{\{f,g\},\mu\}$ satisfies the relations 1.,2.,3. above.

Contravariant pseudo-Hessian manifolds

We will show now that the triple (M,∇,h) satisfying (6.4) are a generalization of a well-known structure, namely, a pseudo-Hessian structure. Recall that a pseudo-Hessian manifold (see [22]) is a triple (M,∇,g) where ∇ is a flat torsionless connection and g is a pseudo-Riemannian metric is given locally by $g=\nabla d\phi$ where ϕ is a local function. This is equivalent to $S:=\nabla g$ is totally symmetric, i.e., (∇,g) satisfying the Codazzi equation

$$\nabla_X g(Y,Z)=\nabla_Y g(X,Z). \tag{6.6}$$

If we put $h=g^{-1}$ and take $X=X_v$, $Y=X_v$ and $Z=X_w$ with $u,v,w\in C^\infty(M)$, on can see easily that this equation is equivalent to $\nabla_{X_u}X_w(v)=\nabla_{X_v}X_w(u)$ and hence, by virtue of Proposition 6.4, g satisfies Codazzi equation if and only if g^{-1} satisfies (6.4). There is a subclass of the class of pseudo-Hessian manifolds, namely, the subclass of affine special real manifolds which appeared in physics. A pseudo-Hessian manifold (M,∇,g) is called affine special real manifold if, in addition, S is parallel. In [1], this subclass has been studied in detail and, in particular, the r-map which associate to any pseudo-Hessian manifold (M,∇,g) a natural pseudo-Kählerian structure on TM has been scrutinized. By virtue of what above, the following definition is natural.

Definition 6.6. We call a triple (M,∇,h) where ∇ is torsionless flat and h satisfies (6.4) a *contravariant pseudo-Hessian manifold*. If, in addition, the tensor field T given by $T(\alpha,\beta,\gamma)=\nabla_{h_\#(\alpha)}h(\beta,\gamma)$ is parallel with respect to the anchored product $\mathcal{D}$ given by (6.1), we call (M,∇,h) contravariant affine special real manifold.

The following theorem shows a similarity between Poisson manifolds and contravariant pseudo-Hessian manifolds.

Theorem 6.7. *Let (M,∇,h) be a contravariant pseudo-Hessian manifold. Then* $\operatorname{Im}h_\#$ *is integrable and defines a singular foliation on M such that for every leaf L we have:*

(i) *For every vector fields X,Y tangent to L, ∇_XY is tangent to L,*

(ii) *L has a natural pseudo-Hessian structure. Moreover, if (M,∇,h) is a contravariant affine special real manifold then L is an affine special real manifold.*

Proof. According to Theorem 6.2, $(T^*M,M,h_\#,\mathcal{D})$ is a left symmetric Lie algebroid and hence $(T^*M,M,h_\#,[\ ,\ ]_{\mathcal{D}})$ is a Lie algebroid. This implies that $\operatorname{Im}h_\#$ is integrable and defines a singular foliation on M. Each leaf L carries a pseudo-Riemannian metric g_L given by $g_L(h_\#(\alpha), h_\#(\beta)) = h(\alpha, \beta)$. On the other hand, (6.3) shows that any leaf L carries an affine structure ∇^L and one can check easily that $\left(\nabla^L, g_L\right)$ is a pseudo-Hessian structure on L which is affine special real when (M, ∇, h) is. □

Example 6.8. Consider $\mathbb{R}^n$ endowed with its canonical affine structure ∇ and denote by $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ its canonical linear coordinates. Let $f \in C^\infty(M)$ such that the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is invertible and put

$$h = \sum_{i,j=1}^{r} h_{ij} \partial_{x_i} \otimes \partial_{x_j},$$

where (h_{ij}) is the inverse of the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. Then $(\mathbb{R}^n, \nabla, h)$ is a contravariant pseudo-Hessian structure. This is a consequence of the following proposition.

Proposition 6.9. *Let (M, ∇) be an affine manifold and let $i : \mathcal{F} \longrightarrow TM$ be a subbundle such that, for any $X, Y \in \Gamma(\mathcal{F})$, $\nabla_X Y \in \Gamma(\mathcal{F})$. Suppose that there exists $\phi \in C^\infty(M)$ such that g given by $g_x(u, v) = \nabla_u d\phi(v)$ for any $x \in M$ and any $u, v \in \mathcal{F}_x$ is nondegenerate symmetric bilinear form on $\mathcal{F}_x$. Then $h_\# = i \circ \# \circ i^*$, where $\# : \mathcal{F}^* \longrightarrow \mathcal{F}$ is the isomorphism associated to g, defines a contravariant pseudo-Hessian structure on (M, ∇).*

Proof. For any $\alpha, \beta, \gamma \in \Omega^1(M)$, put $X = h_\#(\alpha)$, $Y = h_\#(\beta)$ and $Z = h_\#(\gamma)$. There are three vector fields tangent to $\mathcal{F}$. Since $\nabla_X Y$ and $\nabla_X Z$ are tangent to $\mathcal{F}$ then

$$\nabla_{h_\#(\alpha)} h(\beta, \gamma) = -h_\#(\alpha).h(\beta, \gamma) + \prec \beta, \nabla_{h_\#(\alpha)} h_\#(\gamma) \succ + \prec \gamma, \nabla_{h_\#(\alpha)} h_\#(\beta) \succ = -\nabla_X g(Y, Z).$$

Now

$$\begin{aligned}
\nabla_X g(Y, Z) - \nabla_Y g(X, Z) &= X.\nabla d\phi(Y, Z) - Y.\nabla d\phi(X, Z) - \nabla d\phi(\nabla_X Y - \nabla_Y X, Z) \\
&\quad -\nabla d\phi(Y, \nabla_X Z) + \nabla d\phi(X, \nabla_Y Z) \\
&= [X, Y].d\phi(Z) - X.d\phi(\nabla_Y Z) + Y.d\phi(\nabla_X Z) - [X, Y].d\phi(Z) + d\phi(\nabla_{[X,Y]} Z) \\
&\quad -Y.d\phi(\nabla_X Z) + d\phi(\nabla_Y \nabla_X Z) + X.d\phi(\nabla_Y Z) - d\phi(\nabla_X \nabla_Y Z) \\
&= 0.
\end{aligned}$$

□

The following proposition is a generalization of Lemma 2.1 in [12]. The proof we give here is different.

Proposition 6.10. *Let (M, ∇) be an affine manifold and let $i : D \longrightarrow TM$ be a subbundle such that, for any $X, Y \in \Gamma(D)$, $\nabla_X Y \in \Gamma(D)$. Then, for any $m \in M$, there exists a coordinates system $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ on an open set around m such that, for any $p \in U$,*

$$D(p) = \mathrm{span}\left\{\partial_{x_1}(p), \dots, \partial_{x_r}(p)\right\} \quad \textit{and} \quad \nabla_{\partial_{x_i}} \partial_{x_j} = 0,\ i, j = 1, \dots, r.$$

Proof. Denote by ∇^1 the restriction of ∇ to D. Then $\left(D, M, i, \nabla^1\right)$ is a left symmetric algebroid. Moreover, ∇^1 satisfies the hypothesis of Lemma 2.9. Let $(e_1, \dots, e_r)$ be a basis of $D(m)$. By virtue of Lemma 2.9, there exists a family of local vector fields $X_1, \dots, X_r$ tangent to D such that $\nabla^1 X_i = 0$, $X_i(m) = e_i$, $i = 1, \dots, r$. For any, $i, j = 1, \dots, r$, $\left[X_i, X_j\right] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$ and the proposition follows by applying Frobenius's theorem. □

The following theorem shows that any contravariant pseudo-Hessian structure is locally as Proposition 6.9 near any regular point. This can be compared to Darboux–Weinstein near a regular point in Poisson geometry (see [26]).

Theorem 6.11. *Let (M, ∇, h) be a contravariant pseudo-Hessian structure and let $x \in M$ be such the rank of $h_\#$ is constant in a neighborhood of x. Then there exists a chart $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ and a function $f(x, y)$ such that*

$$h = \sum_{i,j=1}^{r} h_{ij} \partial_{x_i} \otimes \partial_{x_j}, \quad \nabla_{\partial_{x_i}} \partial_{x_j} = 0,\ \ i, j = 1, \dots, r,$$

and the matrix (h_{ij}) *is invertible and its inverse is the matrix* $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Proof. By applying Proposition 6.10 to $\operatorname{Im} h_\#$ near x, there exists a chart $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ such that

$$\operatorname{span}\left(\partial_{x_1}, \dots, \partial_{x_r}\right) = \operatorname{Im} h_\# \quad \text{and} \quad \nabla_{\partial_{x_i}} \partial_{x_j} = 0, \; i, j = 1, \dots, r.$$

This implies that $h_\#(dy_i) = 0$ for $i = 1, \dots, n-r$. Note first that, for any $i, j, k = 1, \dots, n$, $\prec \nabla^*_{\partial_{x_k}} dx_j, \partial_{x_j} \succ = 0$ and hence from (6.1),

$$\begin{aligned}\prec \mathcal{D}_{dx_i} dx_j, \partial_{x_k} \succ &= \nabla_{\partial_{x_k}} h(dx_i, dx_j) + \prec \nabla_{h_\#(dx_i)} dx_j, \partial_{x_k} \succ \\ &= \prec dh(dx_i, dx_j), \partial_{x_k} \succ - \prec \nabla^*_{\partial_{x_k}} dx_i, h_\#(dx_j) \succ - \prec \nabla^*_{\partial_{x_k}} dx_j, h_\#(dx_k) \succ \\ &= \prec dh(dx_i, dx_j), \partial_{x_k} \succ .\end{aligned}$$

Thus $\mathcal{D}_{dx_i} dx_j = dh\big(dx_i, dx_j\big) + \alpha_{ij}$ where $\alpha_{ij} \in \ker h_\#$. This implies that

$$\big[h_\#(dx_i), h_\#\big(dx_j\big)\big] = h_\#\big(\mathcal{D}_{dx_i} dx_j\big) - h_\#\big(\mathcal{D}_{dx_j} dx_i\big) = 0.$$

So there exists a coordinates system $(z_1, \dots, z_n)$ such that

$$h_\#(dx_i) = \partial_{z_i}, \; i = 1, \dots, r.$$

We deduce that

$$\partial_{x_i} = \sum_{j=1}^{r} h^{ij} \partial_{z_j}, \; i = 1, \dots, r, \; \big(h^{ij}\big) = \big(dh\big(dx_i, dx_j\big)\big)^{-1}.$$

Thus $h^{ij} = \frac{\partial z_j}{\partial x_i}$. We consider $\sigma = \sum_{j=1}^r z_j dx_j$. We have $d_{\mathcal{F}}\sigma = 0$ so, according to the foliated Poincaré lemma (see [19], pp. 56), there exists a function f such that $h^{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, which completes the proof. □

Remark 6.12. It is important to generalize the theorem above near a singular point. The authors have no ideas how to do it and left the problem open.

7 | CANONICAL CONTRAVARIANT PSEUDO-HESSIAN STRUCTURE ON THE DUAL OF A COMMUTATIVE ASSOCIATIVE ALGEBRA

In this section, we will pursue the study of similarities between contravariant pseudo-Hessian manifolds and Poisson manifolds. Namely, we will show that the dual of any commutative and associative algebra carries a canonical contravariant pseudo-Hessian structure and we will study these structures in detail.

Let (M, ∇, h) be a contravariant pseudo-Hessian manifold. Let $x \in M$ and denote by $\mathfrak{g}_x = \ker h_\#(x)$. Let $\mathcal{D}$ be the right-anchored product associated to h given by (6.1). According to (6.3), for any $\alpha, \beta \in \Omega^1(M)$, $h_\#(\mathcal{D}_\alpha \beta) = \nabla_{h_\#(\alpha)} h_\#(\beta)$. This shows that if $h_\#(\alpha)(x) = 0$ then $h_\#(\mathcal{D}_\alpha \beta)(x) = 0$. Moreover, $\mathcal{D}_\alpha \beta - \mathcal{D}_\beta \alpha = \nabla_{h_\#(\alpha)} \beta - \nabla_{h_\#(\beta)} \alpha$. This implies that if $h_\#(\alpha)(x) = h_\#(\beta)(x) = 0$ then $\mathcal{D}_\alpha \beta(x) = \mathcal{D}_\beta \alpha(x)$. For any $a, b \in \mathfrak{g}_x$ put

$$a.b = (\mathcal{D}_\alpha \beta)(x),$$

where α, β are 2 differential 1-forms satisfying $\alpha(x) = a$ and $\beta(x) = b$. This defines a commutative product on $\mathfrak{g}_x$ and moreover, by using the vanishing of the curvature of $\mathcal{D}$, we get:

Proposition 7.1. $(\mathfrak{g}_x, .)$ *is a commutative associative algebra.*

As the dual of a Lie algebra carries a natural Poisson structure, the dual of a commutative associative algebra carries a contravariant pseudo-Hessian structure. Indeed, let $(\mathcal{A}, .)$ be a finite dimensional commutative associative algebra. We define a symmetric bivector h on $\mathcal{A}^*$ by putting

$$h(\alpha,\beta)(\mu) = \prec \mu, \alpha(\mu).\beta(\mu) \succ, \quad \alpha,\beta \in \Omega^1(\mathcal{A}^*) = C^\infty(\mathcal{A}^*, \mathcal{A}), \mu \in \mathcal{A}^*. \tag{7.1}$$

We denote by ∇^0 the canonical affine connection of $\mathcal{A}^*$ given by $\nabla^0_X Y(\mu) = d_\mu Y(X(\mu))$ where $X, Y : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ are regarded as vector fields on $\mathcal{A}^*$. For any $u \in \mathcal{A}$, we denote by u^* the linear function on $\mathcal{A}^*$ given by $u^*(\mu) = \prec \mu, u \succ$, by X_u the vector field on $\mathcal{A}^*$ given by $X_u = h_\#(du^*)$ and by $L_u : \mathcal{A} \longrightarrow \mathcal{A}$ the left multiplication by u. Let $\mathcal{D}$ be the right-anchored product associated to $\left(\mathcal{A}^*, \nabla^0, h\right)$ and given by (6.1). Finally, denote by T the tensor field on $\mathcal{A}^*$ given by $T(\alpha,\beta,\gamma) = \nabla^0_{h_\#(\alpha)} h(\beta,\gamma)$. A straightforward computation gives the following proposition.

Proposition 7.2. *For any $u, v, w, x \in \mathcal{A}$, we have*

$$h\left(du^*, dv^*\right) = (u.v)^*, \quad X_u = L_u^*, \quad \nabla^0_{X_u} X_v = X_{u.v}, \quad T\left(du^*, dv^*, dw^*\right) = (u.v.w)^*,$$

$$\mathcal{D}_{du^*} dv^* = d(u.v)^* \quad \text{and} \quad \mathcal{D}T\left(du^*, dv^*, dw^*, dx^*\right) = -2(u.v.w.x)^*.$$

As a consequence of these formulas and Theorem 6.7, we get the following result.

Theorem 7.3. $\left(\mathcal{A}^*, \nabla^0, h\right)$ *is a contravariant pseudo-Hessian manifold and the singular foliation associated to* $\operatorname{Im} h_\#$ *is given by the orbits of the linear action Φ of the abelian Lie group $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(u,\mu) = \exp\left(L_u^*\right)(\mu)$. Moreover, $\left(\mathcal{A}^*, \nabla^0, h\right)$ is a contravariant affine special real manifold if and only if $\mathcal{A}^4 = 0$. In particular, the orbits of Φ are pseudo-Hessian manifolds and if $\mathcal{A}^4 = 0$ they are affine special real manifolds.*

Remark 7.4. This theorem is similar to the well-known result asserting that the dual of a Lie algebra carries a natural Poisson structure. However, there is an important difference between the two situations. In the case of a Lie algebra, the symplectic leaves are the orbits of the co-adjoint action of any connected Lie group associated to the Lie algebra and the action preserves the symplectic form of any leaf. In the case of a commutative associative algebra, the pseudo-Hessian leaves are the orbits of the action of $(\mathcal{A}, +)$ and this action preserves the affine structure of any leaf but not its pseudo-Hessian metric unless $\mathcal{A}^3 = 0$. Note that these pseudo-Hessian manifolds are diffeomorphic to a $\mathbb{R}^q \times \mathbb{T}^p$.

The class of associative commutative algebras constitutes a large class of associative algebras so Theorem 7.3 is a powerful tool to build examples of pseudo-Hessian manifolds and affine special real manifolds. Since any pseudo-Hessian structure on a manifold gives rise to a pseudo-Kählerian structure on its tangent bundle we get also a machinery to build examples of pseudo-Kählerian manifolds. In what follows, we will illustrate this by showing that, using Theorem 7.3, we can get interesting examples. Namely, we will show that the Hessian curvature of these manifolds is not trivial in general. Shima introduced the notion of Hessian curvature, which is a finer invariant than Riemannian curvature and is related with the curvature of the associated Kähler metric on the total space of the tangent bundle. Let us recall first the definition of the Hessian curvature and the definitions of some basic notions in a pseudo-Hessian manifold (see [22] for more details).

Let (M, ∇, g) be a pseudo-Hessian manifold. Denote by D the Levi–Civita connection of g and put $\nabla' = 2D - \nabla$ and $\gamma = D - \nabla$. The connection ∇' is called the dual connection of ∇ with respect to g and (M, ∇', g) is also a pseudo-Hessian structure. The Hessian curvature (M, ∇, g) is the tensor Q given by $Q = \nabla\gamma$. The first and the second Koszul forms are given, respectively, by $\alpha(X) = \operatorname{tr}(i_X\gamma)$ and $\beta = \nabla\alpha$.

Let compute now all the mathematical objects above in the case where $M = \left\{\exp\left(L_a^*\right)(\mu), a \in \mathcal{A}\right\}$ is an orbit of the pseudo-Hessian foliation associated to the contravariant pseudo-Hessian manifold $(\mathcal{A}^*, \nabla^0, h)$ appearing in Theorem 7.3. Note that $T_v M = \{X_a(v), a \in \mathcal{A}\}$. As above, we denote by ∇ the affine connection on M, g the pseudo-Riemannian metric, D the Levi–Civita connection and so on. The following proposition is a consequence of an easy and straightforward computation.

Proposition 7.5. *For any $a, b, c \in \mathcal{A}$ and any $v \in \mathcal{A}^*$,*

$$g\left(X_a(v), X_b(v)\right) = \prec v, a.b \succ, \quad \nabla_{X_a} X_b = X_{a.b}, \quad D_{X_a} X_b = \frac{1}{2} X_{a.b}, \quad \nabla'_{X_a} X_b = 0,$$

$$Q\left(X_a, X_b\right) X_c = \frac{1}{2} X_{a.b.c}, \quad \alpha\left(X_a\right) = -\frac{1}{2}\operatorname{tr}(L_a) \quad \text{and} \quad \beta\left(X_a, X_b\right) = \frac{1}{2}\operatorname{tr}(L_{a.b}).$$

In particular, g is a flat pseudo-Riemannian metric and $Q = 0$ if and only if $\mathcal{A}^4 = 0$.

We end this paper by considering examples of commutative associative algebras. For each of them we choose an orbit M and give in an affine system of coordinates (x_i) the pseudo-Hessian metric g and a function ϕ such that $g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. Some examples come from the lists of low dimensional associative algebras obtained in [21].

Example 7.6. All the algebras bellow are identified with $\mathbb{R}^n$ with its canonical basis $(e_i)_{i=1}^n$ and $(e_i^*)_{i=1}^n$ is the dual basis. The action Φ of $\mathcal{A}$ on $\mathcal{A}^*$ is given by $\Phi(a,\mu) = \exp\left(L_a^*\right)(\mu)$ and, for any $a \in \mathcal{A}$, X_a is the vector fields on $\mathcal{A}^*$ given by $X_a = L_a^*$, where L_a is the left multiplication by a. We denote by ∇ the canonical connection on $\mathcal{A}^*$.

1. We take $\mathcal{A} = \mathbb{R}^n$ as a product of n copies of the associative commutative algebra $\mathbb{R}$. The non vanishing product is given by $e_i e_i = e_i$ for $i = 1, \dots, n$. We denote by $(a_i)_{i=1}^n$ the linear coordinates of $\mathcal{A}$ and $(x_i)_{i=1}^n$ the dual coordinates on $\mathcal{A}^*$. We have
$$\Phi\left(\sum_{i=1}^n a_i e_i, \sum_{i=1}^n x_i e_i^*\right) = \sum_{i=1}^n e^{a_i} x_i e_i^*.$$
Moreover, for any $i = 1, \dots, n$, $X_{e_i} = x_i \partial_{x_i}$. The orbit of a point $x \in \mathcal{A}^*$ is $M_x = \left\{\sum_{i=1}^n e^{a_i} x_i e_i^*, a_i \in \mathbb{R}\right\}$. It is a convex cone and one can see easily that if $\phi : \mathcal{A}^* \longrightarrow \mathbb{R}$ is the function given by
$$\phi(u) = \sum_{i=1}^n u_i \ln |u_i|,$$
then the restriction of $\nabla d\phi$ to M_x together with the restriction of ∇ to M_x define the pseudo-Hessian structure on M_x described in Theorem 7.3. Note here that the signature of the pseudo-Hessian metric on M_x is exactly (p,q) where p is the number of x_i such that $x_i > 0$ and q is the number of x_i such that $x_i < 0$. Note that if $x_i > 0$ for $i = 1, \dots, n$ then the metric on M_x is definite positive and we recover the example given in [22], pp. 17.
2. We take $\mathcal{A} = \mathbb{C}$ endowed with its canonical structure of commutative and associative algebra. The non vanishing products are
$$e_1.e_1 = e_1,\ e_1.e_2 = e_2.e_1 = e_2,\ e_2.e_2 = -e_1.$$
We denote here by (x,y) the linear coordinates on $\mathcal{A}$ associated to (e_1,e_2) and (α,β) the dual coordinates on $\mathcal{A}^*$. We have
$$X_{e_1} = \alpha\partial_\alpha + \beta\partial_\beta \quad \text{and} \quad X_{e_2} = \beta\partial_\alpha - \alpha\partial_\beta$$
and it is easy to check that
$$\Phi(xe_1 + ye_2, \alpha e_1^* + \beta e_2^*) = e^x\left((\alpha\cos(y) + \beta\sin(y))e_1^* + (-\alpha\sin(y) + \beta\cos(y))e_2^*\right).$$
We deduce that we have two orbits the origin and $\mathcal{A}^* \setminus \{0\}$. Let describe the pseudo-Hessian structure of $M := \mathcal{A}^* \setminus \{0\}$. The pseudo-Hessian metric g satisfies
$$g\left(X_{e_1}, X_{e_1}\right) = \alpha,\ g\left(X_{e_1}, X_{e_2}\right) = \beta,\ g\left(X_{e_2}, X_{e_2}\right) = -\alpha$$
and hence
$$g = \frac{1}{\alpha^2+\beta^2}\left(\alpha d\alpha^2 + 2\beta d\alpha d\beta - \alpha d\beta^2\right).$$
Thus (M,∇,g) is a Lorentzian Hessian manifold. Moreover, the metric g is flat. Now we look for a function f on M such that $g = \nabla df$, i.e.,
$$\frac{\partial^2 f}{\partial\alpha^2} = \frac{\alpha}{\alpha^2+\beta^2},\ \frac{\partial^2 f}{\partial\beta^2} = \frac{-\alpha}{\alpha^2+\beta^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial\alpha\partial\beta} = \frac{\beta}{\alpha^2+\beta^2}.$$
The function f given by
$$f(\alpha,\beta) = \frac{1}{2}\alpha\ln\left(\alpha^2+\beta^2\right) + \beta\arctan\left(\frac{\alpha}{\beta}\right)$$
satisfies these equations on the open set $\{\beta \neq 0\}$. Note that this function is harmonic.

3. We take $\mathcal{A} = \mathbb{R}^3$ with the commutative associative product given by $e_1e_1 = e_2$ and $e_1e_2 = e_3$ and the others products are zero. We have $\mathcal{A}^3 \neq 0$ and $\mathcal{A}^4 = 0$. We denote by (a, b, c) the linear coordinates of $\mathcal{A}$ and (x, y, z) the dual coordinates of $\mathcal{A}^*$. We have

$$X_{e_1} = y\partial_x + z\partial_y,\ X_{e_2} = z\partial_x \quad \text{and} \quad X_{e_3} = 0$$

and

$$\Phi\big(ae_1 + be_2 + ce_3, xe_1^* + ye_2^* + ze_3^*\big) = \left(x + ay + \left(\frac{1}{2}a^2 + b\right)z, y + az, z\right).$$

The orbits of this action are the plans $\{z = c, c \neq 0\}$, the lines $\{z = 0, y = c, c \neq 0\}$ and the points $\{(c, 0, 0)\}$. The pseudo-Riemannian metric on $M_c = \{z = c, c \neq 0\}$ is given by

$$g_c\big(X_{e_1}, X_{e_1}\big) = y,\ g_c\big(X_{e_1}, X_{e_2}\big) = c \quad \text{and} \quad g_c\big(X_{e_2}, X_{e_2}\big) = 0.$$

This is a Lorentzian metric and one can check easily that, if $\phi(x, y, z) = -\frac{y^3}{6z^2} + \frac{xy}{z}$ then g_c is the restriction of $\nabla d\phi$ to M_c. Note that since $\mathcal{A}^4 = 0$ then $\big(M_c, \nabla_{|M_c}, g_c\big)$ is an affine special real manifold. However, the pseudo-Hessian metric on the line $L_c = \{z = 0, y = c, c \neq 0\}$ is given by the restriction of $\nabla d\phi_1$ where $\phi_1(x, y, z) = \frac{x^2}{2y}$.

4. We take $\mathcal{A} = \mathbb{R}^3$ with the commutative associative product given by

$$e_1e_1 = e_2,\ e_1e_3 = e_1,\ e_2e_3 = e_2,\ e_3e_3 = e_3,$$

the others products are zero. We denote by (a, b, c) the linear coordinates on $\mathcal{A}$ and (x, y, z) the dual coordinates on $\mathcal{A}^*$. We have

$$\Phi\big(ae_1 + be_2 + ce_3, xe_1^* + ye_2^* + ze_3^*\big) = e^c\left(x + ay, y, ax + \frac{1}{2}\big(a^2 + 2b\big)y + z\right).$$

The orbits have dimension 3,2,1 or 0. The three dimensional orbits are $\{y > 0\}$ and $\{y < 0\}$. The two dimensional orbits are $\{y = 0, x > 0\}$ and $\{y = 0, x < 0\}$. The one dimensional orbits are $\{y = x = 0, z > 0\}$ and $\{y = x = 0, z < 0\}$. The origin is the only zero dimensional orbit. Let describe the pseudo-Hessian structure on $M = \{y > 0\}$ or $M = \{y < 0\}$. We have

$$X_{e_1} = y\partial_x + x\partial_z,\ X_{e_2} = y\partial_z \quad \text{and} \quad X_{e_3} = x\partial_x + y\partial_y + z\partial_z,$$

and the pseudo-Hessian metric g on M is satisfies

$$g\big(X_{e_1}, X_{e_1}\big) = y,\ g\big(X_{e_1}, X_{e_2}\big) = 0,$$

$$g\big(X_{e_1}, X_{e_3}\big) = x,\ g\big(X_{e_2}, X_{e_2}\big) = 0,\ g\big(X_{e_2}, X_{e_3}\big) = y \quad \text{and} \quad g\big(X_{e_3}, X_{e_3}\big) = z.$$

Note that the matrix of g in $\big(X_{e_1}, X_{e_2}, X_{e_3}\big)$ is just the passage matrix P from $\big(X_{e_1}, X_{e_2}, X_{e_3}\big)$ to $\big(\partial_x, \partial_y, \partial_z\big)$ and hence the matrix of g in $\big(\partial_x, \partial_y, \partial_z\big)$ is P^{-1}. Thus, in the coordinates (x, y, z), we have

$$g = \frac{1}{y}\left(dx^2 + \frac{x^2 - yz}{y}dy_2 + 2dydz - \frac{2x}{y}dxdy\right).$$

One can check easily that g is the restriction of $\nabla d\phi$ where $\phi(x, y, z) = z\ln|y| + \frac{x^2}{2y}$. This metric is of signature $(+, +, -)$ in $\{y > 0\}$ and $(+, -, -)$ in $\{y < 0\}$.

5. We take $\mathcal{A} = \mathbb{R}^4$ with the commutative associative product given by

$$e_1e_1 = e_2,\ e_1e_2 = e_3,\ e_1e_3 = e_2e_2 = e_4,$$

the others products are zero. We have $\mathcal{A}^3 \neq 0$ and $\mathcal{A}^4 = 0$. We denote by (a, b, c, d) the linear coordinates on $\mathcal{A}$ and (x, y, z, t) the dual coordinates on $\mathcal{A}^*$. We have

$$\begin{aligned}&\Phi\left(ae_1 + be_2 + ce_3 + de_4, xe_1^* + ye_2^* + ze_3^* + te_4^*\right)\\&= \left(x + ay + \left(\frac{1}{2}a^2 + b\right)z + \left(\frac{1}{6}a^3 + ab + c\right)t, y + az + \left(\frac{1}{2}a^2 + b\right)t, z + at, t\right)\end{aligned}$$

and

$$X_{e_1} = y\partial_x + z\partial_y + t\partial_z,\ X_{e_2} = z\partial_x + t\partial_y,\ X_{e_3} = t\partial_x \quad \text{and} \quad X_{e_4} = 0.$$

Let describe the pseudo-Hessian structure of the hyperplan $M_c = \{t = c, c \neq 0\}$ endowed with the coordinates (x, y, z). Since the matrix of g_c in $\left(X_{e_1}, X_{e_2}, X_{e_3}\right)$ is the passage matrix P from $\left(X_{e_1}, X_{e_2}, X_{e_3}\right)$ to $\left(\partial_x, \partial_y, \partial_z\right)$, we get

$$g_c = \frac{1}{c}\left(2dxdz + dy^2 - \frac{2z}{c}dydz + \frac{(z^2 - yc)}{c^2}dz^2\right).$$

The signature of this metric is $(+, +, -)$ if $c > 0$ and $(+, -, -)$ if $c < 0$. One can check easily that g_c is the restriction of $\nabla d\phi$ to M_c, where

$$\phi(x, y, z, t) = \frac{z^4}{12t^3} + \frac{y^2}{2t} - \frac{z^2y}{2t} + \frac{xz}{t}.$$

Since $\mathcal{A}^4 = 0$, M_c is an affine special real manifold.

6. We take $\mathcal{A} = \mathbb{R}^4$ with the commutative associative product given by

$$e_1e_1 = e_1,\ e_1e_2 = e_2,\ e_1e_3 = e_3,\ e_1e_4 = e_4,\ e_2e_2 = e_3,\ e_2e_3 = e_4.$$

We have

$$X_{e_1} = x\partial_x + y\partial_y + z\partial_z + t\partial_t,\ X_{e_2} = y\partial_x + z\partial_y + t\partial_z,\ X_{e_3} = z\partial_x + t\partial_y \quad \text{and} \quad X_{e_4} = t\partial_x.$$

Thus $\{t > 0\}$ and $\{t < 0\}$ are orbits and hence carry a pseudo-Hessian structures. Let us determine the pseudo-Hessian metric. The same argument as above gives that the metric is given by the inverse of the passage matrix from $\left(X_{e_1}, \dots, X_{e_4}\right)$ to $\left(\partial_x, \partial_y, \partial_z, \partial_t\right)$. Thus

$$g = \frac{1}{t}\left(2dxdt + 2dydz - \frac{2z}{t}dydt - \frac{z}{t}dz^2 + \frac{2(z^2 - yt)}{t^2}dzdt + \frac{2zyt - xt^2 - z^3}{t^3}dt^2\right).$$

The signature of this metric is $(+, +, -, -)$. One can check easily that g is the restriction of $\nabla d\phi$ to M, where

$$\phi(x, y, z, t) = -\frac{z^3}{6t^2} + \frac{yz}{t} + x\ln|t|.$$

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REFERENCES

[1] D. V. Alekseevsky and V. Cortés, *Geometric construction of the r-map: from affine special real to special Kähler manifolds*, Comm. Math. Phys. **291** (2009), 579. https://doi.org/10.1007/s00220-009-0803-7.

[2] C. Bai, *Left-symmetric bialgebras and an analogue of the classical yang–Baxter equation*, Commun. Contemp. Math. **10** (2008), no. 2, 221–260.

[3] S. Benayadi and M. Boucetta, *On para-Kähler and Hyper-Kähler Lie algebras*, J. Algebra **436** (2015), 61–101.

[4] R. Blumenhagen, A. Deser, E. Plauschinn et al., *Non-geometric strings, symplectic gravity and differential geometry of Lie algebroids*, J. High Energy Phys. **2013** (2013), 122, available at https://doi.org/10.1007/JHEP02(2013)122.

[5] M. Boucetta, *Solutions of the classical Yang–Baxter equation and non-commutative deformations*, Lett. Math. Phys. **83** (2008), 69–81.

[6] A. C. da Silva and A. Weinstein, *Geometric models for noncommutatve algebras*, Berkeley Mathematics Lecture Notes, vol. 10, Amer. Math. Soc., Providence, 1999.

[7] P. Cartier, *Groupoïdes de Lie et leurs algébroïdes*, Séminaire Bourbaki 2007–2008, no. 987 (French).

[8] J. Cortès, M. De Leòn, J. C. Marrero, M. De Diego, and E. Martinez, *A survey of Lagrangian mechanics and control on Lie algebroids and groupoids*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 3, 509–558.

[9] V. Cruceanu, P. M. Gadea, and J. Munoz Masque, *Para-Hermitian and para-Kähler manifolds*, Quaderni. Inst. Mat. Univ. Messina **1** (1995), 1–72.

[10] N. Elyasi and N. Boroojerdian, *Application of Lie algebroid structures to unification of Einstein and Yang–Mills field equations*, Internat. J. Theoret. Phys., available at https://doi.org/10.1007/s10773-014-2035-7.

[11] R. L. Fernandes, *Lie algebroids, holonomy and characteristic classes*, Adv. in Math. **170** (2002), 119–179.

[12] P. M. D. Furness, *Affine foliations of codimension one*, Q. J. Math. **25** (1974), no. 1, 151–161.

[13] S. Kobayashi and K. Nomizu, *Foundations of differential geometry II*, Interscience Publisher, 1969.

[14] E. Leitchman, X. Tang, and A. Weinsten, *Poisson geometry and deformation quantization near a strictly pseudoconvex boundary*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 681–704.

[15] J. Liu, Y. Sheng, Z. Chen, and C. Bai, *Left symmetric algebroids*, Math. Nachr. **289** (2016), no. 14–15, 1893–1908.

[16] J. Liu, Y. Sheng, and C. Bai, *Left-symmetric bialgebroids and their corresponding Manin triples*, Differential Geom. Appl. **59** (2018) 91–111.

[17] J. Liu, Y. Sheng, and C. Bai, *Pre-symplectic algebroids and their applications*, Lett. Math. Phys. **108** (2018), 779–804.

[18] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Notes Ser., vol. 124, Cambridge Univ. Press, Cambridge, 1987.

[19] C. C. Moore and C. L. Schochet, *Global analysis on foliated space*, MISRP, 2006.

[20] M. Nguiffo Boyom, *Cohomology of Koszul–Vinberg algebroids and Poisson manifolds I, Lie algebroids*, Banach Center Publications, vol. 54, Institute of Mathematics Academy of Sciences Warszawa, 2001.

[21] I. S. Rakhimov, I. M. Rikhsiboev, and W. Basri, *Complete lists of low dimensional complex associative algebras*, arXiv:0910.0932v2 [math.RA].

[22] H. Shima, *The geometry of Hessian structures*, World Scientific Publishing, 2007.

[23] M. Spivak, *A comprehensive introduction to differential geometry*, vol. 1, 3rd ed., Publish or Perish, INC., Houston, Texas, 1999.

[24] H. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.

[25] A. Weinstein, *Lagrangian mechanics and groupoids*, Fields Inst. Comm. **7** (1996), 207–231.

[26] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.

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