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Solutions of the Classical Yang–Baxter Equation and Noncommutative Deformations

MOHAMED BOUCETTA
Faculé des sciences et techniques Gueliz, BP549, Marrakech, Morocco.
e-mail: boucetta@fstg-marrakech.ac.ma

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Abstract. We show that given a finite-dimensional real Lie algebra $\mathcal{G}$ acting on a smooth manifold P then, for any solution of the classical Yang–Baxter equation on $\mathcal{G}$, there is a canonical Poisson tensor on P and an associated canonical torsion-free and flat contravariant connection. Moreover, we prove that the metacurvature of this contravariant connection vanishes if the isotropy Lie subalgebras of the action are trivial. Those results permit to get a large class of smooth manifolds satisfying the necessary conditions, introduced by Eli Hawkins, to the existence of noncommutative deformations.

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1. Introduction and Main Results

In [7] and [8], Hawkins showed that a deformation of the graded algebra of differential forms $\Omega^*(P)$ on a smooth manifold P gives arise to a Poisson tensor on P (which characterizes the deformation) and to a torsion-free and flat contravariant connection whose metacurvature vanishes. Moreover, Hawkins showed that, on a Riemannian manifold P, if the deformation of $\Omega^*(P)$ comes from a deformation of a spectral triple describing the Riemannian manifold P then the Poisson tensor π (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

1. The associated metric contravariant connection $\mathcal{D}$ is flat.
2. The metacurvature of $\mathcal{D}$ vanishes.
3. The Poisson tensor π is compatible with the Riemannian volume ϵ:

$$d(i_\pi \epsilon) = 0.$$

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The metric contravariant connection associated naturally to any couple of pseudo-Riemannian metric and Poisson tensor is an analogue of the Levi-Civita connection. It has appeared first in [3]. The metacurvature, introduced by Hawkins in [8], is a (2,3)-tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any torsion-free and flat contravariant connection.

Throughout this paper, on a smooth manifold, a pseudo-Riemannian metric and a Poisson tensor satisfying the conditions 1–3 are called compatible in the sense of Hawkins.

In this paper, we prove two results which provide a method for constructing a large class of smooth manifolds satisfying the necessary conditions, pointed out by Hawkins in [7] and [8], to the existence of noncommutative deformations of the differential graded algebra of differential forms. Let us state those results and describe this method.

Let $\mathcal{G} \xrightarrow{\Gamma} \mathcal{X}(P)$ be an action of a finite-dimensional real Lie algebra $\mathcal{G}$ on a smooth manifold P, i.e., a morphism of Lie algebras from $\mathcal{G}$ to the Lie algebra of vector fields on P.

Let $r \in \wedge^2 \mathcal{G}$ be a solution of the classical Yang–Baxter equation, i.e.,

$$[r, r] = 0, \tag{1}$$

where $[r, r] \in \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$ is defined by

$$[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)]),$$

and $r : \mathcal{G}^* \longrightarrow \mathcal{G}$ denotes also the linear map given by $\alpha(r(\beta)) = r(\alpha, \beta)$. We denote by π_r the Poisson tensor on P image of r by Γ. Write

$$r = \sum_{i,j} a_{ij} u_i \wedge u_j$$

and put, for $\alpha, \beta \in \Omega^1(P)$,

$$\mathcal{D}^r_\alpha \beta := \sum_{i,j} a_{ij} \alpha(U_i) L_{U_j} \beta, \tag{2}$$

where $U_i = \Gamma(u_i)$. We get a map $\mathcal{D}^r : \Omega^1(P) \times \Omega^1(P) \longrightarrow \Omega^1(P)$ which is a contravariant connection associated to the Poisson tensor π_r.

THEOREM 1.1. *Let $\mathcal{G} \xrightarrow{\Gamma} \mathcal{X}(P)$ be an action of a Lie algebra $\mathcal{G}$ on a smooth manifold P and let $r \in \wedge^2 \mathcal{G}$ be a solution of the classical Yang–Baxter equation such that* Imr *is an unimodular Lie algebra. Then, for any volume form ϵ on P such that $L_{\Gamma(u)}\epsilon = 0$ for each $u \in$* Imr*, π_r is compatible with ϵ, i.e.,*

$$d(i_{\pi_r} \epsilon) = 0.$$

THEOREM 1.2. *Let $\mathcal{G} \xrightarrow{\Gamma} \mathcal{X}(P)$ be an action of a Lie algebra $\mathcal{G}$ on a smooth manifold P and let $r \in \wedge^2\mathcal{G}$ be a solution of the classical Yang–Baxter equation.*

1. *The map $\mathcal{D}^r : \Omega^1(P) \times \Omega^1(P) \longrightarrow \Omega^1(P)$ given by* (2) *depends only on r and Γ and defines a torsion-free and flat contravariant connection associated to the Poisson tensor π_r.*
2. *If P is a pseudo-Riemannian manifold and, for any $u \in$* Imr, *$\Gamma(u)$ is a Killing vector field, then $\mathcal{D}^r$ is the metric contravariant connection associated to the metric and π_r.*
3. *If the isotropy subalgebras of the restriction of Γ to* Imr *are trivial, then the metacurvature of $\mathcal{D}^r$ vanishes.*

There are some interesting implications of Theorems 1.1 and 1.2:

1. Let G be a Lie group, $\mathcal{G}$ its Lie algebra and let $\mathcal{G} \xrightarrow{\Gamma} \mathcal{X}(G)$ be the action of $\mathcal{G}$ on G by left invariant vector fields. For any $\alpha \in \mathcal{G}^*$, let α^l denote the left invariant differential 1-form on G associated to α. For any solution r of the classical Yang–Baxter equation, one can check easily that $\mathcal{D}^r$ is given by
$$\mathcal{D}^r_{\alpha^l}\beta^l = -(ad^*_{r(\alpha)}\beta)^l.$$
Moreover, if Imr is unimodular then, for any right invariant pseudo-Riemannian metric g on G, the left invariant Poisson tensor π_r is compatible with g in the sense of Hawkins.
2. Let (G, ω) be an unimodular Lie group endowed with a left invariant symplectic form. The symplectic form defines on the Lie algebra of G an invertible solution of the classical Yang–Baxter equation. Hence, given any locally free action by isometries of G on a pseudo-Riemannian manifold P, there exists on P a Poisson tensor which is compatible with the metric in the sense of Hawkins.
3. A symplectic nilmanifold is the quotient of a nilpotent Lie group endowed with a left invariant symplectic form by a discrete co-compact subgroup (see [1]), so on any symplectic nilmanifold there exists a Poisson tensor and an associated torsion-free and flat contravariant connection whose metacurvature vanishes. Nilpotent Lie groups admitting left invariant symplectic forms were classified by Medina and Revoy [10].
4. The affine group $K^n \times GL(K^n)$ ($K = \mathbb{R}$ or $\mathbb{C}$) admits many left invariant symplectic forms (see [2]) which implies that its Lie algebra carries many invertible solutions of (1). So any locally free action of the affine group on a manifold gives arise to a Poisson tensor with an associated nontrivial torsion-free and flat contravariant connection whose metacurvature vanishes.
5. Let G be a Lie group with a bi-invariant pseudo-Riemannian metric and r an unimodular solution of the classical Yang–Baxter equation. For any discrete co-compact subgroup Λ of G, G acts on the compact manifold $P := G/\Lambda$

by isometries, so we get a Poisson tensor on the compact pseudo-Riemannian manifold P compatible with the pseudo-Riemannian metric in the sense of Hawkins.

A connected Lie group G admits a bi-invariant Riemannian metric if and only if it is isomorphic to the cartesian product of a compact group and a commutative group (see [11]). Any solution r of (1) on the Lie algebra of G is such that Imr is abelian. So any triple (G, Λ, S), where G is a Lie group with a bi-invariant Riemannian metric, Λ is a discrete and co-compact subgroup of G and S an abelian even dimensional subalgebra of the Lie algebra of G, gives arise to a compact Riemannian manifold with a Poisson tensor compatible with the metric in the sense of Hawkins.

We give now an example of a compact Lorentzian manifold with a Poisson tensor compatible with the Lorentzian metric in the sense of Hawkins and such that the Poisson tensor cannot be constructed locally by commuting Killing vector fields. This shows that Theorem 6.6 in [8] is not true if the metric is Lorentzian.

Our example involves the oscillator groups introduced by Medina and Revoy (see [9]) as the only noncommutative simply connected solvable Lie groups which have a bi-invariant Lorentzian metric. Their discrete co-compact subgroups where classified in [9].

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $0 < \lambda_1 \leq \lambda_2$, the oscillator group of dimension 6 is the Lie group G_λ with underlying manifold $\mathbb{R} \times \mathbb{R} \times \mathbb{C}^2$ and product

$$\begin{aligned}
&(t, s, z_1, z_2) \times (t', s', z_1', z_2') = \\
&= \Bigg(t + t', s + s' + \frac{1}{2}\Bigg(\mathrm{Im}(\bar{z}_1 \exp(\imath t \lambda_1) z_1') + \\
&\qquad + \mathrm{Im}(\bar{z}_2 \exp(\imath t \lambda_2) z_2')\Bigg), z_1 + \exp(\imath t \lambda_1) z_1', z_2 + \exp(\imath t \lambda_2) z_2'\Bigg).
\end{aligned}$$

The associated Lie algebra is

$$\mathcal{G}_\lambda := \mathrm{vect}\{e_{-1}, e_0, e_1, e_2, \check{e}_1, \check{e}_2\}$$

with brackets

$$[e_{-1}, e_j] = \lambda_j \check{e}_j, \quad [e_j, \check{e}_j] = e_0, \quad [e_{-1}, \check{e}_j] = -\lambda_j e_j$$

and the unspecified brackets are either zero or given by antisymmetry.

For $x \in \mathcal{G}_\lambda$, let

$$x = x_{-1} e_{-1} + x_0 e_0 + x_1 e_1 + x_2 e_2 + \check{x}_1 \check{e}_1 + \check{x}_2 \check{e}_2.$$

The nondegenerate symmetric form

$$\mathbf{k}_\lambda(x, x) := 2x_{-1} x_0 + \sum_{j=1}^{2} \frac{1}{\lambda_j}(x_j^2 + \check{x}_j^2)$$

satisfies

$$\mathbf{k}_\lambda([x,y],z)+\mathbf{k}_\lambda(y,[x,z])=0, \quad \text{for any } x,y,z\in\mathcal{G}_\lambda$$

and hence defines a Lorentzian bi-invariant metric on G_λ.
Consider now $r\in\wedge^2\mathcal{G}_\lambda$ given by

$$r=e_0\wedge e_1+e_2\wedge\check{e}_1.$$

It is easy the check that, for any $\alpha,\beta\in\mathcal{G}_\lambda^*$,

$$[r(\alpha),r(\beta)]=(\alpha(e_0)\beta(e_2)-\alpha(e_2)\beta(e_0))\,e_0,$$

and deduce that $[r,r]=0$. Note that Imr is a nilpotent Lie algebra and hence unimodular.
Let π_r denote the left invariant Poisson tensor on G_λ associated to r. The contravariant connection $\mathcal{D}^r$ is given by

$$\mathcal{D}^r_{\alpha^l}\beta^l=-(ad^*_{r(\alpha)}\beta)^l,$$

where $\alpha,\beta\in\mathcal{G}_\lambda^*$ and α^l is the left invariant 1-form on G_λ associated to α.
A direct calculation gives

$$\begin{aligned}ad^*_{r(\alpha)}\beta=&\big(\lambda_1\alpha(e_2)\beta(e_1)-\lambda_1\alpha(e_0)\beta(\check{e}_1)+\lambda_2\alpha(\check{e}_1)\beta(\check{e}_2)\big)\,e^*_{-1}-\\&-\alpha(e_2)\beta(e_0)e_1^*+\alpha(e_0)\beta(e_0)\check{e}_1^*-\alpha(\check{e}_1)\beta(e_0)\check{e}_2^*.\end{aligned}$$

Now, for any $\alpha,\beta,\gamma\in\mathcal{G}_\lambda^*$, we have

$$\begin{aligned}\mathcal{D}^r_{\alpha^l}\pi_r(\beta^l,\gamma^l)&=\pi_{r\#}.\pi_r(\alpha^l,\beta^l)-\pi_r(\mathcal{D}^r_{\alpha^l}\beta^l,\gamma^l)-\pi_r(\beta^l,\mathcal{D}^r_{\alpha^l}\gamma^l)=\\&=r(ad^*_{r(\alpha)}\beta,\gamma)+r(\beta,ad^*_{r(\alpha)}\gamma)=\\&=\alpha(e_0)\,(\beta(e_2)\gamma(e_0)-\beta(e_0)\gamma(e_2)),\end{aligned}$$

and hence $\mathcal{D}^r\pi_r\neq 0$.
The subgroup

$$\Lambda:=\{(t,s,z_1,z_2)\in G_\lambda\,/\,t,s,Re(z_i),\mathrm{Im}(z_i)\in\mathbb{Z}\}$$

is a discrete co-compact subgroups of G_λ if and only if the set $\{\lambda_1,\lambda_2\}$ generates a discrete subgroup of $(\mathbb{R},+)$ (see [9]). The bi-invariant Lorentzian metric on G_λ defines a Lorentzian metric on $P:=G_\lambda/\Lambda$ and we get an action of G_λ on P by isometries. According to Theorems 1.1 and 1.2, we get a Poisson tensor on P which is compatible with the Lorentzian metric. According to the calculation above, this Poisson tensor is not parallel with respect to the metric contravariant connection and hence it cannot be constructed locally from commuting Killing vectors fields.

Section 2 is devoted to a complete proof of Theorems 1.1 and 1.2.

Notations. For a smooth manifold P, $C^\infty(P)$ will denote the space of smooth functions on P, $\Gamma(P,V)$ will denote the space of smooth sections of a vector bundle, $\Omega^p(P):=\Gamma(P,\wedge^p T^*P)$ and $\mathcal{X}^p(P):=\Gamma(P,\wedge^p TP)$. Lower case Greek characters α,β,γ will mostly denote 1-forms. However, π will denote a Poison bivector field and ω will denote a symplectic form.

For a smooth manifold P with a Poisson tensor π, $\pi_\#:T^*P\longrightarrow TP$ will denote the anchor map given by $\beta(\pi_\#(\alpha))=\pi(\alpha,\beta)$, and $[\,,\,]_\pi$ will denote the Koszul bracket given by

$$[\alpha,\beta]_\pi=L_{\pi_\#(\alpha)}\beta-L_{\pi_\#(\beta)}\alpha-d(\pi(\alpha,\beta)).$$

We will denote by P^{reg} the dense open set where the rank of π is locally constant.

2. Proofs of Theorems 1.1 and 1.2

2.1. PRELIMINARIES

Contravariant connections associated to a Poisson structure have recently turned out to be useful in several areas of Poisson geometry. Contravariant connections were defined by Vaismann [13] and were analyzed in detail by Fernandes [5]. This notion appears extensively in the context of noncommutative deformations see [7,8] and [12]. One can consult [5] for a detailed study of contravariant connections.

In this subsection, we recall the definition of the metacurvature of a flat and torsion-free contravariant connection $\mathcal{D}$, and we give a necessary and sufficient condition to the vanishing of the metacurvature in the case where $\mathcal{D}$ is an $\mathcal{F}$-connection (see [5]). Finally, we recall the interpretation of a solution of the Yang–Baxter equation as a symplectic Lie algebra. This interpretation appeared first in [4].

Let (P,π) be a Poisson manifold and $V\xrightarrow{p}P$ a vector bundle over P. A contravariant connection on V with respect to π is a map $\mathcal{D}:\Omega^1(P)\times\Gamma(P,V)\longrightarrow\Gamma(P,V)$, $(\alpha,s)\mapsto\mathcal{D}_\alpha s$ satisfying the following properties:

1. $\mathcal{D}_\alpha s$ is linear over $C^\infty(P)$ in α:

$$\mathcal{D}_{f\alpha+h\beta}s=f\mathcal{D}_\alpha s+h\mathcal{D}_\beta s,\quad f,g\in C^\infty(P);$$

2. $\mathcal{D}_\alpha s$ is linear over $\mathbb{R}$ in s:

$$\mathcal{D}_\alpha(as_1+bs_2)=a\mathcal{D}_\alpha s_1+b\mathcal{D}_\alpha s_2,\quad a,b\in\mathbb{R};$$

3. $\mathcal{D}$ satisfies the following product rule:

$$\mathcal{D}_\alpha(fs)=f\mathcal{D}_\alpha s+\pi_\#(\alpha)(f)s,\quad f\in C^\infty(P).$$

The curvature of a contravariant connection $\mathcal{D}$ is formally identical to the usual definition

$$K(\alpha,\beta)=\mathcal{D}_\alpha\mathcal{D}_\beta-\mathcal{D}_\beta\mathcal{D}_\alpha-\mathcal{D}_{[\alpha,\beta]_\pi}.$$

The connection $\mathcal{D}$ is called flat if K vanishes identically.

A contravariant connection $\mathcal{D}$ is called an $\mathcal{F}$-connection if its satisfies the following properties

$$\pi_\#(\alpha)=0 \qquad \Rightarrow \qquad \mathcal{D}_\alpha=0.$$

We call $\mathcal{D}$ un $\mathcal{F}^{\mathrm{reg}}$-connection if the restriction of $\mathcal{D}$ to P^{reg} is an $\mathcal{F}$-connection.

If $V=T^*P$, one can define the torsion T of $\mathcal{D}$ by

$$T(\alpha,\beta)=\mathcal{D}_\alpha\beta-\mathcal{D}_\beta\alpha-[\alpha,\beta]_\pi.$$

Let us define now an interesting class of contravariant connections, namely contravariant connection associated naturally to a Poisson tensor and a pseudo-Riemannian metric.

Let $(P,<,>)$ be a pseudo-Riemannian manifold and π a Poisson tensor on P.

The metric contravariant connection associated to $(\pi,<,>)$ is the unique contravariant connection $\mathcal{D}$ such that:

1. the metric $<,>$ is parallel with respect to $\mathcal{D}$, i.e.,

$$\pi_\#(\alpha).<\beta,\gamma>=<\mathcal{D}_\alpha\beta,\gamma>+<\beta,\mathcal{D}_\alpha\gamma>;$$

2. $\mathcal{D}$ is torsion-free.

The connection $\mathcal{D}$ is the contravariant analogue of the Levi-Civita connection, so one can define it by the Koszul formula:

$$\begin{aligned}2<\mathcal{D}_\alpha\beta,\gamma>=\pi_\#(\alpha).<\beta,\gamma>+\pi_\#(\beta).<\alpha,\gamma>-\pi_\#(\gamma).<\alpha,\beta>+\\ +<[\gamma,\alpha]_\pi,\beta>+<[\gamma,\beta]_\pi,\alpha>+<[\alpha,\beta]_\pi,\gamma>.\end{aligned} \tag{3}$$

We continue recalling briefly the definition of the metacurvature introduced by Hawkins in [8].

Let (P,π) be a Poisson manifold and $\mathcal{D}$ a torsion-free and flat contravariant connection with respect to π. In [8], Hawkins showed that such a connection defines a bracket $\{,\}$ on the space of differential forms $\Omega^*(P)$ such that:

1. $\{,\}$ is $\mathbb{R}$-bilinear, degree 0 and antisymmetric, i.e.,

$$\{\sigma,\rho\}=-(-1)^{\deg\sigma\deg\rho}\{\rho,\sigma\}.$$

2. The differential d is a derivation with respect to $\{,\}$, i.e.,

$$d\{\sigma,\rho\}=\{d\sigma,\rho\}+(-1)^{\deg\sigma}\{\sigma,d\rho\}.$$

3. $\{,\}$ satisfies the product rule

$$\{\sigma,\rho\wedge\lambda\}=\{\sigma,\rho\}\wedge\lambda+(-1)^{\deg\sigma\deg\rho}\rho\wedge\{\sigma,\lambda\}.$$

4. For any $f, h \in C^{\infty}(P)$ and for any $\sigma \in \Omega^*(P)$ the bracket $\{f, g\}$ coincides with the initial Poisson bracket and

$$\{f, \sigma\} = \mathcal{D}_{df}\sigma.$$

Hawkins called this bracket a generalized Poisson bracket and showed that there exists a (2,3)-tensor $\mathcal{M}$ such that the following assertions are equivalent:

1. The generalized Poisson bracket satisfies the graded Jacobi identity

$$\{\{\sigma, \rho\}, \lambda\} = \{\sigma, \{\rho, \lambda\}\} - (-1)^{\deg\sigma\deg\rho}\{\rho, \{\sigma, \lambda\}\}.$$

2. The tensor $\mathcal{M}$ vanishes identically.

$\mathcal{M}$ is called the metacurvature and is given by

$$\mathcal{M}(df, \alpha, \beta) = \{f, \{\alpha, \beta\}\} - \{\{f, \alpha\}, \beta\} - \{\{f, \beta\}, \alpha\}. \tag{4}$$

It would be helpful if one can wrote down a full global formula for Hawkin's generalized Poisson bracket of two 1-forms. Let α and β be two 1-forms on a Poisson manifold P endowed with a torsion-free and flat contravariant connection $\mathcal{D}$. Suppose that $\beta = gdf$ where $f, g \in C^{\infty}(P)$. Then, we have

$$\begin{aligned}
\{\alpha, fdg\} &= \{\alpha, f\} \wedge dg + f\{\alpha, dg\} \\
&= -\mathcal{D}_{df}\alpha \wedge dg + f\left(d\mathcal{D}_{dg}\alpha - \mathcal{D}_{dg}d\alpha\right) \\
&= -\mathcal{D}_{fdg}d\alpha + d\mathcal{D}_{fdg}\alpha - \mathcal{D}_{df}\alpha \wedge dg - df \wedge \mathcal{D}_{dg}\alpha \\
&= -\mathcal{D}_{fdg}d\alpha + d\mathcal{D}_{fdg}\alpha - \mathcal{D}_{\alpha}(df \wedge dg) - [df, \alpha]_{\pi} \wedge dg - df \wedge [dg, \alpha]_{\pi} \\
&= -\mathcal{D}_{fdg}d\alpha - \mathcal{D}_{\alpha}(d(fdg)) + d\mathcal{D}_{fdg}\alpha - [df, \alpha]_{\pi} \wedge dg - df \wedge [dg, \alpha]_{\pi}.
\end{aligned}$$

This computation shows that a global formula, if it exists, will involve both the contravariant connection and the Poisson tensor. However, Hawkins pointed out in [8, p. 9], that for any parallel 1-form α and any 1-form β, the generalized Poisson bracket of α and β is given by

$$\{\alpha, \beta\} = -\mathcal{D}_{\beta}d\alpha. \tag{5}$$

Thus, one can deduce from (4) that for any parallel 1-forms α, γ and for any 1-form β,

$$\mathcal{M}(\alpha, \beta, \gamma) = -\mathcal{D}_{\beta}\mathcal{D}_{\gamma}d\alpha. \tag{6}$$

The definition of a contravariant connection is similar to the definition of an ordinary (covariant) connection, except that cotangent vectors have taken the place of tangent vectors. So one can translate many definitions, identities and proofs for covariant connections to contravariant connections simply by exchanging the roles of tangent and cotangent vectors and replacing Lie Bracket with Koszul bracket. Nevertheless, there are some differences between those two notions. Fernandes pointed out in [5] that the equation $D\alpha = 0$ cannot be solved locally for a general torsion-free and flat contravariant connection D. However, he showed [5, Proposition 1.9.1] that for a torsion-free and flat $\mathcal{F}$-connection this equation can be solved locally. Let us precise this result.

PROPOSITION 2.1. *Let (P, π) be a Poisson manifold and $\mathcal{D}$ a torsion-free and flat contravariant connection with respect π. Let p be a regular point of π such that the restriction of $\mathcal{D}$ to a neighborhood of p is an $\mathcal{F}$-connection. Then, for any $\beta \in T_p^*P$, there exists a 1-form $\widetilde{\beta}$ over some neighborhood of p such that $\mathcal{D}\widetilde{\beta} = 0$ and $\widetilde{\beta}_p = \beta$.*

By combining (6) and Proposition 2.1, we get the following useful proposition.

PROPOSITION 2.2. *Let (P, π) be a Poisson manifold and $\mathcal{D}$ a torsion-free and flat $\mathcal{F}^{\mathrm{reg}}$-connection with respect to π. Then the metacurvature of $\mathcal{D}$ vanishes if and only if, for any local parallel 1-form on P^{reg}, $D^2 d\alpha = 0$.*

Finally, we give another description of the solutions of the classical Yang–Baxter equation which will be useful latter.

Let $\mathcal{G}$ be a Lie algebra. Note that the data of $r \in \mathcal{G} \wedge \mathcal{G}$ is equivalent to the data of a vectorial subspace $S_r \subset \mathcal{G}$ and a nondegenerate 2-form $\omega_r \in \wedge^2 S_r^*$.

Indeed, for $r \in \mathcal{G} \wedge \mathcal{G}$, we put $S_r = \mathrm{Imr}$ and

$$\omega_r(u, v) = r(r^{-1}(u), r^{-1}(v)), \tag{7}$$

where $u, v \in S_r$ and $r^{-1}(u)$ is any antecedent of u by r.

Conversely, let (S, ω) be a vectorial subspace of $\mathcal{G}$ with a nondegenerate 2-form. The 2-form ω defines an isomorphism $\omega^b : S \longrightarrow S^*$ by $\omega^b(u) = \omega(u, .)$, we denote by $\# : S^* \longrightarrow S$ its inverse and we put

$$r = \# \circ i^*$$

where $i^* : \mathcal{G}^* \longrightarrow S^*$ is the dual of the inclusion $i : S \hookrightarrow \mathcal{G}$.

With this observation in mind, the following well-known result (see [4]) gives another description of the solutions of the classical Yang–Baxter equation.

PROPOSITION 2.3. *Let $\mathcal{G}$ be a Lie algebra and $r \in \mathcal{G} \wedge \mathcal{G}$. The following assertions are equivalent:*

1. *r is a solution of the classical Yang–Baxter equation.*
2. *Imr is a subalgebra of $\mathcal{G}$ and*

$$\omega_r(u, [v, w]) + \omega_r(v, [w, u]) + \omega_r(w, [u, v]) = 0 \tag{8}$$

for any $u, v, w \in \mathrm{Imr}$.

2.2. PROOF OF THEOREM 1.1

Let P be a smooth manifold, $\mathcal{G}$ a Lie algebra with $\Gamma:\mathcal{G}\longrightarrow\mathcal{X}(P)$ a Lie algebras morphism from $\mathcal{G}$ to the Lie algebra of vector fields and $r\in\wedge^2\mathcal{G}$ a solution of the classical Yang–Baxter equation such that Imr is an unimodular Lie algebra.

There exists a basis $(e_1,\ldots,e_n,f_1,\ldots,f_n)$ of Imr such that the symplectic form ω_r, defined by (7), is given by

$$\omega_r=\sum_{i=1}^{n}e_i^*\wedge f_i^*.$$

Since Imr is unimodular, then for any $z\in Imr$, the trace of ad_z is zero. This is equivalent to

$$\sum_{i=1}^{n}(\omega_r([z,e_i],f_i)+\omega_r(e_i,[z,f_i]))=0.$$

According to (8), this relation is equivalent to

$$\sum_{i=1}^{n}\omega_r(z,[e_i,f_i])=0$$

and hence to

$$\sum_{i=1}^{n}[e_i,f_i]=0.$$

Let ϵ be a volume form on P such that $L_{\Gamma(e_i)}\epsilon=L_{\Gamma(f_i)}\epsilon=0$ for $i=1,\ldots,n$. We have

$$\begin{aligned}
d(i_{\pi_r}\epsilon)&=d\left(\sum_{i=1}^{n}i_{\Gamma(e_i)\wedge\Gamma(f_i)}\epsilon\right)=\\
&=\sum_{i=1}^{n}\left(i_{[\Gamma(e_i),\Gamma(f_i)]}\epsilon-i_{\Gamma(e_i)}L_{\Gamma(f_i)}\epsilon-i_{\Gamma(f_i)}L_{\Gamma(e_i)}\epsilon\right)=\\
&=i_{\Gamma(\sum_{i=1}^{n}[e_i,f_i])}\epsilon=0.
\end{aligned}$$

This gives a proof of Theorem 1.1. □

2.3. PROOF OF THEOREM 1.2

Let P be a smooth manifold, $\mathcal{G}$ a Lie algebra with $\Gamma:\mathcal{G}\longrightarrow\mathcal{X}(P)$ a Lie algebras morphism from $\mathcal{G}$ to the Lie algebra of vector fields and $r\in\wedge^2\mathcal{G}$ a solution of the classical Yang–Baxter equation.

Choose a basis $(u_1,\ldots,u_n)$ of $\mathcal{G}$ and write $r=\sum_{i,j}a_{ij}u_i\wedge u_j$. For $\alpha,\beta\in\Omega^1(P)$, we put

$$\mathcal{D}^r_\alpha\beta=\sum_{i,j}a_{ij}\alpha(U_i)L_{U_j}\beta,$$

where $U_i = \Gamma(u_i)$. One can check easily that this formula defines a contravariant connection with respect to π_r which depends only on r and Γ and does not depend on the basis $(u_1, \dots, u_n)$. One can also check that $\mathcal{D}^r$ is torsion-free. If $\Gamma(u)$ is a Killing vector field, for any $u \in \mathrm{Im} r$, then $\mathcal{D}^r$ is the metric contravariant connection associated to the metric and the Poisson tensor π_r.

Let us compute the curvature of $\mathcal{D}^r$ and show that it vanishes identically.

There exists a basis $(u_1, \dots, u_p, v_1, \dots, v_p)$ of $\mathrm{Im} r$ such that $r = \sum_{i=1}^{p} u_i \wedge v_i$. Put $U_i = \Gamma(u_i)$ and $V_i = \Gamma(v_i)$ for $i = 1, \dots, p$. We have

$$\mathcal{D}^r_\alpha \beta = L_{\pi\#(\alpha)}\beta + \sum_{i=1}^{p} A^i(\alpha, \beta)$$

where $A^i(\alpha, \beta) = \beta(U_i) d(\alpha(V_i)) - \beta(V_i) d(\alpha(U_i))$ and $\pi\#$ is the anchor map associated to π_r. With this in mind, we get for any $f, g, h \in C^\infty(P)$,

$$\begin{aligned} K(df, dg, dh) = & \sum_{i=1}^{p} \Big(A^i(df, d\{g,h\}) - A^i(dg, d\{f,h\}) + \\ & + L_{\pi\#(df)} A^i(dg, dh) - L_{\pi\#(dg)} A^i(df, dh) - A^i(d\{f,g\}, dh) \Big) + \\ & + \sum_{i,j=1}^{p} \Big(A^j(df, A^i(dg, dh)) - A^j(dg, A^i(df, dh)). \Big) \end{aligned}$$

A straightforward computation gives

$$\begin{aligned} K(df, dg, dh) = & \sum_{i,j=1}^{p} \big\{ \big(U_j(g)[U_i, V_j](h) - U_j(h)[U_i, V_j](g) + V_j(g)[U_j, U_i](h) - \\ & - V_j(h)\, [U_j, U_i](g) \big)\, d(V_i(f)) + \big(U_j(g)[V_j, V_i](h) - \\ & - U_j(h)[V_j, V_i](g) + V_j(g)[V_i, U_j](h) - V_j(h)[V_i, U_j](g) \big) \times \\ & \times d(U_i(f)) - \big(U_j(f)[U_i, V_j](h) - U_j(h)[U_i, V_j](f) + \\ & + V_j(f)[U_j, U_i](h) - V_j(h)[U_j, U_i](f) \big)\, d(V_i(g)) - \\ & - \big(U_j(f)[V_j, V_i](h) - U_j(h)[V_j, V_i](f) + V_j(f)[V_i, U_j](h) - \\ & - V_j(h)[V_i, U_j](f) \big)\, d(U_i(g)) + \\ & + U_i(h) V_j(g) d\big([U_j, V_i](f)\big) - U_i(h) V_j(f) d\big([U_j, V_i](g)\big) + \\ & + U_i(h) U_j(g) d\big([V_i, V_j](f)\big) - U_i(h) U_j(f) d\big([V_i, V_j](g)\big) + \\ & + V_i(h) U_j(g) d\big([V_j, U_i](f)\big) - V_i(h) U_j(f) d\big([V_j, U_i](g)\big) + \\ & + V_i(h) V_j(g) d\big([U_i, U_j](f)\big) - V_i(h) V_j(f) d\big([U_i, U_j](g)\big) \big\}. \end{aligned}$$

The vanishing of K is a consequence of the equation $[r, r] = 0$ which is equivalent to (8). Note that $\omega_r = \sum_{i=1}^{p} u_i^* \wedge v_i^*$ where $(u_1^*, \dots, u_p^*, v_1^*, \dots, v_p^*)$ is the dual basis of $(u_1, \dots, u_p, v_1, \dots, v_p)$.

Put

$$[u_i, u_j] = \sum_{k=1}^{p} \left(C_{u_i u_j}^{u_k} u_k + C_{u_i u_j}^{v_k} v_k \right)$$

and so on. One can see easily that (8) is equivalent to

$$\begin{cases} C_{v_j v_k}^{u_i} + C_{v_k v_i}^{u_j} + C_{v_i v_j}^{u_k} & = 0, \\ C_{u_j u_k}^{v_i} + C_{u_k u_i}^{v_j} + C_{u_i u_j}^{v_k} & = 0, \\ C_{u_j u_k}^{u_i} - C_{u_k v_i}^{v_j} - C_{v_i u_j}^{v_k} & = 0, \\ C_{v_i v_j}^{v_k} - C_{u_k v_i}^{u_j} - C_{v_j u_k}^{u_i} & = 0. \end{cases} \quad \forall i, j, k.$$

For $i = 1, \ldots, p$, the coefficients of $d(V_i(f))$ and $d(U_i(f))$ in the expression of $K(df, dg, dh)$ are, respectively,

$$\sum_{j,k=1}^{p} \left(C_{u_i v_j}^{u_k} - C_{u_i v_k}^{u_j} + C_{v_k v_j}^{v_i} \right) U_j(g) U_k(h) + \left(C_{u_i v_j}^{v_k} - C_{u_k u_i}^{u_j} + C_{v_j u_k}^{v_i} \right) U_j(g) V_k(h) +$$
$$+ \left(-C_{u_i v_j}^{v_k} + C_{u_k u_i}^{u_j} + C_{u_k v_j}^{v_i} \right) U_j(h) V_k(g) + \left(C_{u_j u_i}^{v_k} - C_{u_k u_i}^{v_j} + C_{u_k u_j}^{v_i} \right) V_j(g) V_k(h),$$
$$\sum_{j,k=1}^{p} \left(C_{v_j v_i}^{u_k} - C_{v_k v_i}^{u_j} + C_{v_k v_j}^{u_i} \right) U_j(g) U_k(h) + \left(C_{v_j v_i}^{v_k} - C_{v_i u_k}^{u_j} + C_{v_j u_k}^{u_i} \right) U_j(g) V_k(h) +$$
$$+ \left(-C_{v_j v_i}^{v_k} + C_{v_i u_k}^{u_j} + C_{u_k v_j}^{u_i} \right) U_j(h) V_k(g) + \left(C_{v_i u_j}^{v_k} - C_{v_i u_k}^{v_j} + C_{u_k u_j}^{u_i} \right) V_j(g) V_k(h).$$

Those coefficients vanish according to the relations above. Similarly, the coefficients of $d(V_i(g))$ and $d(U_i(g)$ vanish also. This shows that K vanishes identically.

Suppose now that the isotropy subalgebras of the restriction of Γ to Imr are trivial. This implies obviously that $\mathcal{D}^r$ is an $\mathcal{F}^{\text{reg}}$-connection and a 1-form β is parallel with respect $\mathcal{D}^r$ if and only if $L_{\Gamma(u)}\beta = 0$ for all $u \in$ Imr. For any parallel 1-form β, we have $L_{\Gamma(u)} d\beta = 0$ and hence $\mathcal{D}^r d\beta = 0$. According to Proposition 2.2, this implies that the metacurvature of $\mathcal{D}^r$ vanishes identically, which achieves the proof of Theorem 1.2. □

Remark 1. In Theorem 1.2, the assumption on the isotropy subalgebras cannot be dropped. For instance, consider the two-dimensional Lie algebra $\mathcal{G} = \text{Vect}\{e_1, e_2\}$ with $[e_1, e_2] = e_1$ and the action $\Gamma : \mathcal{G} \longrightarrow \mathcal{X}(\mathbb{R}^2)$ given by

$$\Gamma(e_1) = \frac{\partial}{\partial x} \quad \text{and} \quad \Gamma(e_2) = x \frac{\partial}{\partial x}.$$

For $r = e_1 \wedge e_2$, π_r is the trivial Poisson tensor and

$$\mathcal{D}_\alpha^r \beta = \alpha \left(\frac{\partial}{\partial x} \right) \beta \left(\frac{\partial}{\partial x} \right) dx.$$

A differential 1-form γ is parallel with respect $\mathcal{D}^r$ if and only if $\gamma = f(x, y)dy$. In this case, for any α and β, we have

$$\mathcal{D}^r_\alpha \mathcal{D}^r_\beta d\gamma = \alpha\left(\frac{\partial}{\partial x}\right)\beta\left(\frac{\partial}{\partial x}\right)\frac{\partial f}{\partial x}dx \wedge dy.$$

This shows that the metacurvature does not vanish.

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