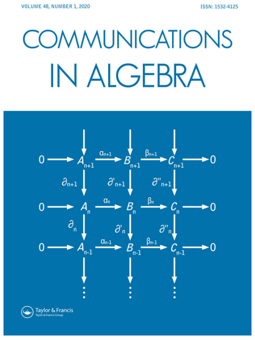

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On k-para-Kähler Lie algebras, a subclass of k-symplectic Lie algebras

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ABSTRACT

k-Para-Kähler Lie algebras are a generalization of para-Kähler Lie algebras ($k=1$) and constitute a subclass of k-symplectic Lie algebras. In this paper, we show that the characterization of para-Kähler Lie algebras as left symmetric bialgebras can be generalized to k-para-Kähler Lie algebras leading to the introduction of two new structures which are different but both generalize the notion of left symmetric algebra. This permits also the introduction of generalized S-matrices. We determine then all the k-symplectic Lie algebras of dimension $(k+1)$ and all the six dimensional 2-para-Kähler Lie algebras.

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1. Introduction

The k-symplectic geometry is a generalization of the symplectic geometry which was developed by A. Awane [3], Awane and Goze [3] and C. Günther in [9] as an attempt to develop a convenient geometric framework to study classical field theories (see [11]). A k-symplectic manifold is a smooth manifold M of dimension $(k+1)n$ endowed with an involutive vector subbundle $E \subset TM$ and a family $(\theta^1, ..., \theta^k)$ of differential closed 2-forms such that: $\text{rank}(E) = nk$, the family $(\theta^1, ..., \theta^k)$ is nondegenerate, i.e., $\cap_{i=1}^k \ker \theta^i = \{0\}$ and E is isotropic with respect all the θ^i. A left invariant k-symplectic structure on a connected Lie group G of dimension $(k+1)n$ is equivalent to its associated infinitesimal structure, namely, the Lie algebra $\mathfrak{g}$ of G, a Lie subalgebra $\mathfrak{h}$ of dimension nk and a family $\{\theta^i \in \wedge^2\mathfrak{g}^*, i = 1, ..., k\}$ of closed nondegenerate 2-forms such that $\theta^i_{|\mathfrak{h}} = 0$ for $i = 1, ..., n$. We call $(\mathfrak{g}, \mathfrak{h}, \theta^1, ..., \theta^k)$ a k-symplectic Lie algebra. If, in addition, there exists a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathfrak{p}$ is isotropic with respect to all the θ^i, we call $(\mathfrak{g}, \mathfrak{h}, \mathfrak{p}, \theta^1, ..., \theta^k)$ a k-para-Kähler Lie algebra. This terminology is justified by the fact that when $k=1$ we recover the classical notion of para-Kähler Lie algebras (see [1, 5–7]).

The purpose of this paper is to study k-para-Kähler Lie algebras aiming the generalization of the results obtained in [5, 8] in the case of para-Kähler Lie algebras. In these papers, para-Kähler Lie algebras were considered as left symmetric bialgebras. Roughly speaking, a para-Kähler Lie algebra is built from two structures of left symmetric algebras on a vector space and its dual which are compatible in some sense. The compatibility condition involves representations of Lie algebras and 1-cocycles. This leads naturally to the notion of exact para-Kähler Lie algebras

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Table 1. Two dimensional 2-left symmetric structures, $(a,b) \in R^2$.

Name of the 2-LSS	First left symmetric product	Second left symmetric product
$\mathbf{b}_{1,\alpha}, \left(\alpha \neq 1, \alpha \neq \frac{1}{2}\right)$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = \alpha e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{1,\frac{1}{2}}$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = \frac{1}{2} e_2$	$e_2 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = \frac{1}{2} ae_2 + be_1$
$\mathbf{b}_{1,1}$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet e_1 = ae_1, e_1 \bullet_2 e_2 = ae_2, e_2 \bullet_2 e_1 = be_1,$ $e_2 \bullet_2 e_2 = be_2$
$\mathbf{b}_2$	$e_2 \bullet_1 e_1 = e_1, e_2 \bullet_1 e_2 = e_1 + e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{3,\alpha}, \alpha \neq 1, \alpha \neq 0,$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_1 = \left(1 - \frac{1}{\alpha}\right) e_1, e_2 \bullet_1 e_2 = e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_{3,1}$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet_2 e_1 = ae_1, e_1 \bullet_2 e_2 = be_1, e_2 \bullet_2 e_1 = ae_2,$ $e_2 \bullet_2 e_2 = be_2$
$\mathbf{b}_4$	$e_1 \bullet_1 e_2 = e_1, e_2 \bullet_1 e_2 = e_1 + e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_5^+$	$e_1 \bullet_1 e_1 = e_2, e_2 \bullet_1 e_1 = -e_1, e_2 \bullet_1 e_2 = -2e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{b}_5^-$	$e_1 \bullet_1 e_1 = -e_2, e_2 \bullet_1 e_1 = -e_1, e_2 \bullet_1 e_2 = -2e_2$	$\bullet_2 = a \bullet_1$
$\mathbf{c}_2$	$e_2 \bullet_1 e_2 = e_2$	$e_1 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_2$
$\mathbf{c}_3^1$	$e_2 \bullet_1 e_2 = e_1$	$e_2 \bullet_2 e_1 = 2ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_3^2$	$e_2 \bullet e_2 = e_1$	$e_1 \bullet_2 e_2 = ae_1, e_2 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_4$	$e_2 \bullet e_2 = e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = ae_1, e_2 \bullet_2 e_1 = ae_1, e_2 \bullet_2 e_2 = be_1 + ae_2$
$\mathbf{c}_5^+$	$e_1 \bullet_1 e_1 = e_2 \bullet_1 e_2 = e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = e_2 \bullet_2 e_1 = be_1 + ae_2$ $e_1 \bullet_2 e_1 = e_2 \bullet_2 e_2 = ae_1 + be_2$
$\mathbf{c}_5^-$	$e_1 \bullet_1 e_1 = -e_2 \bullet_1 e_2 = -e_2, e_1 \bullet_1 e_2 = e_2 \bullet_1 e_1 = e_1$	$e_1 \bullet_2 e_2 = e_2 \bullet_2 e_1 = be_1 + ae_2$ $e_1 \bullet_2 e_1 = -e_2 \bullet_2 e_2 = ae_1 - be_2$

Table 2. Compatible two dimensional 2-left symmetric and (2×2)-left symmetric structures.

Name	2-left symmetric structure	Compatible (2×2)-left symmetric structure	conditions
$\mathbf{bb}_{1,\alpha}$	$\mathbf{b}_{1,\alpha}, \left(\alpha \neq 1, \alpha \neq \frac{1}{2}\right)$	$L_{e_2}^{1,1} = \begin{pmatrix} 0 & 0 \\ 0 & -ac \end{pmatrix}, L_{e_2}^{1,2} = \begin{pmatrix} 0 & 0 \\ 0 & -ad \end{pmatrix}, L_{e_2}^{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, L_{e_2}^{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$	$a \in \mathbb{R}, \ \alpha = 0$
$\mathbf{bb}_{1,1}$	$\mathbf{b}_{1,1}$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1,2\}$	$a = 0, \ b \in \mathbb{R}$
$\mathbf{bb}_2$	$\mathbf{b}_2$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1,2\}$	$a \neq 1$
		$L_{e_2}^{1,1} = L_{e_2}^{1,2} = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, L_{e_2}^{2,1} = L_{e_2}^{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$	$a = 1$
$\mathbf{bb}_{3,1}$	$\mathbf{b}_{3,1}$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1,2\}$	$a \neq 0, \ b \in \mathbb{R}$
$\mathbf{bb}_4$	$\mathbf{b}_4$	$L_{e_1}^{1,1} = \begin{pmatrix} 0 & 0 \\ -ac & 0 \end{pmatrix}, L_{e_1}^{1,2} = \begin{pmatrix} 0 & 0 \\ -a^2c & 0 \end{pmatrix}, L_{e_1}^{2,1} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, L_{e_1}^{2,2} = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}$	$a \in \mathbb{R}$
$\mathbf{cc}_3^1$	$\mathbf{c}_3^1$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1,2\}$	$a \neq 0, \ b \in \mathbb{R}$
		$L_{e_1}^{1,1} = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix}, L_{e_1}^{1,2} = \begin{pmatrix} bc_1 & 0 \\ d_1 & 0 \end{pmatrix}, L_{e_1}^{2,1} = \begin{pmatrix} g_1 & 0 \\ g_2 & 0 \end{pmatrix}, L_{e_1}^{2,2} = \begin{pmatrix} bg_1 & 0 \\ d_2 & 0 \end{pmatrix}$	$a = 0, \ b \in \mathbb{R}$
$\mathbf{cc}_3^2$	$\mathbf{c}_3^2$	$\star_{\alpha,\beta} = 0, \alpha, \beta \in \{1,2\}$	$a \neq 0, \ b \in \mathbb{R}$
		$L_{e_1}^{1,1} = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \end{pmatrix}, L_{e_1}^{1,2} = \begin{pmatrix} bc_1 & 0 \\ d & 0 \end{pmatrix}, L_{e_1}^{2,1} = \begin{pmatrix} g_1 & 0 \\ g_2 & 0 \end{pmatrix}, L_{e_1}^{2,2} = \begin{pmatrix} bg_1 & 0 \\ h & 0 \end{pmatrix}$	$a = 0, \ b \in \mathbb{R}$
$\mathbf{cc}_5^+$	$\mathbf{c}_5^+$	$L_{e_1}^{1,1} = L_{e_2}^{1,1} = \begin{pmatrix} -c & -c \\ -c & -c \end{pmatrix}, L_{e_1}^{1,2} = L_{e_2}^{1,2} = \begin{pmatrix} -c & -c \\ -c & -c \end{pmatrix}$	$a \in \mathbb{R}, \ b \in \mathbb{R}$
		$L_{e_1}^{2,1} = L_{e_2}^{2,1} = \begin{pmatrix} c & c \\ c & c \end{pmatrix}, L_{e_1}^{2,2} = L_{e_2}^{2,2} = \begin{pmatrix} c & c \\ c & c \end{pmatrix}$	

Table 3. Six dimensional 2-para-Kähler Lie algebras.

Structure	Associated 2-para-Kähler Lie algebra	Conditions
$\mathbf{bb}_{1,\alpha}$	$[f_1,f_2]=-f_1, [f_1,f_4]=-af_1, [f_2,f_3]=f_3, [f_3,f_4]=-af_3,$ $[f_2,e_1]=-e_1, [f_2,e_2]=-c(af_2-f_4), [f_4,e_1]=-ae_1, [f_4,e_2]=-d(af_2-f_4).$	$a\in\mathbb{R},$
$\mathbf{bb}_{1,1}$	$[f_1,f_2]=-f_1, [f_1,f_4]=-bf_1, [f_2,f_3]=f_3, [f_2,f_4]=-bf_2+f_4, [f_3,f_4]=-bf_3,$ $[f_2,e_1]=-e_1, [f_2,e_2]=-e_2, [f_4,e_1]=-be_1, [f_4,e_2]=-be_2.$	$b\in\mathbb{R}$
$\mathbf{bb}_2$	$[f_1,f_2]=-f_1, [f_1,f_4]=-af_1, [f_2,f_3]=f_3, [f_2,f_4]=-a(f_1+f_2)+f_3+f_4,$ $[f_3,f_4]=-af_3, [f_2,e_1]=-e_1-e_2, [f_2,e_2]=-e_2, [f_4,e_1]=-a(e_1+e_2), [f_4,e_2]=-ae_2.$	$a\neq 1$
	$[f_1,f_2]=-f_1, [f_1,f_4]=-f_1, [f_2,f_3]=f_3, [f_2,f_4]=-f_1-f_2+f_3+f_4,$ $[f_3,f_4]=-f_3, [f_2,e_1]=-e_1 [f_4,e_2], [f_2,e_2]=-c(f_2-f_4)-e_2, [f_4,e_1]=-e_1-e_2,$	$c\in\mathbb{R}$
$\mathbf{bb}_{3,1}$	$[f_1,f_2]=f_1, [f_1,f_3]=-af_1, [f_1,f_4]=-af_2+f_3, [f_2,f_3]=-bf_1,$ $[f_2,f_4]=-bf_2+f_4, [f_3,f_4]=bf_3-af_4, [f_1,e_1]=-e_2, [f_2,e_2]=-e_2,$ $[f_3,e_1]=-ae_1-be_2, [f_4,e_2]=-ae_1-be_2.$	$a\neq 0, b\in\mathbb{R}$
$\mathbf{bb}_4$	$[f_1,f_2]=f_1, [f_1,f_4]=f_3, [f_2,f_3]=-af_1, [f_2,f_4]=-a(f_1+f_2)+f_3+f_4,$ $[f_3,f_4]=af_3, [f_1,e_1]=-e_2, [f_2,e_1]=-c(af_1-f_3)-e_2, [f_2,e_2]=-e_2,$ $[f_3,e_1]=-ae_2, [f_4,e_1]=-ac(af_1-f_3)-ae_2, [f_4,e_2]=-ae_2.$	$a\in\mathbb{R}$
$\mathbf{cc}_3^1$	$[f_1,f_4]=-2af_1, [f_2,f_4]=-bf_1-af_2+f_3, [f_3,f_4]=-2af_3, [f_2,e_1]=-e_2,$ $[f_4,e_1]=-2ae_1-be_2, [f_4,e_2]=-ae_2.$	$a\neq 0, b\in\mathbb{R}$
	$[f_2,f_4]=-bf_1+f_3, [f_1,e_1]=c_1f_1+g_1f_3, [f_2,e_1]=c_2f_1+g_2f_3-e_2,$ $[f_3,e_1]=bc_1f_1+bg_1f_3, [f_4,e_1]=d_2f_1+hf_3-be_2.$	$b\in\mathbb{R}$
$\mathbf{cc}_3^2$	$[f_1,f_4]=-af_1, [f_2,f_3]=-af_1, [f_2,f_4]=-bf_1-af_2+f_3, [f_2,e_1]=-e_2, [f_3,e_1]=-ae_2$ $[f_4,e_1]=-ae_1-be_2, [f_4,e_2]=-ae_2.$	$a\neq 0, b\in\mathbb{R}$
	$[f_2,f_4]=-bf_1+f_3, [f_1,e_1]=c_1f_1+g_1f_3, [f_2,e_1]=c_2f_1+g_2f_3-e_2,$ $[f_3,e_1]=bc_1f_1+bg_1f_3, [f_4,e_1]=df_1+hf_3-be_2.$	$b\in\mathbb{R}$
$\mathbf{cc}_5^+$	$[f_1,f_3]=-af_1-bf_2+f_4, [f_1,f_4]=-bf_1-af_2+f_3, [f_2,f_3]=-bf_1-af_2+f_3,$ $[f_2,f_4]=-af_1-bf_2+f_4, [f_1,e_1]=-c(f_1+f_2-f_3-f_4)-e_2, [f_1,e_2]=-c(f_1+f_2-f_3-f_4)-e_1,$ $[f_2,e_1]=-c(f_1+f_2-f_3-f_4)-e_1, [f_2,e_2]=-c(f_1+f_2-f_3-f_4)-e_2$ $[f_3,e_1]=-c(f_1+f_2-f_3-f_4)-ae_1-be_2, [f_3,e_2]=-c(f_1+f_2-f_3-f_4)-be_1-ae_2,$ $[f_4,e_1]=-c(f_1+f_2-f_3-f_4)-be_1-ae_2, [f_4,e_2]=-c(f_1+f_2-f_3-f_4)-ae_1-be_2.$	$a\in\mathbb{R},\ b\in\mathbb{R}$
	$\mathfrak{h}=\mathrm{span}\{f_1,f_2,f_3,f_4\}, \theta^1=f_1^*\wedge e_1^*+f_2^*\wedge e_2^*$ and $\theta^2=f_3^*\wedge e_1^*+f_4^*\wedge e_2^*$	

(when one 1-cocycle is a coboundary). Exact para-Kähler Lie algebras are defined from S-matrices in the same way as exact Lie bialgebras are defined by R-matrices.

In this paper, we show that a k-para-Kähler Lie algebra is build from two new algebraic structures compatible in some sense (see Theorems 2.1 and 2.2). We call them k-left symmetric algebra and $(k\times k)$-left symmetric algebra (see Definitions 2.2 and 2.3). As for $k=1$ the compatibility condition involves representations and 1-cocycles and we have naturally the notion of exact k-para-Kähler Lie algebras leading to what we call S_k-matrices (see Theorem 3.1). The notions of k-left symmetric algebra and $(k\times k)$-left symmetric algebra are new and both generalize the notion of left symmetric algebras. We think that these two structures are interesting in their own right. In Proposition 2.3, we give a natural way to build examples of k-left symmetric algebras. We give also all 2-left symmetric algebras in dimension 2 (see Table 1) and we deduce all the six dimensional 2-para-Kähler Lie algebras (see Table 3). Our study permits also the determination of all the k-symplectic Lie algebras of dimension $(k+1)$ (see Theorems 4.1 and 4.2).

Section 2 is devoted to the characterization of k-para-Kähler Lie algebras by means of the two new notions of k-left symmetric algebra and $(k \times k)$-left symmetric algebras. In Section 3, we study exact k-para-Kähler Lie algebras and we introduce the notion of S_k-matrix. Section 4 is devoted to the determination of the k-symplectic Lie algebras of dimension $(k+1)$. In Section 5, we give all the 2-left symmetric algebras of dimension 2 and all the six dimensional 2-para-Kähler Lie algebras.

1.1. Convention

Through-out this paper, we will deal with many representations of Lie algebras. If $\mu : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of a Lie algebra, we denote by $\mu^* : \mathfrak{g} \to \mathrm{End}(V^*)$ its dual representation given by

$$\prec \mu^*(x)(\gamma), y \succ = - \prec \gamma, \mu(x)(y) \succ, \quad x, y \in V, \gamma \in V^*.$$

However, for any endomorphism F between two vector spaces, we denote by F^T its dual.

2. Characterization of k-para-Kähler Lie algebras

A k-symplectic Lie algebra is a real Lie algebra $\mathfrak{g}$ of dimension $nk + n$ with a subalgebra $\mathfrak{h}$ of dimension nk and a family $(\theta^1, ..., \theta^k)$ of 2-forms satisfying:

(i) The family $(\theta^1, ..., \theta^k)$ is nondegenerate, i.e., $\cap_{i=1}^k \ker\theta^i = \{0\}$,
(ii) for $i = 1, ..., k$, θ^i is closed, i.e., $d\theta^i(u, v, w) := \theta^i([u, v], w) + \theta^i([v, w], u) + \theta^i([w, u], v) = 0$,
(iii) $\mathfrak{h}$ is totally isotropic with respect to $(\theta^1, ..., \theta^k)$, i.e., $\theta^i(u, v) = 0$ for any $u, v \in \mathfrak{h}$ and for $i = 1, ..., k$.

According to [4, Theorem 3.1], there exists a basis $B^* = (\omega^{pi}, \omega^i)_{1\le p\le k, 1\le i\le n}$ of $\mathfrak{g}^*$ such that for any $\alpha \in \{1, ..., k\}$,

$$\theta^\alpha = \sum_{i=1}^n \omega^{\alpha i} \wedge \omega^i \quad \text{and} \quad \mathfrak{h} = \ker\omega^1 \cap ... \cap \ker\omega^n.$$

Let $(e_{pi}, e_i)_{1\le p\le k, 1\le i\le n}$ be the dual basis of B^*. Then the vector subspace $\mathrm{span}\{e_1, ..., e_n\}$ is a supplement of $\mathfrak{h}$ and it is totally isotropic with respect to $(\theta^1, ..., \theta^k)$. Thus $\mathfrak{h}$ has an isotropic supplement. We introduce now the main object of this article.

Definition 2.1. Let $(\mathfrak{g}, \theta^1, ..., \theta^k, \mathfrak{h})$ be a k-symplectic Lie algebra. We call it k-para-Kähler if $\mathfrak{h}$ admits an isotropic supplement which is a Lie subalgebra.

When $k=1$, we recover the well-known notion of par-Kähler Lie algebras (see [8]). We proceed now to the study of k-para-Kähler Lie algebras aiming the generalization of the results obtained for $k=1$ in [8].

Let $(\mathfrak{g}, [,], \theta^1, ..., \theta^k, \mathfrak{h})$ be a k-para-Kähler Lie algebra and $\mathfrak{p}$ an isotropic Lie subalgebra supplement of $\mathfrak{h}$.

The linear map $\Theta : \mathfrak{h} \to (\mathfrak{g}/\mathfrak{h})^* \times \cdots \times (\mathfrak{g}/\mathfrak{h})^*, h \mapsto (\Theta_1(h), ..., \Theta_k(h))$ where, for any $p \in \mathfrak{g}$,

$$\Theta_\alpha(h)([p]) = \theta^\alpha(h, p)$$

is well-defined, injective and for dimensional reasons it is an isomorphism. For any $\alpha \in \{1, ..., k\}$, the vector subspace $\mathfrak{h}^\alpha$ of $\mathfrak{h}$ given by

$$\mathfrak{h}^{\alpha} = \{h \in \mathfrak{h}, \Theta_{\beta}(h) = 0, \beta = 1, ..., k, \beta \neq \alpha\}.$$

has dimension n and $\mathfrak{h} = \bigoplus_{\alpha=1}^{k} \mathfrak{h}^{\alpha}$.

The Lie subalgebra $\mathfrak{h}$ carries a product given by

$$\Theta_{\alpha}(h_1 \bullet h_2)([p]) = -\theta^{\alpha}(h_2, [h_1, p]), \tag{1}$$

for any $h_1, h_2 \in \mathfrak{h}$, for any $p \in \mathfrak{g}$ and for any $\alpha = 1, ..., k$,

We have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and, for any $p \in \mathfrak{p}$ and $h \in \mathfrak{h}$, the Lie bracket $[p, h]$ can be written

$$[p, h] = -[h, p] = \phi_p(h) - \phi_h(p), \tag{2}$$

where $\phi_p(h) \in \mathfrak{h}$ and $\phi_h(p) \in \mathfrak{p}$.

For any $\alpha \in \{1, ..., k\}$, we define $i_{\alpha} : \mathfrak{h}^{\alpha} \to \mathfrak{p}^*$ by putting

$$i_{\alpha}(h)(p) = \theta^{\alpha}(h, p).$$

It is obvious that i_{α} is injective and since $\dim \mathfrak{h}^{\alpha} = \dim \mathfrak{p}$ it is bijective. Thus $i_{\alpha}^T : \mathfrak{p} \to (\mathfrak{h}^{\alpha})^*$ is an isomorphism. For any $\beta \in \{1, ..., k\}$ and for any $p, q \in \mathfrak{p}$, the map $h \mapsto -\theta^{\beta}(q, [p, h])$ is an element of $(\mathfrak{h}^{\alpha})^*$ and its image by $(i_{\alpha}^T)^{-1}$ is an element of $\mathfrak{p}$ we denote by $p \star_{\alpha,\beta} q$. Thus, for any $\alpha, \beta \in \{1, ..., k\}$, we have a product $\star_{\alpha,\beta} : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}, (p, q) \mapsto p \star_{\alpha,\beta} q$ where, for any $h \in \mathfrak{h}^{\alpha}$,

$$\theta^{\alpha}(p \star_{\alpha,\beta} q, h) = -\theta^{\beta}(q, [p, h]). \tag{3}$$

Finally, for any $\alpha, \beta \in \{1, ..., k\}$, we endow $\mathfrak{p}^*$ with the product $\bullet_{\alpha\beta}$ obtained by putting, for any $a, b \in \mathfrak{p}^*$,

$$a \bullet \alpha\beta\, b = i_{\beta}(i_{\alpha}^{-1}(a) \bullet i_{\beta}^{-1}(b)). \tag{4}$$

The formulas (1), (3) and (4) define, respectively, a product on $\mathfrak{h}$, a family of products on $\mathfrak{p}$ and a family of products on $\mathfrak{p}^*$ which depend on $(\theta^1, ..., \theta^k)$, the Lie bracket and $\mathfrak{p}$. We will use these products to describe the k-symplectic Lie algebra in a useful way. Let us give now the properties of these products. Recall that a left symmetric algebra is an algebra $(A, \bullet)$ such that for any $a, b, c \in A$,

$$\mathrm{ass}(a, b, c) = \mathrm{ass}(b, a, c) \quad \text{where} \quad \mathrm{ass}(a, b, c) = (a \bullet b) \bullet c - a \bullet (b \bullet c).$$

Proposition 2.1. *We have:*

1. *$(\mathfrak{h}, \bullet)$ is a left symmetric algebra, the product $\bullet$ is Lie-admissible, i.e., for any $u, v \in \mathfrak{h}, [u, v] = u \bullet v - v \bullet u$, and for any $\alpha = 1, ..., k, \mathfrak{h} \bullet \mathfrak{h}^{\alpha} \subset \mathfrak{h}^{\alpha}$.*
2. *For any $\alpha, \beta \in \{1, ..., k\}$ with $\alpha \neq \beta$ and for any $p_1, p_2 \in \mathfrak{p}$, we have*

$$[p_1, p_2] = p_1 \star_{\alpha,\alpha} p_2 - p_2 \star_{\alpha,\alpha} p_1, p_1 \star_{\alpha,\beta} p_2 = p_2 \star_{\alpha,\beta} p_1.$$

3. *For any $\alpha, \beta, \gamma, \bullet_{\alpha\beta} = \bullet_{\alpha\gamma}$ and if we denote $\bullet_{\alpha\beta} = \bullet_{\alpha}$, we have, for any $a, b, c \in \mathfrak{p}^*$,*

$$a \bullet_{\alpha} (b \bullet_{\beta} c) - (a \bullet_{\alpha} b) \bullet_{\beta} c = b \bullet_{\beta} (a \bullet_{\alpha} c) - (b \bullet_{\beta} a) \bullet_{\alpha} c. \tag{5}$$

In particular, $\bullet_{\alpha}$ is a left symmetric product on $\mathfrak{p}^*$.

Proof.

1. We have

$$\begin{aligned}\Theta_{\alpha}(h_1 \bullet h_2)([p]) - \Theta_{\alpha}(h_2 \bullet h_1)([p]) &= \theta^{\alpha}(h_1, [h_2, p]) - \theta^{\alpha}(h_2, [h_1, p]) \\ &= \theta^{\alpha}([h_1, h_2], p) = \Theta_{\alpha}([h_1, h_2])([p])\end{aligned}$$

and hence $\bullet$ is Lie-admissible. On the other hand,

$$\begin{aligned}\Theta_\alpha([h_1,h_2]\bullet h_3)([p]) &= -\theta^\alpha(h_3,[[h_1,h_2],p])\\ &= -\theta^\alpha(h_3,[h_1,[h_2,p]]) - \theta^\alpha(h_3,[h_2,[p,h_1]])\\ &= \Theta_\alpha(h_1\bullet(h_2\bullet h_3))([p]) - \Theta_\alpha(h_2\bullet(h_1\bullet h_3))([p]).\end{aligned}$$

So

$$[h_1,h_2]\bullet h_3 = h_1\bullet(h_2\bullet h_3) - h_2\bullet(h_1\bullet h_3).$$

This shows that $\bullet$ is left symmetric. It is obvious that $\mathfrak{h}\bullet\mathfrak{h}^\alpha\subset\mathfrak{h}^\alpha$.

2. Now consider $\alpha,\beta\in\{1,\dots,k\}, p,q\in\mathfrak{p}$ and $h\in\mathfrak{h}^\alpha$ then

$$\begin{aligned}0 &= \theta^\beta([p,q],h) + \theta^\beta([q,h],p) + \theta^\beta([h,p],q)\\ &= \theta^\beta([p,q]_\mathfrak{p},h) + \theta^\alpha(q\star_{\alpha,\beta}p,h) - \theta^\alpha(p\star_{\alpha,\beta}q,h).\end{aligned}$$

So if $\alpha\neq\beta$ we get $p\star_{\alpha,\beta}q = q\star_{\alpha,\beta}p$ and if $\alpha=\beta$ we get

$$[p,q]_\mathfrak{p} = p\star_{\alpha,\alpha}q - q\star_{\alpha,\alpha}p.$$

3. For any $a,b\in\mathfrak{p}^*$ and any $q\in\mathfrak{p}$,

$$\begin{aligned}\prec a\bullet_{\alpha\beta}b,q\succ &= \prec i_\beta(i_\alpha^{-1}(a)\bullet i_\beta^{-1}(b)),q\succ\\ &= \theta^\beta(i_\alpha^{-1}(a)\bullet i_\beta^{-1}(b),q)\\ &\overset{(1)}{=} -\theta^\beta(i_\beta^{-1}(b),[i_\alpha^{-1}(a),q])\\ &\overset{(2)}{=} \prec b,\phi_q(i_\alpha^{-1}(a))\succ\\ &= \prec a\bullet_{\alpha\gamma}b,q\succ.\end{aligned}$$

We have

$$\begin{aligned}a\bullet_\alpha(b\bullet_\beta c) - (a\bullet_\alpha b)\bullet_\beta c &= a\bullet_{\alpha\beta}(b\bullet_{\beta\beta}c) - (a\bullet_{\alpha\beta}b)\bullet_{\beta\beta}c\\ &= i_\beta\left[i_\alpha^{-1}(a)\bullet(i_\beta^{-1}(b)\bullet i_\beta^{-1}(c))\right] - i_\beta\left[(i_\alpha^{-1}(a)\bullet i_\beta^{-1}(b))\bullet i_\beta^{-1}(c)\right],\\ b\bullet_\beta(a\bullet_\alpha c) - (b\bullet_\beta a)\bullet_\alpha c &= b\bullet_{\beta\beta}(a\bullet_{\alpha\beta}c) - (b\bullet_{\beta\alpha}a)\bullet_{\alpha\beta}c\\ &= i_\beta\left[i_\beta^{-1}(b)\bullet(i_\alpha^{-1}(a)\bullet i_\beta^{-1}(c))\right] - i_\beta\left[(i_\beta^{-1}(b)\bullet i_\alpha^{-1}(a))\bullet i_\beta^{-1}(c)\right].\end{aligned}$$

Hence the two expressions are equal due to item 1. This completes the proof. □

We consider $\Phi(\mathfrak{p},k) = \mathfrak{p}\oplus(\mathfrak{p}^*)^k$ and we endow $(\mathfrak{p}^*)^k$ with the product $\circ$ given by

$$(a_1,\dots,a_k)\circ(b_1,\dots,b_k) = \left(\sum_{\alpha=1}^k a_\alpha\bullet_\alpha b_1,\dots,\sum_{\alpha=1}^k a_\alpha\bullet_\alpha b_k\right). \tag{6}$$

We define $\phi:(\mathfrak{p}^*)^k\otimes\mathfrak{p}^*\to\mathfrak{p}^*$ and $\psi:\mathfrak{p}\otimes\mathfrak{p}^k\to\mathfrak{p}^k$ by

$$\begin{cases}\phi((a_1,\dots,a_k),b) = \phi_{(a_1,\dots,a_k)}b = \sum_{\alpha=1}^k \mathrm{L}^\alpha_{a_\alpha}b,\\ \psi(q,(p_1,\dots,p_k)) = \psi_q(p_1,\dots,p_k) = \sum_{\alpha=1}^k\left(\mathrm{L}_q^{\alpha,1}p_\alpha,\dots,\mathrm{L}_q^{\alpha,k}p_\alpha\right).\end{cases} \tag{7}$$

where $\mathrm{L}_a^\alpha : \mathfrak{p}^* \to \mathfrak{p}^*, b \mapsto a \bullet_\alpha b$ and $\mathrm{L}_q^{\alpha,\beta} : \mathfrak{p} \to \mathfrak{p}, p \mapsto q \star_{\alpha,\beta} p$, and we endow $\Phi(\mathfrak{p}, k)$ with the bracket $[,]_n$

$$\begin{cases} [a,b]_n = a \circ b - b \circ a, & \text{if} a, b \in (\mathfrak{p}^*)^k \\ [p,q]_n = [p,q], & \text{if} p, q \in \mathfrak{p} \\ [a,p]_n = \phi_a^*(p) - \psi_p^* a, & \text{if} a \in (\mathfrak{p}^*)^k, p \in \mathfrak{p} \end{cases} \tag{8}$$

where

$$\prec b, \phi_a^*(p) \succ = - \prec \phi_a b, p \succ \quad \text{and} \quad \prec \psi_p^* a, (p_1, ..., p_k) \succ = - \prec a, \psi_p(p_1, ..., p_k) \succ .$$

Finally, we define also a family of 2-forms ρ^α on $\Phi(\mathfrak{p}, k), \alpha = 1, ..., k$ by

$$\rho^\alpha(p + (a_1, ..., a_k), q + (b_1, ..., b_k)) = \prec a_\alpha, q \succ - \prec b_\alpha, p \succ . \tag{9}$$

Theorem 2.1. *Let $(\mathfrak{g}, [,], \theta^1, ..., \theta^k, \mathfrak{h}, \mathfrak{p})$ be a k-para-Kähler Lie algebra. Then $F : \mathfrak{g} \to \Phi(\mathfrak{p}, k), (h_1 + ... + h_k + p) \mapsto (p, i_1(h_1), ..., i_k(h_k))$ is a linear isomorphism which sends the Lie bracket on $\mathfrak{g}$ to $[,]_n$ and the θ^α to the ρ^α and hence $(\Phi(\mathfrak{p}, k), [,]_n, (\mathfrak{p}^*)^k, \rho^1, ..., \rho^k)$ is a k-para-Kähler Lie algebra.*

Proof. It is obvious that F is an isomorphism. Let us show that F sends $[,]$ to $[,]_n$ and the θ^α to the ρ^α. Let $h_\alpha \in \mathfrak{h}^\alpha$ and $h_\beta \in \mathfrak{h}^\beta$. Then

$$\begin{aligned} F([h_\alpha, h_\beta]) &= F(h_\alpha \bullet h_\beta) - F(h_\beta \bullet h_\alpha) \\ &\overset{(4)}{=} F(i_\beta^{-1}[i_\alpha(h_\alpha) \bullet_\alpha i_\beta(h_\beta)]) - F(i_\alpha^{-1}[i_\beta(h_\beta) \bullet_\beta i_\alpha(h_\alpha)]) \\ &\overset{(6)}{=} F(h_\alpha) \circ F(h_\beta) - F(h_\beta) \circ F(h_\beta). \end{aligned}$$

It is obvious from (2) and Proposition 2.1 that, for any $p, q \in \mathfrak{p}, F([p,q]) = [F(p), F(q)]_n$. Let $h = h_1 + ... + h_k \in \mathfrak{h}$ and $p \in \mathfrak{p}$. We have

$$\begin{aligned} F([h,p]) &= \sum_{\alpha=1}^k F([h_\alpha, p]) \\ &= \sum_{\alpha=1}^k F([h_\alpha, p]_\mathfrak{p}) - \sum_{\alpha=1}^k F([p, h_\alpha]_\mathfrak{h}) \\ &\overset{(2)}{=} \sum_{\alpha=1}^k \phi_{h_\alpha}(p) - \sum_{\alpha=1}^k (i_1(\phi_p(h_\alpha)), ..., i_k(\phi_p(h_\alpha))). \end{aligned}$$

We use here the convention that $i_\alpha(h_\beta) = 0$ when $h_\beta \in \mathfrak{h}^\beta$ and $\alpha \neq \beta$. For any $a \in \mathfrak{p}^*$ and $\beta \in \{1, ..., k\}$, we have

$$\begin{aligned} \prec a, \phi_{h_\alpha}(p) \succ &= \theta^\beta(i_\beta^{-1}(a), \phi_{h_\alpha}(p)) \\ &= \theta^\beta(i_\beta^{-1}(a), [h_\alpha, p]) \\ &\overset{(1)}{=} -\theta^\beta(h_\alpha \bullet i_\beta^{-1}(a), p) \\ &\overset{(4)}{=} - \prec i_\alpha(h_\alpha) \bullet_\alpha a, p \succ \\ &= \prec a, (\mathrm{L}_{F(h_\alpha)^\alpha}^\alpha)^* p \succ . \end{aligned}$$

Thus $\phi_{h_\alpha}(p) = (\mathrm{L}^\alpha_{F(h_\alpha)^\alpha})^* p$. On the other hand, for any $q \in \mathfrak{p}$,

$$\begin{aligned} \prec i_\beta(\phi_p(h_\alpha)), q \succ &= \theta^\beta([h_\alpha, p], q) \\ &\overset{(3)}{=} \theta^\alpha(p \star_{\alpha,\beta} q, h_\alpha) \\ &= - \prec i_\alpha(h_\alpha), p \star_{\alpha,\beta} q \succ \\ &= \prec (\mathrm{L}^{\alpha,\beta}_p)^*(i_\alpha(h_\alpha)), q \succ . \end{aligned}$$

Thus $i_\beta(\phi_p(h_\alpha)) = (\mathrm{L}^{\alpha,\beta}_p)^*(i_\alpha(h_\alpha))$. Therefore

$$\begin{aligned} F([h,p]) &= \sum_{\alpha=1}^k \phi_{h_\alpha}(p) - \sum_{\alpha=1}^k \big(i_1(\phi_p(h_\alpha)), \dots, i_k(\phi_p(h_\alpha))\big). \\ &= \sum_{\alpha=1}^k (\mathrm{L}^\alpha_{F(h_\alpha)^\alpha})^* p - \sum_{\alpha=1}^k \Big((\mathrm{L}^{\alpha,1}_p)^*(i_\alpha(h_\alpha)), \dots, (\mathrm{L}^{\alpha,k}_p)^*(i_\alpha(h_\alpha))\Big) \\ &= \phi^*_{F(h)}(F(p)) - \psi^*_{F(p)}(F(h)) \end{aligned}$$

and we get that $F([h,p]) = [F(h), F(p)]_n$. To conclude, one can check easily that F sends θ^α to ρ^α. □

To study the converse, we introduce two algebraic structures which appeared naturally in our study above.

Definition 2.2. A k-left symmetric algebra is a real vector space $\mathcal{A}$ endowed with k left symmetric products $\bullet_1, \dots, \bullet_k$ such that one of the following equivalent assertions hold:

1. For any $\alpha, \beta \in \{1, \dots, k\}$ and for any $a, b, c \in \mathcal{A}$,
$$a \bullet_\alpha (b \bullet_\beta c) - (a \bullet_\alpha b) \bullet_\beta c = b \bullet_\beta (a \bullet_\alpha c) - (b \bullet_\beta a) \bullet_\alpha c. \tag{10}$$
2. $(\mathcal{A}^k, \circ)$ is a left symmetric algebra where $\circ$ is given by
$$(a_1, \dots, a_k) \circ (b_1, \dots, b_k) = \left(\sum_{\alpha=1}^k a_\alpha \bullet_\alpha b_1, \dots, \sum_{\alpha=1}^k a_\alpha \bullet_\alpha b_k \right). \tag{11}$$

In this case the map $\phi : \mathcal{A}^k \times \mathcal{A} \to \mathcal{A}$ given by
$$\phi((a_1, \dots, a_k), b) = \phi_{(a_1, \dots, a_k)} b = \sum_{\alpha=1}^k \mathrm{L}^\alpha_{a_\alpha} b \tag{12}$$
defines a representation of the Lie algebra $(\mathcal{A}^k, [,])$ in $\mathcal{A}$ where $[a,b] = a \circ b - b \circ a$.

Indeed, the two relations (10) and (11) are equivalent. In fact, for any $a, b, c \in \mathcal{A}^k$ we have

$$\begin{aligned} (a \circ b) \circ c &= \left(\sum_{\alpha=1}^k (a_\alpha \bullet_\alpha b_1), \dots, \sum_{\alpha=1}^k (a_\alpha \bullet_\alpha b_k) \right) \circ c \\ &= \left(\sum_{\alpha=1}^k \sum_{\beta=1}^k (a_\alpha \bullet_\alpha b_\beta) \bullet_\beta c_1, \dots, \sum_{\alpha=1}^k \sum_{\beta=1}^k (a_\alpha \bullet_\alpha b_\beta) \bullet_\beta c_k \right), \\ a \circ (b \circ c) &= a \circ \left(\sum_{\beta=1}^k (b_\beta \bullet_\beta c_1), \dots, \sum_{\beta=1}^k (b_\beta \bullet_\beta c_k) \right) \\ &= \left(\sum_{\alpha=1}^k \sum_{\beta=1}^k a_\alpha \bullet_\alpha (b_\beta \bullet_\beta c_1), \dots, \sum_{\alpha=1}^k \sum_{\beta=1}^k a_\alpha \bullet_\alpha (b_\beta \bullet_\beta c_k) \right). \end{aligned}$$

Then

$$\mathrm{ass}(a,b,c) = \Big(\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}[(a_\alpha \bullet_\alpha b_\beta)\bullet_\beta c_1 - a_\alpha \bullet_\alpha (b_\beta \bullet_\beta c_1)\ ], \ldots, \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left[(a_\alpha \bullet_\alpha b_\beta)\bullet_\beta c_k - a_\alpha \bullet_\alpha (b_\beta \bullet_\beta c_k)\right]\Big)$$

where $\mathrm{ass}(a,b,c) = (a\circ b)\circ c - a\circ(b\circ c)$. Similarly, we have

$$\mathrm{ass}(b,a,c) = \left(\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left[(b_\beta \bullet_\beta a_\alpha)\bullet_\alpha c_1 - b_\beta \bullet_\beta (a_\alpha \bullet_\alpha c_1)\right], \ldots, \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left[(b_\beta \bullet_\beta a_\alpha)\bullet_\alpha c_k - b_\beta \bullet_\beta (a_\alpha \bullet_\alpha c_k)\right]\right)$$

But $\circ$ is left symmetric if and only if, for any $\alpha,\beta \in \{1,\ldots,k\}$ and any $a=(0,\ldots,a_\alpha,\ldots,0), b=(0,\ldots,b_\alpha,\ldots,0)$ and $c\in\mathcal{A}^k$, $\mathrm{ass}(a,b,c)=\mathrm{ass}(b,a,c)$ which gives the equivalence. Moreover, if (10) or (11) holds then, for any $a,b\in\mathcal{A}^k$ and any $c\in\mathcal{A}$, we have

$$\begin{aligned}
\phi([a,b],c) &= \phi(a\circ b,c) - \phi(b\circ a,c)\\
&= \sum_{\alpha=1}^{k}(\phi((a_\alpha \bullet_\alpha b_1,\ldots,a_\alpha \bullet_\alpha b_k),c) - \phi((b_\alpha \bullet_\alpha a_1,\ldots,b_\alpha \bullet_\alpha a_k),c))\\
&= \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left(L^{\beta}_{a_\alpha \bullet_\alpha b_\beta}c - L^{\beta}_{b_\alpha \bullet_\alpha a_\beta}c\right)\\
&\overset{(10)}{=} \sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\left(L^{\alpha}_{a_\alpha}(L^{\beta}_{b_\beta}c) - L^{\beta}_{b_\beta}(L^{\alpha}_{a_\alpha}c)\right)\\
&= \left[\sum_{\alpha=1}^{k}L^{\alpha}_{a_\alpha}, \sum_{\beta=1}^{k}L^{\beta}_{b_\beta}\right]c\\
&= [\phi(a,.),\phi(b,.)](c).
\end{aligned}$$

This shows that ϕ defines a representation of the Lie algebra $(\mathcal{A}^k,[,])$ in $\mathcal{A}$.

Definition 2.3. A $(k\times k)$-left symmetric algebra is a vector space $\mathcal{B}$ endowed with a k × k-matrix $(\star_{\alpha,\beta})_{1\le\alpha,\beta\le k}$ of products such that:

1. For any α,β and for any $p,q\in\mathcal{B}$,
$$p\star_{\alpha,\alpha}q - q\star_{\alpha,\alpha}p = p\star_{\beta,\beta}q - q\star_{\beta,\beta}p = [p,q].$$
2. $\star_{\alpha,\beta}$ are commutative when $\alpha\neq\beta$,
3. the map $\psi:\mathcal{B}\otimes\mathcal{B}^k\to\mathcal{B}^k$ given by
$$\psi(q,(p_1,\ldots,p_k)) = \psi_q(p_1,\ldots,p_k) = \begin{pmatrix} \mathrm{L}_q^{1,1} & \ldots & \mathrm{L}_q^{k,1}\\ \vdots & & \vdots\\ \mathrm{L}_q^{1,k} & \ldots & \mathrm{L}_q^{k,k}\end{pmatrix}\begin{pmatrix}p_1\\ \vdots\\ p_k\end{pmatrix} = \sum_{\alpha=1}^{k}\left(\mathrm{L}_q^{\alpha,1}p_\alpha,\ldots,\mathrm{L}_q^{\alpha,k}p_\alpha\right) \tag{13}$$

satisfies

$$\psi_{[p,q]} = [\psi_p,\psi_q]. \tag{14}$$

We will show now that if $\mathcal{B}$ is $(k \times k)$-left symmetric algebra then, for any $\alpha = 1, \ldots, k$, $\star_{\alpha,\alpha}$ is Lie admissible. In order to do so, we will use the Bianchi identity. Indeed, let $(A, \star)$ be an algebra and $[,]$ the associated bracket given by $[a, b] = a \star b - b \star a$. The curvature of $\star$ is given by

$$R(a, b) = \mathrm{L}_{[a,b]} - [\mathrm{L}_a, \mathrm{L}_b],$$

where $\mathrm{L}_a b = a \star b$. The Bianchi identity is given by

$$\oint_{a,b,c} R(a, b)c = \oint_{a,b,c} [[a, b], c],$$

where $\oint$ denotes the cyclic sum. Thus $\star$ is Lie admissible if and only if $\oint R(a, b)c = 0$.

Proposition 2.2. *Let $\mathcal{B}$ be $(k \times k)$-left symmetric algebra. Then, the relation (14) is equivalent to*

$$\mathrm{L}^{\alpha,\gamma}_{[p,q]} = \sum_{\beta=1}^{k} \left[\mathrm{L}^{\beta,\gamma}_{p} \circ \mathrm{L}^{\alpha,\beta}_{q} - \mathrm{L}^{\beta,\gamma}_{q} \circ \mathrm{L}^{\alpha,\beta}_{p} \right], \tag{15}$$

for any $p, q \in \mathfrak{p}$ and any α, γ. Moreover, $\star_{\alpha,\alpha}$ is Lie admissible and $[,]$ is a Lie bracket and hence ψ is a representation of the Lie algebra $(\mathcal{B}, [,])$.

Proof. The fact that (14) is equivalent to (15) can be deduced by a direct computation from (13). Indeed, for $\alpha = 1, \ldots, k$ and for any $p, q, p_\alpha \in \mathfrak{p}$,

$$\begin{aligned}
\psi([p, q], (0, \ldots, 0, p_\alpha, 0, \ldots, 0)) &= \left(\mathrm{L}^{\alpha,1}_{[p,q]} p_\alpha, \ldots, \mathrm{L}^{\alpha,k}_{[p,q]} p_\alpha \right), \\
\psi_p \circ \psi_q((0, \ldots, 0, p_\alpha, 0, \ldots, 0)) &= \psi_p \left(\mathrm{L}^{\alpha,1}_{q} p_\alpha, \ldots, \mathrm{L}^{\alpha,k}_{q} p_\alpha \right) \\
&= \sum_{\beta=1}^{k} \left(\mathrm{L}^{\beta,1}_{p} \circ \mathrm{L}^{\alpha,\beta}_{q} p_\alpha, \ldots, \mathrm{L}^{\beta,k}_{p} \circ \mathrm{L}^{\alpha,\beta}_{q} p_\alpha \right), \\
\psi_q \circ \psi_p((0, \ldots, 0, p_\alpha, 0, \ldots, 0)) &= \sum_{\beta=1}^{k} \left(\mathrm{L}^{\beta,1}_{q} \circ \mathrm{L}^{\alpha,\beta}_{p} p_\alpha, \ldots, \mathrm{L}^{\beta,k}_{q} \circ \mathrm{L}^{\alpha,\beta}_{p} p_\alpha \right).
\end{aligned}$$

where $(0, \ldots, 0, p_\alpha, 0, \ldots, 0) \in \mathfrak{p}^k$ with p_α its α-component and zero elsewhere. Thus, the equivalence of (14) and (15) follows.

Now we will use the Bianchi identity to show that $\star_{\alpha,\alpha}$ is Lie admissible. Indeed, by virtue of (15), the curvature of $\star_{\alpha,\alpha}$ is given by

$$R^\alpha(p, q)r = \mathrm{L}^{\alpha,\alpha}_{[p,q]} r - \left[\mathrm{L}^{\alpha,\alpha}_{p}, \mathrm{L}^{\alpha,\alpha}_{q} \right](r) = \sum_{\alpha \neq \beta = 1}^{k} \left[\mathrm{L}^{\beta,\alpha}_{p} \circ \mathrm{L}^{\alpha,\beta}_{q} r - \mathrm{L}^{\beta,\alpha}_{q} \circ \mathrm{L}^{\alpha,\beta}_{p} r \right].$$

One can deduce easily that

$$\oint_{p,q,r} R^\alpha(p, q)r = 0.$$

□

Example 1.

1. Note that the notions of 1-left symmetric algebra and (1×1)-left symmetric algebra are the same and correspond to the classical notion of left symmetric algebra.
2. If $(\mathcal{A}, \bullet)$ is a left-symmetric algebra, then $(\mathcal{A}, \bullet_1 = \bullet, \ldots, \bullet_k = \bullet)$ is a k-left symmetric algebra.

3. If $\bullet_1, \ldots, \bullet_k$ are left symmetric products on $\mathcal{B}$ such that $a \bullet_\alpha b - b \bullet_\alpha a = a \bullet_\beta b - b \bullet_\beta a$ for any α, β then $(\mathcal{B}, (\star_{\alpha,\beta})_{1 \le \alpha \le \beta \le k})$ is $(k \times k)$-left symmetric algebra where $\star_{\alpha,\beta} = 0$ if $\alpha \neq \beta$ and $\star_{\alpha,\alpha} = \bullet_\alpha$.

It turns out that as a para-Kähler Lie algebra is built from two compatible left symmetric algebras (see [5, 8]), a k-para-Kähler Lie algebras is built from a k-left symmetric algebra and a $(k \times k)$-left symmetric algebra compatible in some sense.

Let $\mathfrak{p}$ be a vector space of dimension n such that:

1. $\mathfrak{p}$ carries a structure $([,]_\mathfrak{p}, (\star_{\alpha,\beta})_{1 \le \alpha, \beta \le k})$ of $(k \times k)$-left symmetric algebra,
2. $\mathfrak{p}^*$ carries a structure $(\bullet_1, \ldots, \bullet_k)$ of k-left symmetric algebra.

Define on $\Phi(\mathfrak{p}, k) = \mathfrak{p} \oplus (\mathfrak{p}^*)^k$ the bracket

$$[a,b] = a \circ b - b \circ a, [p,q] = [p,q]_\mathfrak{p} \quad \text{and} \quad [a,p] = \phi_a^*(p) - \psi_p^* a, \quad a,b \in (\mathfrak{p}^*)^k, p,q \in \mathfrak{p}, \tag{16}$$

and the family $(\rho^1, \ldots, \rho^k)$ of 2-forms given by (9).

The vector space $(\mathfrak{p}^*)^k$ has a structure of Lie algebra $[,]$ and ϕ is a representation of this Lie algebra and $\mathfrak{p}$ has a structure of Lie algebra $[,]_\mathfrak{p}$ and ψ is a representation of this Lie algebra structure. We denote by $\phi^T : \mathfrak{p} \to \mathfrak{p}^k \otimes \mathfrak{p}$ and $\psi^T : (\mathfrak{p}^*)^k \to (\mathfrak{p}^*) \otimes (\mathfrak{p}^*)^k$ the dual of $\phi : (\mathfrak{p}^*)^k \otimes \mathfrak{p}^* \to \mathfrak{p}^*$ and $\psi : \mathfrak{p} \otimes \mathfrak{p}^k \to \mathfrak{p}$.

The following theorem is a generalization of a result first obtained in [5, Theorem 4.1] and recovered in [8, Proposition 3.3].

Theorem 2.2. *$(\Phi(\mathfrak{p}, k), [,])$ is a Lie algebra if and only if*

1. *$\phi^T : \mathfrak{p} \to \mathfrak{p}^k \otimes \mathfrak{p}$ is a 1-cocycle of $(\mathfrak{p}, [,]_\mathfrak{p})$ with respect to the representation $\psi \otimes \mathrm{ad}$, i.e.,*

$$\phi^T([p,q]_\mathfrak{p})((a_1,\ldots,a_k),b) = \phi^T(p)((a_1,\ldots,a_k),\mathrm{ad}_q^* b) + \phi^T(p)(\psi_q^*(a_1,\ldots,a_k),b) - \phi^T(q)((a_1,\ldots,a_k),\mathrm{ad}_p^* b)$$
$$-\phi^T(q)(\psi_p^*(a_1,\ldots,a_k),b).$$

2. *$\psi^T : (\mathfrak{p}^*)^k \to (\mathfrak{p}^*) \otimes (\mathfrak{p}^*)^k$ is a 1-cocycle of $((\mathfrak{p}^*)^k, [,])$ with respect to the representation $\phi \otimes \mathrm{ad}$ and $[,]$ is given by $[a,b] = a \circ b - b \circ a$, i.e.,*

$$\psi^T([a,b])(p,(q_1,\ldots,q_k)) = \psi^T(a)(p,\mathrm{ad}_b^*(q_1,\ldots,q_k)) + \psi^T(a)(\phi_b^* p,(q_1,\ldots,q_k)) - \psi^T(b)(p,\mathrm{ad}_a^*(q_1,\ldots,q_k))$$
$$-\psi^T(b)(\phi_a^* p,(q_1,\ldots,q_k)).$$

In this case $(\Phi(\mathfrak{p}, k), [,], (\mathfrak{p}^*)^k, \rho^1, \ldots, \rho^k)$ is a k-para-Kähler Lie algebra. Moreover, all k-para-Kähler Lie algebras are obtained in this way.

Proof. We will show that the bracket given by (16) satisfies the Jacobi identity if and only if the equations in the statement of the theorem hold. Note first that since the bracket on $\mathfrak{p}$ and $(\mathfrak{p}^*)^k$ are Lie brackets then the bracket given by (16) satisfies the Jacobi identity for any $p, q, r \in \mathfrak{p}$ and any $a, b, c \in (\mathfrak{p}^*)^k$. Let $a \in (\mathfrak{p}^*)^k$ and $p, q \in \mathfrak{p}$. We have

$$[[p,q],a] = \psi^*_{[p,q]_\mathfrak{p}} a - \phi_a^*([p,q]_\mathfrak{p}),$$
$$[[q,a],p] = \left[\psi_q^* a, p\right] - [\phi_a^* q, p] = \phi^*_{\psi_q^* a} p - \psi_p^* \psi_q^* a - [\phi_a^* q, p]_\mathfrak{p},$$
$$[[a,p],q] = \psi_q^* \psi_p^* a - \phi^*_{\psi_p^* a} q + [\phi_a^* p, q]_\mathfrak{p}.$$

According to Proposition 2.2, ψ is a representation and so is ψ^* and hence

$$J(p,q,a) := [[p,q],a] + [[q,a],p] + [[a,p],q] = \phi^*_{\psi^*_q a}p - \phi^*_{\psi^*_p a}q - \phi^*_a([p,q]_{\mathfrak{p}}) + [\phi^*_a p,q]_{\mathfrak{p}} - [\phi^*_a q,p]_{\mathfrak{p}} \in \mathfrak{p}.$$

Now, for any $b \in \mathfrak{p}^*$, we have

$$\begin{aligned}
\prec b, J(p,q,a) \succ &= -\prec \phi(\psi^*_q a,b),p \succ + \prec \phi(\psi^*_p a,b),q \succ + \prec \phi(a,b),[p,q]_{\mathfrak{p}} \succ + \prec b,[\phi^*_a p,q]_{\mathfrak{p}} \succ \\
&\quad - \prec b,[\phi^*_a q,p]_{\mathfrak{p}} \succ \\
&= -\prec (\psi^*_q a,b),\phi^T(p) \succ + \prec (\psi^*_p a,b),\phi^T(q) \succ + \prec (a,b),\phi^T([p,q]_{\mathfrak{p}}) \succ + \prec \mathrm{ad}^*_q b,\phi^*_a p \succ \\
&\quad - \prec \mathrm{ad}^*_p b,\phi^*_a q \succ \\
&= -\prec (\psi^*_q a,b),\phi^T(p) \succ + \prec (\psi^*_p a,b),\phi^T(q) \succ + \prec (a,b),\phi^T([p,q]_{\mathfrak{p}}) \succ \\
&\quad - \prec (a,\mathrm{ad}^*_q b),\phi^T(p) \succ + \prec (a,\mathrm{ad}^*_p b),\phi^T(q) \succ .
\end{aligned}$$

The vanishing of $J(p,q,a)$ gives the first relation in the theorem. A similar computation gives the second relation. □

A $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ and a k-left symmetric left algebra structure on $\mathfrak{p}^*$ are called compatible if they satisfy the conditions of Theorem 2.2.

Example 2.

1. Any k-left symmetric algebra structure on $\mathfrak{p}^*$ is compatible with the trivial $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$.
2. Any $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ is compatible with the trivial k-left symmetric algebra structure on $\mathfrak{p}^*$.

We end this section by giving a way of building k-left symmetric algebras. The following is a generalization of a construction given by S. Gelfand.

Proposition 2.3. *Let $(A,.)$ be a commutative associative algebra and D_1, D_2 two derivations of $(A,.)$ which commute. Then the two products*

$$a \bullet_1 b = a.D_1 b \quad \text{and} \quad a \bullet_2 b = a.D_2 b$$

are left symmetric and compatible in the sense of (10).

Proof. We have, for any $a, b \in A$, the left multiplication of $\bullet_i$ is $\mathrm{L}^i_a = \mathrm{L}_a \circ D_i$, where L_a is the left multiplication of the product on A. By virtue of (10), $\bullet_1$ and $\bullet_2$ are compatible if and only if

$$Q := [\mathrm{L}_a \circ D_1, \mathrm{L}_b \circ D_2] - \mathrm{L}_{a.D_1 b} \circ D_2 + \mathrm{L}_{b.D_2 a} \circ D_1 = 0.$$

Let us compute:

$$\begin{aligned}
Q &= [\mathrm{L}_a \circ D_1, \mathrm{L}_b \circ D_2] - \mathrm{L}_{a.D_1 b} \circ D_2 + \mathrm{L}_{b.D_2 a} \circ D_1 \\
&= \mathrm{L}_a \circ D_1 \circ \mathrm{L}_b \circ D_2 - \mathrm{L}_b \circ D_2 \circ \mathrm{L}_a \circ D_1 - \mathrm{L}_a \circ \mathrm{L}_{D_1 b} \circ D_2 + \mathrm{L}_b \circ \mathrm{L}_{D_2 a} \circ D_1 \\
&= \mathrm{L}_a \circ D_1 \circ \mathrm{L}_b \circ D_2 - \mathrm{L}_b \circ D_2 \circ \mathrm{L}_a \circ D_1 - \mathrm{L}_a \circ [D_1, \mathrm{L}_b] \circ D_2 + \mathrm{L}_b \circ [D_2, \mathrm{L}_a] \circ D_1 \\
&= \mathrm{L}_a \circ \mathrm{L}_b \circ D_1 \circ D_2 - \mathrm{L}_b \circ \mathrm{L}_a \circ D_2 \circ D_1 \\
&= \mathrm{L}_{ab} \circ [D_1, D_2] = 0
\end{aligned}$$

which completes the proof. □

Example 3. We consider R^4 endowed with the associative commutative product

$$e_1.e_1 = e_1, e_1.e_2 = e_2.e_1 = e_2, e_1.e_3 = e_3.e_1 = e_3, e_1.e_4 = e_4.e_1 = e_4.$$

We consider the two derivations

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These two derivations commute and, according to Proposition 2.3, they define a 2-left symmetric structure on R^4 by

$$e_1 \bullet_1 e_i = e_i, i = 2, 3, 4, \quad e_1 \bullet_2 e_3 = ae_2 \quad \text{and} \quad e_1 \bullet_2 e_4 = be_2 + ce_3.$$

3. Exact k-para-Kähler Lie algebras

In this section, we start with a k-left symmetric algebra structure on $\mathfrak{p}^*$ and we look for a compatible $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ such that ψ^T is a coboundary leading to the generalization of the results obtained in the case $k=1$ in [5, 8].

Let $\mathfrak{p}$ be a vector space of dimension n. Suppose that $\mathfrak{p}^*$ is endowed with a k-left symmetric algebra structure $(\bullet_1, ..., \bullet_k)$ and we consider $\phi : (\mathfrak{p}^*)^k \times \mathfrak{p}^* \to \mathfrak{p}^*$ the associated representation given by

$$\phi(a, \rho) = \mathrm{L}_a \rho = \sum_{\alpha=1}^{k} \mathrm{L}^\alpha_{a_\alpha} \rho = \sum_{\alpha=1}^{k} a_\alpha \bullet_\alpha \rho, \quad a = (a_1, ..., a_k) \in (\mathfrak{p}^*)^k, \rho \in \mathfrak{p}^*.$$

The left symmetric product on $(\mathfrak{p}^*)^k$ is given by

$$a \circ b = \mathrm{L}_a b = (\phi(a, b_1), ..., \phi(a, b_k)) \quad \text{and} \quad [a, b] = a \circ b - b \circ a.$$

Let $\mathbf{r} \in \mathfrak{p}^* \otimes (\mathfrak{p}^*)^k$ and we define $\psi : \mathfrak{p} \otimes (\mathfrak{p})^k \to \mathfrak{p}^k$ by

$$\prec a, \psi(p, u) \succ = -\mathbf{r}(\phi^*_a p, u) - \mathbf{r}(p, \mathrm{ad}^*_a u), \quad p \in \mathfrak{p}, u \in \mathfrak{p}^k, a \in (\mathfrak{p}^*)^k.$$

If we define $\mathbf{r} : \mathfrak{p} \to (\mathfrak{p}^*)^k$ by $\prec \mathbf{r}(p), u \succ = \mathbf{r}(p, u)$, we get

$$\prec a, \psi(p, u) \succ = \mathrm{L}_a(\mathbf{r})(p, u) + \prec a, \mathrm{L}^*_{\mathbf{r}(p)} u \succ, \tag{17}$$

where

$$\mathrm{L}_a(\mathbf{r})(p, u) = -\mathbf{r}(\mathrm{L}^*_a p, u) - \mathbf{r}(p, \mathrm{L}^*_a u).$$

Note that

$$\mathbf{r}(p, u) = \sum_{\alpha=1}^{k} r(p, (0, ..., u_\alpha, ..., 0)) = \sum_{\alpha=1}^{k} \mathbf{r}_\alpha(p, u_\alpha) = \sum_{\alpha=1}^{k} \big(\mathbf{a}_\alpha(p, u_\alpha) + \mathbf{s}_\alpha(p, u_\alpha)\big),$$

where $\mathbf{r}_\alpha = \mathbf{a}_\alpha + \mathbf{s}_\alpha \in \mathfrak{p}^* \otimes \mathfrak{p}^*, \mathbf{a}_\alpha$ its skew-symmetric part and $\mathbf{s}_\alpha$ its symmetric part. On the other hand, we define the family of products $\star_{\alpha,\beta}$ on $\mathfrak{p}$ by

$$\psi(p, u) = \sum_{\alpha=1}^{k} \psi(p, (0, ..., u_\alpha, ..., 0)) = \sum_{\alpha=1}^{k} (p \star_{\alpha,1} u_\alpha, ..., p \star_{\alpha,k} u_\alpha).$$

Let us see now under which conditions ψ defines a $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ such that ϕ^T is a 1-cocycle of $(\mathfrak{p}, [,]_\mathfrak{p})$ with respect to the representation $\psi \otimes \mathrm{ad}$.

Let us start by studying under which conditions the family of products $\star_{\alpha,\beta}$ is adapted to our purpose.

Proposition 3.1. *For any $\alpha,\beta\in\{1,\ldots,k\}$ with $\alpha\neq\beta$ and for any $\rho\in\mathfrak{p}^*$ and $p,q\in\mathfrak{p}$,*

$$\prec\rho,p\star_{\alpha,\alpha}q\succ=\mathrm{L}_\rho^\alpha(\mathbf{r}_\alpha)(p,q)+\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q\succ\quad\textit{and}\quad\prec\rho,p\star_{\alpha,\beta}q\succ=\mathrm{L}_\rho^\beta(\mathbf{r}_\alpha)(p,q).$$

Thus $\star_{\alpha,\beta}$ is commutative when $\alpha\neq\beta$ if and only if $\mathrm{L}_\rho^\beta(\mathbf{a}_\alpha)=0$ and

$$\prec\rho,p\star_{\alpha,\beta}q\succ=\mathrm{L}_\rho^\beta(\mathbf{s}_\alpha)(p,q).$$

Moreover, $[p,q]_\mathfrak{p}=p\star_{\alpha,\alpha}q-q\star_{\alpha,\alpha}p$ is independent of α if and only if $\mathrm{L}_\rho^\alpha(\mathbf{a}_\alpha)=\mathrm{L}_\rho^\beta(\mathbf{a}_\beta)$ for any α,β. In this case

$$\prec\rho,[p,q]_\mathfrak{p}\succ=2\mathrm{L}_\rho^\alpha(\mathbf{a}_\alpha)(p,q)+\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q-\mathrm{L}^*_{\mathbf{r}(q)}p\succ.$$

Proof. For any $\alpha\in\{1,\ldots,k\}$ and $\rho\in\mathfrak{p}^*$, we define $\rho^\alpha\in(\mathfrak{p}^*)^k$ by $\rho_i^\alpha=\rho\delta_{i\alpha}$, where δ denotes the Kronecker symbol.

Put

$$\psi(p,h)=\sum_{\alpha=1}^k\psi(p,(0,\ldots,h_\alpha,\ldots,0))=\sum_{\alpha=1}^k(p\star_{\alpha,1}h_\alpha,\ldots,p\star_{\alpha,k}h_\alpha).$$

We have, for any $\rho\in\mathfrak{p}^*$ and $p,q\in\mathfrak{p}$ and $\alpha,\beta\in\{1,\ldots,k\}$

$$\prec\rho,p\star_{\alpha,\beta}q\succ=\prec\rho^\beta,\psi(p,q^\alpha)\succ=\mathrm{L}_{\rho^\beta}(\mathbf{r})(p,q^\alpha)+\prec\rho^\beta,\mathrm{L}^*_{\mathbf{r}(p)}q^\alpha\succ.$$

But $\prec\rho^\beta,\mathrm{L}^*_{\mathbf{r}(p)}q^\alpha\succ=\delta_{\alpha\beta}\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q\succ$ and

$$\mathrm{L}_{\rho^\beta}(\mathbf{r})(p,q^\alpha)=-\mathbf{r}((\mathrm{L}_\rho^\beta)^*p,q^\alpha)-\mathbf{r}(p,\mathrm{L}^*_{\rho^\beta}q^\alpha)=-\mathbf{r}_\alpha((\mathrm{L}_\rho^\beta)^*p,q)-\mathbf{r}_\alpha(p,(\mathrm{L}_\rho^\beta)^*q).$$

Thus

$$\prec\rho,p\star_{\alpha,\alpha}q\succ=\mathrm{L}_\rho^\alpha(\mathbf{r}_\alpha)(p,q)+\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q\succ,$$

and if $\alpha\neq\beta$,

$$\prec\rho,p\star_{\alpha,\beta}q\succ=\mathrm{L}_\rho^\beta(\mathbf{r}_\alpha)(p,q).$$

So we get

$$\prec\rho,[p,q]_\mathfrak{p}\succ=2\mathrm{L}_\rho^\alpha(\mathbf{a}_\alpha)(p,q)+\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q-\mathrm{L}^*_{\mathbf{r}(q)}p\succ,\rho\in\mathfrak{p}^*,p,q\in\mathfrak{p}.$$

So we must have $\mathrm{L}_\rho^\alpha(\mathbf{a}_\alpha)=\mathrm{L}_\rho^\beta(\mathbf{a}_\beta)$ for any α,β. Thus $\star_{\alpha,\beta}$ is commutative when $\alpha\neq\beta$ if and only if $\mathrm{L}_\rho^\beta(\mathbf{a}_\alpha)=0$ and

$$\prec\rho,p\star_{\alpha,\beta}q\succ=\mathrm{L}_\rho^\beta(\mathbf{s}_\alpha)(p,q).$$

□

We suppose that, for any $\alpha,\beta\in\{1,\ldots,k\}$ with $\alpha\neq\beta$ and for any $\rho\in(\mathfrak{p})^*$,

$$\mathrm{L}_\rho^\beta(\mathbf{a}_\alpha)=0\quad\text{and}\quad\mathrm{L}_\rho^\alpha(\mathbf{a}_\alpha)=\mathrm{L}_\rho^\beta(\mathbf{a}_\beta).$$

So we get

$$\mathrm{L}_a(\mathbf{a})(p,u)=\sum_{\alpha,\beta}\mathrm{L}^\alpha_{a_\alpha}(\mathbf{a}_\beta)(p,u_\beta)=\sum_{\alpha=1}^k\mathrm{L}^\alpha_{a_\alpha}(\mathbf{a}_\alpha)(p,u_\alpha).$$

So

$$\prec\rho,[p,q]_\mathfrak{p}\succ=2\mathrm{L}(\mathbf{a})(\rho,p,q)+\prec\rho,\mathrm{L}^*_{\mathbf{r}(p)}q-\mathrm{L}^*_{\mathbf{r}(q)}p\succ,\rho\in\mathfrak{p}^*,p,q\in\mathfrak{p}.\qquad(18)$$

where $\mathrm{L}(\mathbf{a}) \in \mathfrak{p} \otimes \mathfrak{p}^* \otimes \mathfrak{p}^*$ is given by

$$\mathrm{L}(\mathbf{a})(\rho, p, q) = \mathrm{L}^{\alpha}_{\rho}(\mathbf{a}_{\alpha})(p, q), \quad \alpha = 1, \dots, k. \tag{19}$$

The second step is to see under which conditions ψ defines a $(k \times k)$-left symmetric algebra structure on $\mathfrak{p}$ and ϕ^T is a 1-cocycle of $(\mathfrak{p}, [,]_{\mathfrak{p}})$ with respect to the representation $\psi \otimes \mathrm{ad}$, i.e.,

$$\begin{cases} P := \prec a, \psi([p,q]_{\mathfrak{p}}, u) \succ - \prec a, \psi(p, \psi(q,u)) \succ + \prec a, \psi(q, \psi(p,u)) \succ = 0, \\ Q := \phi^T([p,q]_{\mathfrak{p}})(a, \rho) - \phi^T(p)(a, \mathrm{ad}^*_q \rho) - \phi^T(p)(\psi^*_q a, \rho) + \phi^T(q)(a, \mathrm{ad}^*_p \rho) + \phi^T(q)(\psi^*_p a, \rho) = 0 \end{cases}$$

for any $a \in (\mathfrak{p}^*)^k, u \in \mathfrak{p}^k$ and $p, q \in \mathfrak{p}, \rho \in \mathfrak{p}^*$ and

$$\psi^*_p a = -(\mathrm{L}_a(\mathbf{r}))(p) + \mathrm{L}_{\mathbf{r}(p)} a = -(\mathrm{L}_a(\mathbf{r}))(p) + \mathbf{r}(p) \circ a \quad \text{and} \quad \phi^T(p)(a, \rho) = \prec \mathrm{L}_a \rho, p \succ .$$

Let us compute Q.

$$\begin{aligned} Q &= \phi^T([p,q]_{\mathfrak{p}})(a, \rho) - \phi^T(p)(a, \mathrm{ad}^*_q \rho) - \phi^T(p)(\psi^*_q a, \rho) + \phi^T(q)(a, \mathrm{ad}^*_p \rho) + \phi^T(q)(\psi^*_p a, \rho) \\ &= \prec \mathrm{L}_a \rho, [p,q]_{\mathfrak{p}} \succ - \prec \mathrm{L}_a \mathrm{ad}^*_q \rho, p \succ + \prec \mathrm{L}_{(\mathrm{L}_a(\mathbf{r}))(q)} \rho, p \succ - \prec \mathrm{L}_{\mathbf{r}(q) \circ a} \rho, p \succ \\ &\quad + \prec \mathrm{L}_a \mathrm{ad}^*_p \rho, q \succ - \prec \mathrm{L}_{(\mathrm{L}_a(\mathbf{r}))(p)} \rho, q \succ + \prec \mathrm{L}_{\mathbf{r}(p) \circ a} \rho, q \succ . \end{aligned}$$

Then

$$\begin{aligned} Q &= 2\mathrm{L}(\mathbf{a})(\mathrm{L}_a \rho, p, q) - \prec \mathrm{L}_a \rho, \mathrm{L}^*_{\mathbf{r}(q)} p \succ + \prec \mathrm{L}_a \rho, \mathrm{L}^*_{\mathbf{r}(p)} q \succ - \prec \rho, \left[q, \mathrm{L}^*_a p\right]_{\mathfrak{p}} \succ \\ &\quad + \prec \mathrm{L}_{(\mathrm{L}_a(\mathbf{r}))(q)} \rho, p \succ - \prec \mathrm{L}_{\mathbf{r}(q) \circ a} \rho, p \succ - \prec \mathrm{L}_{(\mathrm{L}_a(\mathbf{r}))(p)} \rho, q \succ + \prec \mathrm{L}_{\mathbf{r}(p) \circ a} \rho, q \succ \\ &\quad + \prec \rho, \left[p, \mathrm{L}^*_a q\right]_{\mathfrak{p}} \succ \\ &= 2\mathrm{L}(\mathbf{a})(\mathrm{L}_a \rho, p, q) + 2\mathrm{L}(\mathbf{a})(\rho, p, \mathrm{L}^*_a q) + 2\mathrm{L}(\mathbf{a})(\rho, \mathrm{L}^*_a p, q) \\ &\quad + \prec \rho, \mathrm{L}^*_{\mathbf{r}(p)} \mathrm{L}^*_a q \succ - \prec \rho, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} p \succ - \prec \rho, \mathrm{L}^*_{\mathbf{r}(q)} \mathrm{L}^*_a p \succ + \prec \rho, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q \succ \\ &\quad + \prec \rho, \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(q)} p \succ - \prec \rho, \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(p)} q \succ \\ &\quad - \prec \rho, \mathrm{L}^*_{(\mathrm{L}_a(\mathbf{r}))(q)} p \succ + \prec \rho, \mathrm{L}^*_{\mathbf{r}(q) \circ a} p \succ + \prec \rho, \mathrm{L}^*_{(\mathrm{L}_a(\mathbf{r}))(p)} q \succ - \prec \rho, \mathrm{L}^*_{\mathbf{r}(p) \circ a} q \succ \\ &= 2\mathrm{L}(\mathbf{a})(\mathrm{L}_a \rho, p, q) + 2\mathrm{L}(\mathbf{a})(\rho, p, \mathrm{L}^*_a q) + 2\mathrm{L}(\mathbf{a})(\rho, \mathrm{L}^*_a p, q) + \prec \rho, A(p,q) - A(q,p) \succ , \end{aligned}$$

where

$$A(p,q) = \mathrm{L}^*_{\mathbf{r}(p)} \mathrm{L}^*_a q + \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q - \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(p)} q + \mathrm{L}^*_{(\mathrm{L}_a(\mathbf{r}))(p)} q - \mathrm{L}^*_{\mathbf{r}(p) \circ a} q.$$

But

$$\mathbf{r}(\mathrm{L}^*_a p) + (\mathrm{L}_a(\mathbf{r}))(p) = a \circ \mathbf{r}(p) \tag{20}$$

and hence

$$A(p,q) = \left[\mathrm{L}^*_{\mathbf{r}(p)}, \mathrm{L}^*_a\right] q - \mathrm{L}^*_{\left[\mathbf{r}(p), a\right]} q = 0.$$

Finally,

$$Q = 2\mathrm{L}(\mathbf{a})(\mathrm{L}_a \rho, p, q) + 2\mathrm{L}(\mathbf{a})(\rho, p, \mathrm{L}^*_a q) + 2\mathrm{L}(\mathbf{a})(\rho, \mathrm{L}^*_a p, q).$$

Now let us compute P. We split it in three parts.

$$
\begin{aligned}
P_1 &= \prec a, \psi([p,q]_{\mathfrak{p}}, u) \succ \\
&= \mathrm{L}_a(\mathbf{r})([p,q]_{\mathfrak{p}}, u) + \prec a, \mathrm{L}^*_{\mathbf{r}([p,q]_{\mathfrak{p}})} u \succ \\
&= -\mathbf{r}(\mathrm{L}^*_a[p,q]_{\mathfrak{p}}, u) - \mathbf{r}([p,q]_{\mathfrak{p}}, \mathrm{L}^*_a u) + \prec a, \mathrm{L}^*_{\mathbf{r}([p,q]_{\mathfrak{p}})} u \succ , \\
P_2 &= - \prec a, \psi(p, \psi(q,u)) \succ \\
&= - \prec (\mathrm{L}_a(\mathbf{r}))(p), \psi(q,u) \succ + \prec \mathbf{r}(p) \circ a, \psi(q,u) \succ \\
&= -\mathrm{L}_{(\mathrm{L}_a(\mathbf{r}))(p)}(\mathbf{r})(q,u) - \prec \mathrm{L}_a(\mathbf{r})(p), \mathrm{L}^*_{\mathbf{r}(q)} u \succ + \mathrm{L}_{\mathbf{r}(p)\circ a}(\mathbf{r})(q,u) + \prec \mathbf{r}(p) \circ a, \mathrm{L}^*_{\mathbf{r}(q)} u \succ \\
&\overset{(20)}{=} -\mathrm{L}_{a\circ \mathbf{r}(p)}(\mathbf{r})(q,u) + \mathrm{L}_{\mathbf{r}(\mathrm{L}^*_a p)}(\mathbf{r})(q,u) - \mathrm{L}_a(\mathbf{r})(p, \mathrm{L}^*_{\mathbf{r}(q)} u) - \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(p)\circ a} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(p)\circ a} u) \\
&\quad - \prec a, \mathrm{L}^*_{\mathbf{r}(p)} \circ \mathrm{L}^*_{\mathbf{r}(q)} u \succ .
\end{aligned}
$$

So

$$
\begin{aligned}
P_1 &= -\mathbf{r}(\mathrm{L}^*_a[p,q]_{\mathfrak{p}}, u) - \mathbf{r}([p,q]_{\mathfrak{p}}, \mathrm{L}^*_a u) + \prec a, \mathrm{L}^*_{\mathbf{r}([p,q]_{\mathfrak{p}})} u \succ , \\
P_2 &= \mathbf{r}(\mathrm{L}^*_{a\circ \mathbf{r}(p)} q, u) + \mathbf{r}(q, \mathrm{L}^*_{a\circ \mathbf{r}(p)} u) - \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} u) \\
&\quad + \mathbf{r}(\mathrm{L}^*_a p, \mathrm{L}^*_{\mathbf{r}(q)} u) + \mathbf{r}(p, \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(q)} u) - \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(p)\circ a} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(p)\circ a} u) - \prec a, \mathrm{L}^*_{\mathbf{r}(p)} \circ \mathrm{L}^*_{\mathbf{r}(q)} u \succ \\
&= \mathbf{r}(\mathrm{L}^*_{[a,\mathbf{r}(p)]} q, u) + \mathbf{r}(q, \mathrm{L}^*_{[a,\mathbf{r}(p)]} u) - \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} u) \\
&\quad + \mathbf{r}(\mathrm{L}^*_a p, \mathrm{L}^*_{\mathbf{r}(q)} u) + \mathbf{r}(p, \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(q)} u) - \prec a, \mathrm{L}^*_{\mathbf{r}(p)} \circ \mathrm{L}^*_{\mathbf{r}(q)} u \succ \\
P_3 &= \prec a, \psi(q, \psi(p,u)) \succ \\
&= -\mathbf{r}(\mathrm{L}^*_{[a,\mathbf{r}(q)]} p, u) - \mathbf{r}(p, \mathrm{L}^*_{[a,\mathbf{r}(q)]} u) + \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} p, u) + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} u) \\
&\quad - \mathbf{r}(\mathrm{L}^*_a q, \mathrm{L}^*_{\mathbf{r}(p)} u) - \mathbf{r}(q, \mathrm{L}^*_a \mathrm{L}^*_{\mathbf{r}(p)} u) + \prec a, \mathrm{L}^*_{\mathbf{r}(q)} \circ \mathrm{L}^*_{\mathbf{r}(p)} u \succ
\end{aligned}
$$

Thus if we put $\Delta(\mathbf{r})(p,q) = \mathbf{r}([p,q]_{\mathfrak{p}}) - [\mathbf{r}(p), \mathbf{r}(q)]$ then

$$
\begin{aligned}
P &= -\mathbf{r}(\mathrm{L}^*_a[p,q]_{\mathfrak{p}}, u) - \mathbf{r}([p,q]_{\mathfrak{p}}, \mathrm{L}^*_a u) + \prec a, \mathrm{L}^*_{\Delta(\mathbf{r})(p,q)} u \succ \\
&\quad + \mathbf{r}(\mathrm{L}^*_{[a,\mathbf{r}(p)]} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(p)} \mathrm{L}^*_a u) - \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q, u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} u) \\
&\quad + \mathbf{r}(\mathrm{L}^*_a p, \mathrm{L}^*_{\mathbf{r}(q)} u) - \mathbf{r}(\mathrm{L}^*_{[a,\mathbf{r}(q)]} p, u) + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(q)} \circ \mathrm{L}^*_a u) + \mathbf{r}(\mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} p, u) + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} u) \\
&\quad - \mathbf{r}(\mathrm{L}^*_a q, \mathrm{L}^*_{\mathbf{r}(p)} u) \\
&= \mathbf{r}(s,u) - \mathbf{r}([p,q]_{\mathfrak{p}}, \mathrm{L}^*_a u) + \prec a, \mathrm{L}^*_{\Delta(\mathbf{r})(p,q)} u \succ + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(q)} \circ \mathrm{L}^*_a u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(p)} \mathrm{L}^*_a u) \\
&\quad + \mathbf{r}(\mathrm{L}^*_a p, \mathrm{L}^*_{\mathbf{r}(q)} u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} u) + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} u) - \mathbf{r}(\mathrm{L}^*_a q, \mathrm{L}^*_{\mathbf{r}(p)} u),
\end{aligned}
$$

with

$$
s = -\mathrm{L}^*_a[p,q]_{\mathfrak{p}} + \mathrm{L}^*_{[a,\mathbf{r}(p)]} q - \mathrm{L}^*_{[a,\mathbf{r}(q)]} p - \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a p)} q + \mathrm{L}^*_{\mathbf{r}(\mathrm{L}^*_a q)} p.
$$

Let us simplify the part of P given by

$$
T = -\mathbf{r}([p,q]_{\mathfrak{p}}, \mathrm{L}^*_a u) + \mathbf{r}(p, \mathrm{L}^*_{\mathbf{r}(q)} \circ \mathrm{L}^*_a u) - \mathbf{r}(q, \mathrm{L}^*_{\mathbf{r}(p)} \mathrm{L}^*_a u) + \prec a, \mathrm{L}^*_{\Delta(\mathbf{r})(p,q)} u \succ .
$$

We have

$$T = -\mathbf{r}([p,q]_{\mathfrak{p}}, L^*_a u) + \mathbf{r}(p, L^*_{\mathbf{r}(q)} \circ L^*_a u) - \mathbf{r}(q, L^*_{\mathbf{r}(p)} L^*_a u) + \prec a, L^*_{\Delta(\mathbf{r})(p,q)} u \succ$$
$$= - \prec \mathbf{r}([p,q]_{\mathfrak{p}}), L^*_a u \succ - \prec [\mathbf{r}(q), \mathbf{r}(p)], L^*_a u \succ + \prec a, L^*_{\Delta(\mathbf{r})(p,q)} u \succ$$
$$= \prec [a, \Delta(\mathbf{r})(p,q)], u \succ .$$

On the other hand,

$$\mathbf{r}(p, L^*_{\mathbf{r}(L^*_a q)} u) - \mathbf{r}(L^*_a q, L^*_{\mathbf{r}(p)} u) = \prec \mathbf{r}(p), L^*_{\mathbf{r}(L^*_a q)} u \succ - \prec \mathbf{r}(L^*_a q), L^*_{\mathbf{r}(p)} u \succ = \prec [\mathbf{r}(p), \mathbf{r}(L^*_a q)], u \succ .$$

So

$$P = \mathbf{r}(s,u) + \prec [a, \Delta(\mathbf{r})(p,q)], u \succ + \prec [\mathbf{r}(p), \mathbf{r}(L^*_a q)], u \succ - \prec [\mathbf{r}(q), \mathbf{r}(L^*_a p)], u \succ .$$

From (18), we have

$$\prec \rho, [p,q]_{\mathfrak{p}} \succ = 2L(\mathbf{a})(\rho, p, q) + \prec \rho, L^*_{\mathbf{r}(p)} q - L^*_{\mathbf{r}(q)} p \succ, \rho \in \mathfrak{p}^*, p, q \in \mathfrak{p}. \quad (21)$$

where $L(\mathbf{a})$ is given by (19). Let us simplify the expression of s. We have

$$-L^*_a [p,q]_{\mathfrak{p}} = 2L(\mathbf{a})(L_a \bullet, p, q) - L^*_a L^*_{\mathbf{r}(p)} q + L^*_a L^*_{\mathbf{r}(q)} p,$$
$$-L^*_{\mathbf{r}(L^*_a p)} q = [q, L^*_a p]_{\mathfrak{p}} - L^*_{\mathbf{r}(q)} L^*_a p + 2L(\mathbf{a})(\bullet, L^*_a p, q),$$
$$L^*_{\mathbf{r}(L^*_a q)} p = -[p, L^*_a q] + L^*_{\mathbf{r}(p)} L^*_a q + 2L(\mathbf{a})(\bullet, p, L^*_a q).$$

So

$$s = 2L(\mathbf{a})(L_a \bullet, p, q) + 2L(\mathbf{a})(\bullet, L^*_a p, q) + 2L(\mathbf{a})(\bullet, p, L^*_a q) - [L^*_a p, q]_{\mathfrak{p}} - [p, L^*_a q]_{\mathfrak{p}}.$$

So, we get

$$P = \prec [a, \Delta(r)(p,q)] + \mathbf{r}(P(a)(L(\mathbf{a}))) - \Delta(\mathbf{r})(L^*_a p, q) - \Delta(\mathbf{r})(p, L^*_a q), u \succ ,$$

where

$$\prec \rho, P(a)(L(\mathbf{a})) \succ = 2L(\mathbf{a})(L_a \rho, p, q) + 2L(\mathbf{a})(\rho, L^*_a p, q) + 2L(\mathbf{a})(\rho, p, L^*_a q).$$

So far, we have proved the following theorem.

Theorem 3.1. *Let $\mathfrak{p}$ be a vector space of dimension n such that $\mathfrak{p}^*$ is endowed with a k-left symmetric algebra structure $(\bullet_1, \ldots, \bullet_k)$ and $\mathbf{r} = (\mathbf{s}_1 + \mathbf{a}_1, \ldots, \mathbf{s}_k + \mathbf{a}_k) \in \mathfrak{p}^* \otimes (\mathfrak{p}^*)^k$ such that, for any $\alpha \neq \beta$ and for any $\rho \in \mathfrak{p}^*$,*

$$L^\alpha_\rho(\mathbf{a}_\beta) = 0 \quad \text{and} \quad L^\alpha_\rho(\mathbf{a}_\alpha) = L^\beta_\rho(\mathbf{a}_\beta) =: L(\mathbf{a})(\rho, ., .).$$

Then ψ given by (17) defines a $(k \times k)$-left symmetric structure on $\mathfrak{p}$ compatible with the k-left symmetric structure of $(\mathfrak{p}^*)^k$ if and only if, for any $a \in (\mathfrak{p}^*)^k$ and $p, q \in \mathfrak{p}$,

$$[a, \Delta(\mathbf{r})(p,q)] + L_a(\Delta(\mathbf{r}))(p,q) = 0, \quad \Delta(\mathbf{r})(p,q) = \mathbf{r}([p,q]_{\mathfrak{p}}) - [\mathbf{r}(p), \mathbf{r}(q)]$$

and, for any $a \in (\mathfrak{p}^*)^k, \rho \in \mathfrak{p}^*, p, q \in \mathfrak{p}$,

$$L(\mathbf{a})(L_a \rho, p, q) + L(\mathbf{a})(\rho, L^*_a p, q) + L(\mathbf{a})(\rho, p, L^*_a q) = 0.$$

An important consequence of this theorem is the introduction of the generalization of S-matrices (see [5, 8]).

Definition 3.1. Let $\mathbf{r} = (\mathbf{r}^1, \ldots, \mathbf{r}^k)$ be a family of symmetric elements of $\mathcal{A} \otimes \mathcal{A}$ where $\mathcal{A}$ has a structure of k-left symmetric algebra $(\bullet_1, \ldots, \bullet_k)$. We call $\mathbf{r}$ a S_k-matrix if, for any $\alpha = 1, \ldots, k, p, q \in \mathcal{A}^*$,

$$\mathbf{r}^\alpha([p, q]_*) = \sum_{\beta=1}^{k} \left[\mathbf{r}^\beta(p) \bullet_\beta \mathbf{r}^\alpha(q) - \mathbf{r}^\beta(q) \bullet_\beta \mathbf{r}^\alpha(p)\right],$$

where

$$[p, q]_* = \sum_{\beta=1}^{k} \left[(\mathrm{L}^\beta_{\mathbf{r}^\beta(p)})^* q - (\mathrm{L}^\beta_{\mathbf{r}^\beta(q)})^* p\right].$$

Example 4.

1. Let $(\mathcal{A}, \bullet)$ be a left symmetric algebra and $\mathbf{r} \in \mathcal{A} \otimes \mathcal{A}$ be a classical S-matrix, i.e., $\mathbf{r}$ satisfies

$$\mathbf{r}\left(\mathrm{L}^*_{\mathbf{r}(p)} q - \mathrm{L}^*_{\mathbf{r}(p)} q\right) = \mathbf{r}(p) \bullet \mathbf{r}(q) - \mathbf{r}(q) \bullet \mathbf{r}(p),$$

for any $p, q \in \mathcal{A}^*$ (see [5, 8]). For any $k \geq 1$, endow $\mathcal{A}$ with the k-left symmetric structure given by $\bullet_\alpha = \mu_\alpha \bullet$, where $\mu_\alpha \in R$. Then $\mathbf{r}^k = (\mathbf{r}, \ldots, \mathbf{r})$ is a S_k-matrix.

2. Consider the 2-left symmetric on R^4 given in Example 3, then one can check by a direct computation that, for $\mathrm{r}_{2,4}, \mathrm{r}_{2,2}, \mathrm{r}_{4,4}, \mathrm{s}_{1,1}, \mathrm{s}_{1,2} \in R$,

$$\mathbf{r}^1 = r_{2,4} e_2 \odot e_4 + r_{2,2} e_2 \odot e_2 + r_{4,4} e_4 \odot e_4 \quad \text{and} \quad \mathbf{r}^2 = s_{1,1} e_1 \otimes e_1 + s_{1,2} e_1 \odot e_2$$

constitute a S_2-matrix on R^4 ($\odot$ is the symmetric product).

Let $(\mathfrak{p}^*, \bullet_1, \ldots, \bullet_k)$ be a k-left symmetric algebra and $\mathbf{r} = (\mathbf{r}^1, \ldots, \mathbf{r}^k) \in \mathfrak{p}^* \otimes (\mathfrak{p}^*)^k$. We call $\mathbf{r}$ a quasi-S_k-matrix if, for any α, β and for any $\rho \in \mathfrak{p}^*, a \in (\mathfrak{p}^*)^k, p, q \in \mathfrak{p}$,

$$\mathrm{L}^\alpha_\rho(\mathbf{a}_\beta) = 0, \left[a, \Delta(\mathbf{r})(p, q)\right] + \mathrm{L}_a(\Delta(\mathbf{r}))(p, q) = 0, \quad \Delta(\mathbf{r})(p, q) = \mathbf{r}([p, q]_\mathfrak{p}) - \left[\mathbf{r}(p), \mathbf{r}(q)\right].$$

According to Theorem 2.2, $(\Phi(\mathfrak{p}, k) = (\mathfrak{p}^*)^k \oplus \mathfrak{p}, (\mathfrak{p}^*)^k, [,]^\mathbf{r}, \theta^1, \ldots, \theta^k)$ is a k-para-Kähler Lie algebra where

$$[a + p, b + q]^\mathbf{r} = \left\{[a, b] + \psi^*_p b - \psi^*_q a\right\} + \left\{\phi^*_a q - \phi^*_b p + [p, q]_\mathfrak{p}\right\}, a, b \in (\mathfrak{p}^*)^k, p, q \in \mathfrak{p}$$

and

$$\begin{cases} [a, b] = a \circ b - b \circ a, a \circ b = \displaystyle\sum_{\alpha=1}^{k} (a_\alpha \bullet_\alpha b_1, \ldots, a_\alpha \bullet_\alpha b_k), \\ \prec \rho, \phi^*_a p \succ = -\displaystyle\sum_{\alpha=1}^{k} \prec a_\alpha \bullet_\alpha \rho, p \succ, \\ \psi^*_p a = \mathbf{r}(\phi^*_a p) + \left[\mathbf{r}(p), a\right], \\ [p, q]_\mathfrak{p} = \phi^*_{\mathbf{r}(p)} q - \phi^*_{\mathbf{r}(q)} p = \displaystyle\sum_{\beta=1}^{k} \left[(\mathrm{L}^\beta_{\mathbf{r}_\beta(p)})^* q - (\mathrm{L}^\beta_{\mathbf{r}_\beta(q)})^* p\right], \\ \theta^i(a + p, b + q) = \prec a_i, q \succ - \prec b_i, p \succ. \end{cases}$$

Note also that $\Phi(\mathfrak{p}, k)$ has another Lie algebra structure, namely,

$$[a + p, b + q]^\triangleright = [a, b] + \phi^*_a q - \phi^*_b p.$$

We consider now the bracket on $\Phi(\mathfrak{p}, k)$ given by

$$[a + p, b + q]^{\triangleright, \mathbf{r}} = [a + p, b + q]^\triangleright + \Delta(\mathbf{r})(p, q).$$

It can be verified that both $[,]^\triangleright$ and $[,]^{\triangleright, \mathbf{r}}$ are Lie brackets on $\Phi(\mathfrak{p}, k)$.

Theorem 3.2. *The linear map* $K : (\Phi(\mathfrak{p},k),[,]^{\triangleright,\mathbf{r}}) \to (\Phi(\mathfrak{p},k),[,]^{\mathbf{r}}), a+p \mapsto a-\mathbf{r}(p)+p$ *is an isomorphism of Lie algebras.*

Proof. It is clear that K is bijective and that for any $a,b \in (\mathfrak{p}^*)^k, K([a,b]^{\triangleright,\mathbf{r}}) = [K(a),K(b)]^{\mathbf{r}}$. Now for any $p,q \in \mathfrak{p}$,

$$\begin{aligned} K([p,q]^{\triangleright,\mathbf{r}}) &= \Delta(\mathbf{r})(p,q), \\ \left[K(p),K(q)\right]^{\mathbf{r}} &= \left[p-\mathbf{r}(p),q-\mathbf{r}(q)\right]^{\mathbf{r}} \\ &= [p,q]_{\mathfrak{p}} - \psi_p^*\mathbf{r}(q) + \psi_q^*\mathbf{r}(p) - \phi_{\mathbf{r}(p)}^*q + \phi_{\mathbf{r}(q)}^*p + \left[\mathbf{r}(p),\mathbf{r}(q)\right] \\ &= -\mathbf{r}\left(\phi_{\mathbf{r}(q)}^*p\right) - \left[\mathbf{r}(p),\mathbf{r}(q)\right] + \mathbf{r}\left(\phi_{\mathbf{r}(p)}^*q\right) + \left[\mathbf{r}(q),\mathbf{r}(p)\right] + \left[\mathbf{r}(p),\mathbf{r}(q)\right] \\ &= \Delta(\mathbf{r})(p,q). \end{aligned}$$

On the other hand,

$$\begin{aligned} K([a,p]^{\triangleright,\mathbf{r}}) &= K(\phi_a^*p) = \phi_a^*p - \mathbf{r}(\phi_a^*p) \\ \left[K(a),K(p)\right]^{\mathbf{r}} &= \left[a,p-\mathbf{r}(p)\right]^{\mathbf{r}} \\ &= -\left[a,\mathbf{r}(p)\right] + \phi_a^*p - \psi_p^*a \\ &= -\left[a,\mathbf{r}(p)\right] + \phi_a^*p - \mathbf{r}(\phi_a^*p) - \left[\mathbf{r}(p),a\right] \\ &= \phi_a^*p - \mathbf{r}(\phi_a^*p). \end{aligned}$$

This completes the proof. □

Proposition 3.2 . *Let* $(\mathfrak{p}^*, \bullet_1, \ldots, \bullet_k)$ *be a k-left symmetric Lie algebra and* $\mathbf{r} = (\mathbf{r}_1,\ldots,\mathbf{r}_k)$ *be a quasi-*S_k*-matrix. Then* $(\Phi(\mathfrak{p},k),[,]^{\triangleright,\mathbf{r}},(\mathfrak{p}^*)^k,\theta_{\mathbf{r}}^1,\ldots,\theta_{\mathbf{r}}^k)$ *is a k-para-Kähler Lie algebra where*

$$\theta_{\mathbf{r}}^i(a+p,b+q) = \theta^i(a+p,b+q) - 2\mathbf{s}_i(p,q),$$

where $\mathbf{s}_i$ *is the symmetric part of* $\mathbf{r}_i$.

4. *k*-Symplectic Lie algebras of dimension $(k+1)$

In [3], there is a study of k-symplectic Lie algebras of dimension $(k+1)$. In this section, by using Theorem 2.2, we give a description of these Lie algebras which completes the results obtained in [3].

Let $(\mathfrak{g},\mathfrak{h},\theta^1,\ldots,\theta^k)$ be a k-symplectic Lie algebra of dimension $(k+1)$. Since $\mathfrak{h}$ has codimension 1 then $\mathfrak{g}$ is indeed a k-para-Kähler Lie algebra and according to Theorem 2.2, there exists a basis $(f_1,\ldots,f_k,e)$ of $\mathfrak{g}$ and $(a_1,\ldots,a_k) \in R^k$ such that $(f_1,\ldots,f_k)$ is a basis of $\mathfrak{h}$ and for any $i,j=1,\ldots,k$

$$[f_i,f_j] = a_if_j - a_jf_i, [e,f_i] = a_ie + D(f_i), \theta^i = f_i^* \wedge e^* \tag{22}$$

where $D : \mathfrak{h} \to \mathfrak{h}$ is a linear endomorphism. This bracket must satisfy the Jacobi identity. We will solve the obtained equations in what follows. We distinguish two cases: $k=2$ and $k \geq 3$. Note first that if we define $\ell \in \mathfrak{h}^*$ by $\ell(f_i) = a_i$ the bracket above satisfies

$$[x,y] = \ell(x)y - \ell(y)x \quad \text{and} \quad [e,x] = \ell(x)e + D(x) \tag{23}$$

for any $x,y \in \mathfrak{h}$ and one can see easily that the Jacobi identity is equivalent to

$$\ell(y)D(x) - \ell(x)D(y) + \ell(D(y))x - \ell(D(x))y = 0 \tag{24}$$

for any $x,y \in \mathfrak{h}$.

Let us start with the case $k=2$. We consider $sl(2, R)$ with its basis $\left\{h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$, where

$$[h,g] = 2g, [h,f] = -2f \quad \text{and} \quad [g,f] = h.$$

We consider the Lie algebra $\mathfrak{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$.

In the basis

$$\left\{ u_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

we have

$$[u_1, u_2] = u_2, [u_1, u_3] = -u_3 \quad \text{and} \quad [u_2, u_3] = 0.$$

Theorem 4.1. *Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ be a 2-symplectic Lie algebra of dimension 3. Then one of the following situations holds:*

1. *$\mathfrak{h}$ is an abelian ideal and there exists a basis (e, f, g) of $\mathfrak{g}$ and D an endomorphism of $\mathfrak{h}$ such that $[h, e] = D(h)$ for any $h \in \mathfrak{h}, \theta^1 = e^* \wedge f^*$ and $\theta^2 = e^* \wedge g^*$.*
2. *$(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ is isomorphic to $(sl(2, R), \mathfrak{h}_0, \rho^1, \rho^2)$ with $\mathfrak{h}_0 = \text{span}\{h, g\}, \rho^1 = h^* \wedge f^* + bg^* \wedge f^*$ and $\rho^2 = g^* \wedge f^*$.*
3. *$(\mathfrak{g}, \mathfrak{h}, \theta^1, \theta^2)$ is isomorphic to $(\mathfrak{sol}, \mathfrak{h}_0, \rho^1, \rho^2)$ with $\mathfrak{h}_0 = \text{span}\{u_1, u_2\}, \rho_1 = u_1^* \wedge u_3^* + bu_2^* \wedge u_3^*$ and $\theta^2 = cu_1^* \wedge u_3^* + u_2^* \wedge u_3^*$ for some b,c $\in R$.*

Proof. According to what was said above, $\mathfrak{g} = \mathfrak{h} \oplus Re$ and there exists a basis (f_1, f_2) of $\mathfrak{h}$, a linear endomorphism $D : \mathfrak{h} \to \mathfrak{h}$ and $\ell \in \mathfrak{h}^*$ such that

$$\ell(f_1) = a_1, \ell(f_2) = a_2, \theta^1 = f_1^* \wedge e^* \quad \text{and} \quad \theta^2 = f_2^* \wedge e^*$$

and the Lie bracket is given by (23) and ℓ and D satisfy (24).

If $\ell = 0$ then this equation is satisfied and $\mathfrak{g}$ is an almost abelian algebra.

If $\ell \neq 0$ we can suppose that $a_1 \neq 0$. Put $g_2 = a_2 f_1 - a_1 f_2 \in \ker\ell$ and the Eq. (24) is equivalent to

$$a_1 D(g_2) + \ell(D(f_1))g_2 - \ell(D(g_2))f_1 = 0.$$

Put $D(f_1) = d_{11}f_1 + d_{21}g_2$ and $D(g_2) = d_{12}f_1 + d_{22}g_2$ then the equation above is equivalent to $d_{11} = -d_{22}$. So in the basis (f_1, g_2, e), we have

$$[e, f_1] = a_1 e + d_{11}f_1 + d_{21}g_2, [e, g_2] = d_{12}f_1 - d_{11}g_2 \quad \text{and} \quad [f_1, g_2] = a_1 g_2$$

and

$$\theta^1 = f_1^* \wedge e^* + a_2 g_2^* \wedge e^* \quad \text{and} \quad \theta^2 = -a_1 g_2^* \wedge e^*.$$

We distinguish two cases:

- $d_{12} \neq 0$. If we put

$$(h, g, f) = \left(\frac{2}{a_1} f_1, g_2, -\frac{1}{a_1 d_{12}} \left(2e + \frac{2d_{11}}{a_1} f_1 + \frac{d_{12}}{a_1} g_2 \right) \right)$$

we get the desired isomorphism between $\mathfrak{g}$ and $sl(2, R)$.

- $d_{12} = 0$. If we put

$$(u_1, u_2, u_3) = \left(-\frac{1}{d_{11}}e, g_2, a_1 e + d_{11}f_1 + \frac{d_{21}}{2}g_2\right)$$

we get the desired isomorphism between $\mathfrak{g}$ and $\mathfrak{sol}$. □

Theorem 4.2. *Let $(\mathfrak{g}, \mathfrak{h}, \theta^1, \ldots, \theta^k)$ be a k-symplectic Lie algebra such that $\dim \mathfrak{h} = k \geq 3$. Then one of the following situation holds:*

1. *$\mathfrak{h}$ is an abelian ideal and there exists a basis $(e, f_1, \ldots, f_k)$ of $\mathfrak{g}$ and an endomorphism D of $\mathfrak{h}$ such that $\mathfrak{h} = \mathrm{span}\{f_1, \ldots, f_k\}, [e, h] = D(h)$ for any $h \in \mathfrak{h}$ and, for $\alpha = 1, \ldots, k, \theta^\alpha = f_\alpha^* \wedge e^*$.*
2. *There exists $(f_1, \ldots, f_k, e)$ a basis of $\mathfrak{g}$, a family of constants $(a_1, \ldots, a_k) \in R^k$, $a_1 \neq 0$, $(b_2, \ldots, b_k) \in R^{k-1}$ and $\lambda \in R$ such that $\mathfrak{h} = \mathrm{span}\{f_1, \ldots, f_k\}$,*

$$\theta^1 = f_1^* \wedge e^* - \sum_{i=2}^{k} a_i f_i^* \wedge e^* \quad \text{and} \quad \theta^i = a_1 f_i^* \wedge e^*, i = 2, \ldots, k,$$

and the non vanishing Lie brackets are given by

$$[e, f_1] = a_1 e + \lambda f_1 + \sum_{l=2}^{k} b_l f_l, [e, f_i] = -\lambda f_i, [f_1, f_i] = a_1 f_i, i = 2, \ldots, k.$$

These Lie algebras are solvable nonunimodular.

Proof. According to what was said above, $\mathfrak{g} = \mathfrak{h} \oplus Re$ and there exists a basis $(f_1, \ldots, f_k)$ of $\mathfrak{h}$, a homomorphism $D : \mathfrak{h} \to \mathfrak{h}$ and $\ell \in \mathfrak{h}^*$ such that

$$\ell(f_i) = a_i \quad \text{and} \quad \theta^i = f_i^* \wedge e^*, \quad i = 1, \ldots, k$$

and the Lie bracket is given by (23) where ℓ and D satisfy (24).

If $\ell = 0$ then this equation is satisfied and $\mathfrak{g}$ is an almost abelian algebra.

Suppose that $\ell \neq 0$ and we can suppose $a_1 \neq 0$. Then for $x, y \in \ker \ell$

$$\ell(D(y))x - \ell(D(x))y = 0$$

and hence $\ker \ell$ is invariant by D. If we take $x \in \ker \ell$ and $y \notin \ker \ell$ then

$$\ell(y)D(x) + \ell(D(y))x = 0$$

and hence

$$D(x) = -\frac{\ell(D(y))}{\ell(y)}x.$$

If we choose y_0 such that $\ell(y_0) \neq 0$, we get

$$D(y_0) = \lambda y_0 + x_0 \quad \text{and} \quad D(x) = -\lambda x, x, x_0 \in \ker \ell.$$

Put $g_1 = f_1$ and for $i = 2, \ldots, k$, we put $g_i = a_1 f_i - a_i f_1 \in \ker \ell$. Thus

$$D(g_1) = \lambda g_1 + \sum_{i=2}^{k} b_i g_i, \theta^1 = g_1^* \wedge e^* - \sum_{i=2}^{k} a_i g_i^* \wedge e^* \quad \text{and} \quad D(g_i) = -\lambda g_i, \theta^i = a_1 g_i^* \wedge e^*,$$

$$i = 2, \ldots, k.$$

and the Lie brackets are

$$[e, g_1] = a_1 e + \lambda g_1 + \sum_{l=2}^{k} b_l g_l, [e, g_i] = -\lambda g_i, [g_1, g_i] = a_1 g_i, [g_i, g_j] = 0, i, j = 2, ..., k.$$

This completes the proof. □

5. Six-dimensional 2-para-Kähler Lie algebras

In this section, by using Theorem 2.2, we give all six dimensional 2-para-Kähler Lie algebras. We proceed as follows:

1. In Table 1, we determine all 2-left symmetric algebras by a direct computation using the classification of real two-dimensional left symmetric algebras given in [10].
2. In Table 2, we give for each 2-left symmetric algebra in Table 1 its compatible 2×2-left symmetric algebras.
3. In Table 3, we give for each couple of compatible structures in Table 2 the corresponding 2-para-Kähler Lie algebra.
4. All our computations were checked by using the software Maple.

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