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ORIGINAL ARTICLE

Riemannian geometry of Lie algebroids

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Abstract We introduce Riemannian Lie algebroids as a generalization of Riemannian manifolds and we show that most of the classical tools and results known in Riemannian geometry can be stated in this setting. We give also some new results on the integrability of Riemannian Lie algebroids.

1. Introduction

Lie groupoids and Lie algebroids are now a central notion in differential geometry and constitute an active domain of research. They have many applications in various part of mathematics (see for instance [4–6,14]). Roughly speaking, a Lie algebroid is a structure where one replaces the tangent bundle with a new vector bundle with similar properties. In this spirit, many geometrical notions which involves the tangent bundle were generalized to the context of Lie algebroids. For instance, covariant derivatives were generalized by Fernandes [9], Lagrangian mechanics were generalized by Weinstein [18] (see also [6]). Actually, a Riemannian metric on a manifold is a notion which involves the Lie algebroid structure of the tangent bundle and the Koszul formula, which defines the Levi-Civita connection, is an illustration of this fact. A Riemannian metric on a Lie algebroid is a natural extension of the classical notion of Riemannian metric on a manifold and this notion appeared first in the context of Lie algebroids associated to Poisson structures (see [2,3,12,13]).

In this paper, we present some basic concepts related to a Riemannian structure on a Lie algebroid, namely, we will show that most of the classical tools and results known in Riemannian geometry can be stated in this setting. In Section 2, we present some basic facts on connections on Lie algebroids based on recent results of [7]. In Section 3, we define the Levi-Civita connection associated to a Riemannian Lie algebroid and we show the existence of two tensors similar to those introduced by O'Neill in the context of Riemannian submersions [15] (see [1] for a detailed presentation). Section 4 is devoted to the study of the geodesic flow of a Riemannian Lie algebroid. As the classical case, we define the Sasaki metric and we compute the divergence of the geodesic flow with respect to this metric. This divergence does not vanish in general contrast to Liouville theorem and, in fact, it is a modular cocycle and its class is the modular class of the Lie algebroid. We state the first and the second variation formulas and introduce Jacobi sections along a geodesic. This section can be thought of as a completion of subSection 4.2 in [18] and Section 5 in [11]. In Section 5, we study the curvature of a Riemannian Lie algebroid and generalize some classical results, namely, Mayers theorem. Section 6 is devoted to the study of integrability of Riemannian Lie algebroids, for instance, we show that the vanishing of one of O'Neill's tensors implies the integrability

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and we give a large class of Riemannian Lie algebroids which satisfy this condition.

2. Background on Lie algebroids

In this section we review some basic facts related to Lie algebroids and to connections in the context of Lie algebroids (see [6,7,9] for a detailed presentation).

2.1. Canonical Poisson structure on the dual of a Lie algebroid

Definition 2.1. A Lie algebroid A over a smooth manifold M is a vector bundle $p : A \to M$ together with a Lie algebra structure $[\ ,\ ]$ on the space of sections $\Gamma(A)$ and a bundle map $\# : A \to TM$ called *anchor* such that, for any sections $a, b \in \Gamma(A)$ and for every smooth function $f \in C^\infty(M)$, we have the Leibniz identity

$$[a, fb] = f[a, b] + \#(a)(f)b. \tag{1}$$

An immediate consequence of this definition is that:

1. the induced map $\# : \Gamma(A) \to \mathcal{X}(M)$ is a Lie algebra homomorphism;
2. for any $x \in M$, there is an induced Lie bracket say $[\ ,\ ]_x$ on
$$\mathcal{G}_x = \mathrm{Ker}(\#_x) \subset A_x$$

which makes it into a Lie algebra.

The following theorem describes the local structure of a Lie algebroid (for a proof see [9]). Let n and r denote, respectively, the dimension of M and the rank of the vector bundle $A \to M$.

Theorem 2.2 (Local splitting). *Let $x_0 \in M$ be a point where $\#_{x_0}$ has rank q. There exists a system of coordinates $(x_1, \ldots, x_q, y_1, \ldots, y_{n-q})$ valid in a neighborhood U of x_0 and a basis of sections $\{a_1, \ldots, a_r\}$ of A over U, such that*

$$\#(a_i) = \partial_{x_i} \quad (i = 1, \ldots, q),$$
$$\#(a_i) = \sum_j b^{ij}\partial_{y_j} \quad (i = q+1, \ldots, r),$$

where $b^{ij} \in C^\infty(U)$ are smooth functions depending only on the y's and vanishing at x_0: $b^{ij} = b^{ij}(y^s)$, $b^{ij}(x_0) = 0$. Moreover, for any $i, j = 1, \ldots, r$,

$$[a_i, a_j] = \sum_u C^u_{ij} a_u,$$

where $C^u_{ij} \in C^\infty(U)$ vanish if $u \leqslant q$ and satisfy $\sum_{u>q} \frac{\partial C^u_{ij}}{\partial x_s} b^{ut} = 0$.

From this theorem we deduce that the image of $\#$ defines a smooth generalized distribution in M, in the sense of Sussman [16], which is integrable. This foliation is called *characteristic foliation* of A. We call A *transitive Lie algebroid* if $\#$ is surjective, so the leaves are the connected components of M.

We denote by A_L the restriction of A to a leaf L. From (1) one can deduce easily that the bracket $[\ ,\ ]$ induces a bracket on the space of sections of $p_L : A_L \to L$ and hence a transitive Lie algebroid structure. When x run over L the $\mathcal{G}'_x$s are all isomorphic and fit into a Lie algebra bundle $\mathcal{G}_L$ over L (see [14]). Hence, we get an exact sequence of Lie algebroids over L

$$0 \to \mathcal{G}_L \to A_L \to TL. \tag{2}$$

The dual A^* of a Lie algebroid $p : A \to M$ carries a natural Poisson structure which can be described as follows.

For any function $F \in C^\infty(A^*)$ and for any section $\xi \in \Gamma(A^*)$, we define a section $F_\xi \in \Gamma(A)$ by putting, for any $x \in M$ and for any $\mu_x \in A^*_x$,

$$\langle \mu_x, F_\xi(x)\rangle = \frac{d}{dt}_{|t=0} F(\xi(x) + t\mu_x).$$

Now, for any functions $F, H \in C^\infty(A^*)$, we define the bracket $\{F, H\}$ by putting, for any section $\xi \in \Gamma(A^*)$,

$$\{F, H\} \circ \xi = \langle \xi, [F_\xi, H_\xi]\rangle + \#(F_\xi)(H \circ \xi - \langle \xi, H_\xi\rangle) - \#(H_\xi)(F \circ \xi - \langle \xi, F_\xi\rangle). \tag{3}$$

One checks that this bracket defines a Poisson structure and for any $f, g \in C^\infty(M)$ and for any $a, b \in \Gamma(A)$, we have

$$\{f \circ p, g \circ p\} = 0, \quad \{f \circ p, a\} = -\#(a)(f) \circ p \quad \text{and} \quad \{a, b\} = [a, b]. \tag{4}$$

If one chooses local coordinates $(x_1, \ldots, x_n)$ over a neighborhood U of M and a basis of local sections $(a_1, \ldots, a_r)$ over U, we have structure functions $b^{si}, C^u_{st} \in C^\infty(U)$ defined by

$$\#(a_s) = \sum_{i=1}^n b^{si}\partial_{x_i} \quad (s = 1, \ldots, r),$$
$$[a_s, a_t] = \sum_{u=1}^r C^u_{st} a_u \quad (s, t = 1, \ldots, r).$$

Let $(\xi_1, \ldots, \xi_r)$ denote the linear coordinates on the fibers of A^* associated with the dual basis $(a^1, \ldots, a^r)$. One can see easily that

$$\{x_i, x_j\} = 0, \quad \{x_i, \xi_s\} = -b^{si} \quad \text{and} \quad \{\xi_s, \xi_t\} = \sum_u C^u_{st}\xi_u. \tag{5}$$

Example 2.3.

1. The basic example of a Lie algebroid over M is the tangent bundle itself, with the identity mapping as anchor. The associated Poisson structure on T^*M is defined by the symplectic form $d\lambda$ where λ is the Liouville form.
2. Every finite dimensional Lie algebra is a Lie algebroid over a one point space. The associated Poisson structure on the dual is the Lie–Poisson structure.
3. Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket.
4. Let (P, π) be a Poisson manifold. Then there is a natural Lie algebra structure on $\Omega^1(P)$ which makes T^*P into a Lie algebroid over P (see [17]).

2.2. Connections on Lie algebroids

We develop now the basic theory of connections on Lie algebroids. This notion, which is a natural extension of the usual concept of covariant connection, have recently turned out to be useful in the study of Lie algebroids. It appeared first in the context of Poisson geometry (see [9,10,17]).

Let $p : A \to M$ be a Lie algebroid with anchor map $\#$. An A-connection on a vector bundle $E \to M$ is an operator $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ satisfying:

1. $\nabla_{a+b}s = \nabla_a s + \nabla_b s$ for any $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$;
2. $\nabla_a(s_1 + s_2) = \nabla_a s_1 + \nabla_a s_2$ for any $a \in \Gamma(A)$ and $s_1, s_2 \in \Gamma(E)$;
3. $\nabla_{fa}s = f\nabla_a s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$;
4. $\nabla_a(fs) = f\nabla_a s + \#(a)(f)s$ for any $a \in \Gamma(A)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

From this definition, one can deduce immediately that, for any leaf L, ∇ induces an A_L-connection on $E_L \to L$.

Given an A-connection on a vector bundle E over M, most of the classical constructions (related to a classical covariant derivative) extend to Lie algebroids, provided we use the appropriate notion of paths on A.

Definition 2.4. Let $p : A \to M$ be a Lie algebroid with anchor $\#$.

1. An A-path is a smooth path $\alpha : [t_0, t_1] \to A$ such that

$$\#(\alpha(t)) = \frac{d}{dt}p(\alpha(t)), \quad t \in [t_0, t_1].$$

We call the curve $\gamma : [t_0, t_1] \to M$ given by $\gamma(t) = p(\alpha(t))$ *base path of* α.

2. An A-path α is called *vertical* if $\#(\alpha(t)) = 0$ for any $t \in [t_0, t_1]$.

Remark 2.5. Even if, for a vertical A-path, the base path is reduced to a constant curve, vertical A-paths play a non trivial role in the study of connections on a Lie algebroid.

2.3. Parallel transport

Let $p : A \to M$ be a Lie algebroid, $E \to M$ a vector bundle and ∇ an A-connection on E. Fix an A-path $\alpha : [t_0, t_1] \to A$. An α-section of E is a smooth map $s : [t_0, t_1] \to E$ such that the projections on M of α and s define the same base path. We denote by $\Gamma(E)_\alpha$ the space of α-sections of E. Then there is exists an unique map

$$\nabla^\alpha : \Gamma(E)_\alpha \to \Gamma(E)_\alpha$$

satisfying:

1. $\nabla^\alpha(c_1 s_1 + c_2 s_2) = c_1\nabla^\alpha s_1 + c_2\nabla^\alpha s_2$, $c_1, c_2 \in \mathbb{R}$;
2. $\nabla^\alpha fs = f's + f\nabla^\alpha s$ where $f : [t_0, t_1] \to \mathbb{R}$ is a smooth function;
3. if $\tilde{s}$ is a local section of E which extends s and $\#(\alpha(t)) \neq 0$ then

$$\nabla^\alpha s(t) = \nabla_{\alpha(t)}\tilde{s};$$

4. if $\tilde{s}$ is a local section of E which extends s and α is vertical then

$$\nabla^\alpha s(t) = \nabla_{\alpha(t)}\tilde{s} + \frac{d}{dt}s(t).$$

An α-section s is called parallel along α if $\nabla^\alpha s = 0$. One has then the notion of parallel transport along α, denoted by

$$\tau_\alpha^t : E_{\gamma(t_0)} \to E_{\gamma(t)},$$

and $\tau_\alpha^t(s_0) = s(t)$ where s is the unique parallel α-section satisfying $s(0) = s_0$.

If $\alpha_0 \in A_x$ and s is a section of E in a neighborhood of x, one can check easily that

$$\nabla_{\alpha_0}s = \frac{d}{dt}_{|t=0}(\tau_\alpha^t)^{-1}(s(\gamma(t))), \tag{6}$$

where α is any A-path satisfying $\alpha(0) = \alpha_0$.

2.4. Linear A-connections, geodesics and compatibility with the Lie algebroid structure

Let $p : A \to M$ be a Lie algebroid with anchor $\#$. We shall call A-connections on the vector bundle $A \to M$ *linear A-connections*.

Let $\mathcal{D}$ be a linear A-connection. An A-path $\alpha : [t_0, t_1] \to A$ is a *geodesic* of $\mathcal{D}$ if $\mathcal{D}^\alpha\alpha = 0$. Let $(x_1, \ldots, x_n)$ be a local system of coordinates on an open set U and $(a_1, \ldots, a_r)$ a basis of local sections over U. The structure functions $b^{si}, C_{st}^u \in C^\infty(U)$ are given by

$$\#a_s = \sum_{i=1}^n b^{si}\partial_{x_i} \quad (s = 1, \ldots, r),$$

$$[a_s, a_t] = \sum_{u=1}^r C_{st}^u a_u \quad (s, t = 1, \ldots, r).$$

We define the Christoffel symbols of $\mathcal{D}$ according to $(a_1, \ldots, a_r)$ as usually by

$$\mathcal{D}_{a_s}a_t = \sum_{u=1}^r \Gamma_{st}^u a_u.$$

The A-path α is a geodesic if, for $i = 1, \ldots, n$ and $j = 1, \ldots, r$,

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^r \alpha_j(t) b^{ji}(x_1(t), \ldots, x_n(t)), \\ \dot{\alpha}_j(t) = -\sum_{s,u=1}^r \alpha_s(t)\alpha_u(t)\Gamma_{su}^j(x_1(t), \ldots, x_n(t)), \end{cases} \tag{7}$$

where $\alpha(t) = \sum_{i=1}^r \alpha_i(t)a_i$ is the local expression of α and $p(\alpha(t)) = (x_1(t), \ldots, x_n(t))$ is the local expression of its base path.

Exactly as in the classical case, one has existence and uniqueness of geodesics with given initial base point $x \in M$ and "initial speed" $a_0 \in A_x$. Actually, there exists a vector field G on A such that the geodesics of $\mathcal{D}$ are the integral curves of G. We call G the *geodesic vector field* associated to $\mathcal{D}$ and $\mathcal{D}$ is called complete if G is complete.

We introduce now two natural notions of compatibility between linear A-connections and the structures of Lie algebroids.

Definition 2.6.

1. A linear A-connection $\mathcal{D}$ is *strongly compatible* with the Lie algebroid structure if, for any A-path α, the parallel transport τ_α preserves $\mathrm{Ker}\#$.

2. A linear A-connection $\mathcal{D}$ is *weakly compatible* with the Lie algebroid structure if, for any vertical A-path α, the parallel transport τ_α preserves $\mathrm{Ker}\#$.

The following proposition gives an useful characterization of the these notions of compatibility.

Proposition 2.7.

1. *A linear A-connection $\mathcal{D}$ is strongly compatible with the Lie algebroid structure if and only if, for any leaf L and for any sections $\alpha \in \Gamma(A_L)$ and $\beta \in \Gamma(\mathcal{G}_L)$, $\mathcal{D}_\alpha\beta \in \Gamma(\mathcal{G}_L)$.*
2. *A linear A-connection $\mathcal{D}$ is weakly compatible with the Lie algebroid structure if and only if, for any leaf L and for any sections $\alpha \in \Gamma(\mathcal{G}_L)$ and $\beta \in \Gamma(\mathcal{G}_L)$, $\mathcal{D}_\alpha\beta \in \Gamma(\mathcal{G}_L)$.*

Proof. This is a consequence of (6). $\square$

Example 2.8. Let $p: A \to M$ be a Lie algebroid and ∇ be a TM-connection on A. Associated with ∇ there is an obvious linear A-connection

$$\mathcal{D}^0_a b = \nabla_{\#(a)} b$$

which is clearly weakly compatible with the Lie algebroid structure. A bit more subtle is the following linear A-connection

$$\mathcal{D}^1_a b = \nabla_{\#(b)} a + [a, b]$$

which is strongly compatible with the Lie algebroid structure. These connections play a fundamental role in the theory of characteristic classes (see for instance [9]).

Remark 2.9. In [9] there is a notion of compatibility between linear A-connections and the Lie algebroid structure which is stronger than the notion of compatibility given in Definition 2.6.

2.5. Variations of A-paths, homotopy and curvature of A-connections

We give an interpretation of the torsion and the curvature of an A-connection which leads naturally to the notion of homotopy of A-paths. This notion plays a crucial role in the integrability of Lie algebroids (see [7]).

Let $p: A \to M$ be a Lie algebroid with anchor $\#$ and $E \to M$ a vector bundle. The curvature of an A-connection ∇ on E is formally identical to the usual definition

$$R(a, b)s = \nabla_a\nabla_b s - \nabla_b\nabla_a s - \nabla_{[a,b]}s,$$

where $a, b \in \Gamma(A)$ and $s \in \Gamma(E)$. The connection ∇ is called flat if R vanishes identically.

If $\mathcal{D}$ is a linear A-connection the torsion of $\mathcal{D}$ is given by

$$T_{\mathcal{D}}(a, b) = \mathcal{D}_a b - \mathcal{D}_b a - [a, b].$$

In the classical case ($A = TM$), the curvature and the torsion can be interpreted by using variations of paths. We will show now that we have a similar interpretation in the general case. First, let us give the appropriate notion of variation of paths.

A *variation* of A-paths is a smooth map $\alpha : [0,1] \times [0,1] \to A$, $(\epsilon, t) \mapsto \alpha(\epsilon, t)$ such that:

(i) for any $\epsilon \in [0,1]$, the map $t \mapsto \alpha(\epsilon, t)$ is an A-path,
(ii) the base variation $\gamma(\epsilon, t) = p(\alpha(\epsilon, t))$ lies entirely in a fixed leaf L of the characteristic foliation.

Let α be a variation of A-paths. A *transverse variation* to α is a smooth map $\beta : [0,1] \times [0,1] \to A$ such that α and β have the same base variation γ and $\#(\beta) = \frac{\partial\gamma}{\partial\epsilon}$.

It is clear that if $\#$ is injective, there is an unique transverse variation to a given variation of A-paths. However, if $\#$ is not injective, a given variation of A-paths admits many transverse variations to it. There is a way which permit the control of transverse variations to a fixed variation of A-path. Let us explain this important fact which is at the origin of the notion of homotopy of A-paths used in [7].

First, let us fix some notations. Let α and β be, respectively, a variation of A-paths and a transverse variation and let γ denote the commune base path. Let ∇ be an A-connection on a vector bundle $E \to M$ and let $s : [0,1] \times [0,1] \to E$ be a section over γ. For any $\epsilon \in [0,1]$, $t \mapsto \alpha(\epsilon, t)$ is an A-path and $\nabla_t s$ denotes the derivative of $t \mapsto s(\epsilon, t)$ along this A-path. On the other hand, for any $t \in [0,1]$, $\epsilon \mapsto \beta(\epsilon, t)$ is an A-path and $\nabla_\epsilon s$ denotes the derivative of $\epsilon \mapsto s(\epsilon, t)$ along this A-path.

The first claim in the following proposition is a reformulation of a part of Proposition 1.3 in [7].

Proposition 2.10. *With the notation above the following assertions hold.*

1. *For any linear A-connection $\mathcal{D}$, the variation*

$$\Delta(\alpha, \beta) = \mathcal{D}_t\beta - \mathcal{D}_\epsilon\alpha - T_{\mathcal{D}}(\alpha, \beta)$$

does not depend on $\mathcal{D}$ and satisfies $\#(\Delta(\alpha, \beta)) = 0$.

2. *for any A-connection ∇ on E and for any section s of E over γ*

$$\nabla_t\nabla_\epsilon s - \nabla_\epsilon\nabla_t s = R(\alpha, \beta)s + \nabla_{\Delta(\alpha,\beta)}s.$$

Proof.

1. Fix $(\epsilon_0, t_0) \in [0,1] \times [0,1]$ and choose a local coordinates $(x_1, \ldots, x_q, y_1, \ldots, y_{n-q})$ near $x_0 = \gamma(\epsilon_0, t_0)$ and a basis of sections $(a_1, \ldots, a_r)$ as in Theorem 2.2 ($q = \mathrm{rank}\#_{x_0}$). In these coordinates, we have

$$\begin{cases} \alpha(\epsilon, t) = \sum_{i=1}^r \alpha^i(\epsilon, t)a_i, \\ \beta(\epsilon, t) = \sum_{i=1}^r \beta^i(\epsilon, t)a_i, \\ \gamma(\epsilon, t) = (x_1(\epsilon, t), \ldots, x_q(\epsilon, t), c_1, \ldots, c_{n-q}), \\ \frac{\partial\gamma}{\partial t} = \sum_{j=1}^q \frac{\partial x_j}{\partial t}\partial_{x_j} = \sum_{i=1}^q \alpha^i(\epsilon, t)\partial_{x_i}, \\ \frac{\partial\gamma}{\partial\epsilon} = \sum_{j=1}^q \frac{\partial x_j}{\partial\epsilon}\partial_{x_j} = \sum_{i=1}^q \beta^i(\epsilon, t)\partial_{x_i}, \end{cases} \tag{8}$$

where $c_1, \ldots, c_{n-q}$ are constant. Now

$$\mathcal{D}_t\beta = \sum_{i=1}^r \frac{\partial\beta^i}{\partial t}a_i + \sum_{i,j=1}^r \alpha^j\beta^i\mathcal{D}_{a_j}a_i \quad \text{and}$$

$$\mathcal{D}_\epsilon\alpha = \sum_{i=1}^r \frac{\partial\alpha^i}{\partial\epsilon}a_i + \sum_{i,j=1}^r \alpha^i\beta^j\mathcal{D}_{a_j}a_i.$$

Hence

$$\mathcal{D}_t\beta - \mathcal{D}_\epsilon\alpha = \sum_{i=1}^{r}\left(\frac{\partial\beta^i}{\partial t} - \frac{\partial\alpha^i}{\partial\epsilon}\right)a_i + T_{\mathcal{D}}(\alpha,\beta) + \sum_{i,j=1}^{r}\alpha^i\beta^j[a_i,a_j].$$

Now, form (8), we have $\frac{\partial\beta^i}{\partial t} = \frac{\partial\alpha^i}{\partial\epsilon}$ for any $i = 1,\ldots,q$, so

$$\mathcal{D}_t\beta - \mathcal{D}_\epsilon\alpha - T_{\mathcal{D}}(\alpha,\beta) = \sum_{i=q+1}^{r}\left(\frac{\partial\beta^i}{\partial t} - \frac{\partial\alpha^i}{\partial\epsilon}\right)a_i + \sum_{i,j=1}^{r}\alpha^i\beta^j[a_i,a_j]. \quad (9)$$

One can see that the right hand of this equality lies in Ker$\#$ and does not depend on $\mathcal{D}$.

2. We choose a local trivialization $(x_1,\ldots,x_q,y_1,\ldots,y_{n-q},a_1,\ldots,a_r)$ as above, we trivialize E near x_0 by a local basis of sections $(e_1,\ldots,e_\mu)$ and put

$$s(\epsilon,t) = \sum_{j=1}^{\mu}s^j(\epsilon,t)e_j.$$

We have

$$\nabla_t s = \sum_{j=1}^{\mu}\frac{\partial s^j}{\partial t}e_j + \sum_{i,j}\alpha^i s^j\nabla_{a_i}e_j.$$

$$\nabla_\epsilon\nabla_t s = \sum_{j=1}^{\mu}\frac{\partial^2 s^j}{\partial\epsilon\partial t}e_j + \sum_{i,j}\left(\beta^i\frac{\partial s^j}{\partial t} + \frac{\partial\alpha^i}{\partial\epsilon}s^j + \alpha^i\frac{\partial s^j}{\partial\epsilon}\right)\nabla_{a_i}e_j + \sum_{i,j,k}\beta^k\alpha^i s^j\nabla_{a_k}\nabla_{a_i}e_j.$$

$$\nabla_t\nabla_\epsilon s = \sum_{j=1}^{\mu}\frac{\partial^2 s^j}{\partial t\partial\epsilon}e_j + \sum_{i,j}\left(\alpha^i\frac{\partial s^j}{\partial\epsilon} + \frac{\partial\beta^i}{\partial t}s^j + \beta^i\frac{\partial s^j}{\partial t}\right)\nabla_{a_i}e_j + \sum_{i,j,k}\alpha^k\beta^i s^j\nabla_{a_k}\nabla_{a_i}e_j.$$

$$\nabla_t\nabla_\epsilon s - \nabla_\epsilon\nabla_t s - R(\alpha,\beta)s = \sum_{i,j}\left(\frac{\partial\beta^i}{\partial t} - \frac{\partial\alpha^i}{\partial\epsilon}\right)s^j\nabla_{a_i}e_j + \sum_{i,j,k}\alpha^k\beta^i s^j\nabla_{[a_k,a_i]}e_j.$$

The above computation and (9) give the desired formula. □

From the expression of $\Delta(\alpha,\beta)$ given by (9) and from (8), we have

$$\Delta(\alpha,\beta) = 0 \iff \begin{cases} \frac{\partial\alpha^i}{\partial\epsilon} - \frac{\partial\beta^i}{\partial t} = \sum\limits_{l,k=1}^{r}\alpha^l\beta^k C^i_{lk} & i = q+1,\ldots,r, \\ \alpha^j = \frac{\partial x_j}{\partial t},\ \beta^j = \frac{\partial x_j}{\partial\epsilon} & j = 1,\ldots,q. \end{cases} \quad (10)$$

Now by using the standard results about linear differential systems one can deduce easily the following useful proposition (compare to Proposition 1.1 in [7]).

Proposition 2.11. *Let $p : A \to M$ be a Lie algebroid. Then, for a given variation of A-paths α and for given $\beta_0 : [0,1] \to A$ such that $\#(\beta_0)(\epsilon) = \frac{\partial p\circ\alpha}{\partial\epsilon}(\epsilon,0)$ there exists an unique transverse variation β to α such that*

$$\Delta(\alpha,\beta) = 0 \quad \text{and} \quad \beta(\epsilon,0) = \beta_0(\epsilon) \quad \text{forany}\ \epsilon \in [0,1].$$

Following [7], we can now define the homotpoy of A-paths with fixed end-points. Let α_0 and α_1 be two A-paths on a Lie algebroid $p : A \to M$ such that $p(\alpha_0(0)) = p(\alpha_1(0))$ and $p(\alpha_0(1)) = p(\alpha_1(1))$. An A-homotopy with fixed end-points from α_0 to α_1 is a variation of A-paths α such that:

(i) $p(\alpha(\epsilon,0)) = p(\alpha(0,0))$ and $p(\alpha(\epsilon,1)) = p(\alpha(0,1))$ for any $\epsilon \in [0,1]$, $\alpha(0,.) = \alpha_0$ and $\alpha(1,.) = \alpha_1$,

(ii) the unique transverse variation β to α satisfying $\Delta(\alpha,\beta) = 0$ and $\beta(\epsilon,0) = 0$ satisfies also $\beta(\epsilon,1) = 0$.

The following Lemma will be useful latter.

Lemma 2.12. *Let $\alpha_0 : [0,1] \to A$ be an A-path and $\beta_0 : [0,1] \to A$ an α_0-section such that $\beta_0(0) = \beta_0(1) = 0$. Then there exists an A-homotopy α with fixed end-points such that $\alpha(0,.) = \alpha_0$ and the corresponding transverse variation β satisfies $\beta(0,.) = \beta_0$.*

Proof. Consider the base path $\gamma_0 : [0,1] \to M$ of α_0 and choose an homotopy $\gamma : [0,1] \times [0,1] \to M$ with fixed end points such that γ lies in the same leaf as γ_0, $\gamma(0,.) = \gamma_0$ and $\frac{\partial\gamma}{\partial\epsilon}(0,t) = \#(\beta_0(t))$. We choose also $\beta : [0,1] \times [0,1] \to A$ such that $\beta(0,t) = \beta_0(t)$ for any $t \in [0,1]$, $\beta(\epsilon,0) = \beta(\epsilon,1) = 0$ for any $\epsilon \in [0,1]$ and $\frac{\partial\gamma}{\partial\epsilon}(\epsilon,t) = \#(\beta(\epsilon,t))$ for any (ϵ,t). From (10), one can deduce that there exists an unique variation $\alpha : [0,1] \times [0,1] \to A$ such that the base path of α is γ, $\frac{\partial\gamma}{\partial t}(\epsilon,t) = \#(\alpha(\epsilon,t))$, $\alpha(0,.) = \alpha_0$ and $\Delta(\alpha,\beta) = 0$. This variation is clearly an A-homotopy with fixed end-points and satisfies the required properties. □

2.6. The modular class of a Lie algebroid

We recall briefly the definition of the modular class of a Lie algebroid. For a detailed presentation see [8].

The *canonical representation* of a Lie algebroid A is the flat A-connexion $\mathcal{D}^A$ on the line bundle $L^A = \wedge^{\mathrm{top}}A \otimes \wedge^{\mathrm{top}}T^*M$ defined by

$$\mathcal{D}^A_a(\lambda\otimes v) = [a,\lambda]\otimes v + \lambda\otimes\mathcal{L}_{\#(a)}v,$$

where $a \in \Gamma(A)$, $\lambda \in \Gamma(\wedge^{\mathrm{top}}A)$ and $v \in \Gamma(\wedge^{\mathrm{top}}T^*M)$. If $\lambda\otimes v$ is a nowhere-vanishing section of L^A, the 1-form $\theta_{\lambda\otimes v} \in \Gamma(A^*)$ given by

$$\mathcal{D}^A_a(\lambda\otimes v) = \theta_{\lambda\otimes v}(a)\lambda\otimes v \quad (11)$$

is a d_A-cocycle and its class is independent of the choice of the section $\lambda\otimes v$. The section $\theta_{\lambda\otimes v}$ is called a *modular cocycle* of A and its class is called the *modular class* of A.

3. Riemannian metrics on Lie algebroids

In this section, we introduce the notion of Riemannian metric on a Lie algebroid which is a natural extension of the notion of Riemannian metric on a manifold. We show that most of the classical notions associated to a Riemannian metric can be defined in this context, namely, Levi-Civita connection, geodesics, geodesic flow, Sasaki metric, first and second variation formulas, Jacobi fields, the exponential...We show also that the Riemannian curvature of a Riemannian metric on a Lie algebroid satisfies formulas which are formally identical to the O'Neill formulas for Riemannian submersions.

3.1. The Levi-Civita connection of a Riemannian metric on a Lie

A Riemannian metric on a Lie algebroid $p : A \to M$ is the data, for any $x \in M$, of a scalar product $\langle\ ,\ \rangle_x$ on the fiber A_x such that, for any local sections $a,b \in \Gamma(A)$, the function $\langle a,b\rangle$ is smooth.

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The most interesting fact about Riemannian metrics on Lie algebroids is the existence on the analogous of the Levi-Civita connection. Indeed, if $\langle\ ,\ \rangle$ is a Riemannian metric on a Lie algebroid $p : A \to M$, then the formula

$$2\langle \mathcal{D}_a b, c\rangle = \#(a).\langle b, c\rangle + \#(b).\langle a, c\rangle - \#(c).\langle a, b\rangle + \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle$$

defines a linear A-connection which is characterized by the two following properties:

(i) $\mathcal{D}$ is metric, i.e., $\#(a).\langle b, c\rangle = \langle \mathcal{D}_a b, c\rangle + \langle b, \mathcal{D}_a c\rangle$,

(ii) $\mathcal{D}$ is torsion free, i.e., $\mathcal{D}_a b - \mathcal{D}_b a = [a, b]$.

We call $\mathcal{D}$ the *Levi-Civita A-connection* associated to the Riemannian metric $\langle\ ,\ \rangle$.

In a system of coordinates $(x_1, \ldots, x_n)$ over a trivializing neighborhood U of M where A admits a basis of local sections $(a_1, \ldots, a_r)$ the Levi-Civita A-connection is determined by the Christoffel's symbols defined by $\mathcal{D}_{a_i} a_j = \sum_k \Gamma_{ij}^k a_k$. A direct computation gives

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{n} g^{kl}\left(b^{iu}\partial_{x_u}(g_{jl}) + b^{ju}\partial_{x_u}(g_{il}) - b^{lu}\partial_{x_u}(g_{ij})\right) + \frac{1}{2}\sum_{l=1}^{r}\sum_{u=1}^{r} g^{kl}\left(C_{ij}^u g_{ul} + C_{li}^u g_{uj} + C_{lj}^u g_{ui}\right), \tag{12}$$

where the structure functions $b^{si}, C_{st}^u \in C^\infty(U)$ are given by

$$\#a_s = \sum_{i=1}^{n} b^{si}\partial_{x_i} \quad (s = 1, \ldots, r),$$

$$[a_s, a_t] = \sum_{u=1}^{r} C_{st}^u a_u \quad (s, t = 1, \ldots, r),$$

$g_{ij} = \langle a_i, a_j\rangle$ and (g^{ij}) denotes the inverse matrix of (g_{ij}).

Remark 3.1. There are two extremal cases:

1. The Lie algebroid A is the tangent bundle TM of a manifold and we recover the classical notion of Riemannian manifold.
2. The Lie algebroid A is a Lie algebra $\mathcal{G}$ considered as a Lie algebroid over a point. In this case a Riemannian metric on $\mathcal{G}$ is a scalar product $\langle\ ,\ \rangle$ and the Levi-Civita $\mathcal{G}$-connection is the product $\mathcal{D} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ given by

$$2\langle \mathcal{D}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

Actually $\mathcal{D}$ is the infinitesimal data associated to the Levi-Civita connection of the left invariant metric associated to $\langle\ ,\ \rangle$ on any Lie group with $\mathcal{G}$ as a Lie algebra.

The general setting is a combination of these two extremal cases. Indeed, let $\langle\ ,\ \rangle$ be a Riemannian metric on a Lie algebroid $p : A \to M$ with anchor $\#$. For any leaf L of the characteristic foliation and for any $x \in L$,

$$A_x = \mathcal{G}_x \oplus \mathcal{G}_x^\perp,$$

where $\mathcal{G}_x^\perp$ is the orthogonal to $\mathcal{G}_x$ with respect $\langle\ ,\ \rangle_x$. The restriction of the anchor $\#$ to $\mathcal{G}_x^\perp$ is an isomorphism into $T_x L$ and hence induces a scalar product on $T_x L$

$$\langle u, v\rangle_L = \langle a, b\rangle,$$

where $a, b \in \mathcal{G}_x^\perp$ and $\#(a) = u$ and $\#(b) = v$. Thus $\langle\ ,\ \rangle$ induces a Riemannian metric $\langle\ ,\ \rangle_L$ on L. We call it the *induced Riemannian metric* on L. On the other hand, the scalar product $\langle\ ,\ \rangle_x$ induces a scalar product on $\mathcal{G}_x$ and we denote by $\widehat{\mathcal{D}}$ the Levi-Civita $\mathcal{G}_x$-connection associated with $(\mathcal{G}_x, \langle\ ,\ \rangle_x)$.

Let us precise more this situation. Fix a leaf L and consider $p_L : A_L \to L$. We have

$$A_L = \mathcal{G}_L \oplus \mathcal{G}_L^\perp.$$

We call the elements of $\Gamma(\mathcal{G}_L)$ *vertical sections* and the elements of $\Gamma(\mathcal{G}_L^\perp)$ *horizontal sections*. For any section a, we denote by a^v and a^h, respectively, its horizontal and vertical component. Note that the bracket of a vertical section with every section is a vertical section. Thus, in the Riemannian point of view, the short exact sequence

$$0 \to \mathcal{G}_L \to A_L \to TL$$

is formally identical to a Riemannian submersion. So we can introduce the O'Neill tensors [15] (see [1] for a detailed presentation).

We denote by T and H the elements of $\Gamma(A^* \otimes A^* \otimes A)$ whose values on sections a, b are given by

$$T_a b = (\mathcal{D}_{a^v} b^v)^h + (\mathcal{D}_{a^v} b^h)^v \quad \text{and} \quad H_a b = (\mathcal{D}_{a^h} b^v)^h + (\mathcal{D}_{a^h} b^h)^v.$$

The following properties of T and H follow immediately from the definition: for any $a, b \in \Gamma(A)$,

$$H_{a^h} b^h = \frac{1}{2}[a^h, b^h]^v, \tag{13}$$

$$\mathcal{D}_{a^v} b^h = T_{a^v} b^h + (\mathcal{D}_{a^v} b^h)^h, \tag{14}$$

$$\mathcal{D}_{a^h} b^v = (\mathcal{D}_{a^h} b^v)^v + H_{a^h} b^v, \tag{15}$$

$$\mathcal{D}_{a^h} b^h = H_{a^h} b^h + (\mathcal{D}_{a^h} b^h)^h. \tag{16}$$

Moreover, for any $u, v \in \mathcal{G}_x$,

$$\mathcal{D}_u v = \widehat{\mathcal{D}}_u v + T_u v. \tag{17}$$

The following proposition is an immediate consequence of (16).

Proposition 3.2. *Let $\gamma : [t_0, t_1] \to L$ be a smooth path and let $\gamma^h : [t_0, t_1] \to \mathcal{G}_L^\perp$ be the unique A-path with the base path γ. Then γ is a geodesic with respect to the induced Riemannian metric on L if and only if γ^h is a geodesic of the Levi-Civita A-connexion.*

The following proposition gives an interpretation of the tensors T and H.

Proposition 3.3.

1. *The Levi-Civita A-connection is strongly compatible with the Lie algebroid structure if and only if $T = H = 0$.*
2. *The Levi-Civita A-connection is weakly compatible with the Lie algebroid structure if and only if $T = 0$.*

Proof. This is a consequence of Proposition 2.7, (14)–(16). $\square$

3.2. Geodesic flow of a Riemannian Lie algebroid

The Riemannian structure on a Lie algebroid A gives arise to an identification between A and A^*. Thus A inherits a Poisson structure from the canonical Poisson structure of A^*. As the classical case (when $A = TM$), the Hamiltonian vector field associated to the energy function on A is the geodesic flow of the Riemannian Lie algebroid. In this section, we give a complete proof of this fact and we generalize all the classical notions related to the geodesic flow, namely, the Sasaki metric, the first and second variation formulas, the Jacobi fields and the exponential. We show that the divergence of the geodesic flow according to the Sasaki metric is a modular cocycle and the modular class of the Lie algebroid is the obstruction to the vanishing of this divergence.

Let $p : A \to M$ be a Lie algebroid and $\langle\ ,\ \rangle$ a Riemannian metric on A. The Riemannian metric defines a bundle isomorphism between A and A^* which transport the Lie–Poisson structure on A^* into a Poisson structure say $\pi_{\langle\ ,\ \rangle}$ in A. Let $E : A \to \mathbb{R}$ be the energy function given by $E(a) = \frac{1}{2}\langle a, a\rangle$ and let X_E denote the hamiltonian vector field associated to E with respect to $\pi_{\langle\ ,\ \rangle}$. The following result is a generalization of a well-known result in Riemannian geometry.

Theorem 3.4. *The geodesics of the Levi-Civita A-connection associated to $\langle\ ,\ \rangle$ are the integral curves of the hamiltonian vector field X_E.*

Proof. Let $(x_1, \dots, x_n)$ be a system of coordinates over an open set U of M where A admits a basis of local sections $(a_1, \dots, a_r)$. The structure functions $b^{si}, C^u_{st} \in C^\infty(U)$ are given by

$$\#a_s = \sum\nolimits_{i=1}^n b^{si}\partial_{x_i} \quad (s = 1, \dots, r),$$
$$[a_s, a_t] = \sum\nolimits_{u=1}^r C^u_{st} a_u \quad (s, t = 1, \dots, r).$$

We denote by $(\mu_1, \dots, \mu_r)$ the linear coordinates on the fibers of A associated to $(a_1, \dots, a_r)$ and by $(\xi_1, \dots, \xi_r)$ its dual coordinates on A^*. Recall that the Poisson brackets on A^* are given by

$$\{x_i, x_j\} = 0, \quad \{x_i, \xi_s\} = -b^{si} \quad \text{and} \quad \{\xi_s, \xi_t\} = \sum_u C^u_{st}\xi_u.$$

Put $g_{ij} = \langle a_i, a_j\rangle$ and denote by (g^{ij}) the inverse matrix of (g_{ij}). The isomorphism $\langle\ ,\ \rangle^\# : A^* \to A$, the energy function and X_E are given, respectively, by

$$(x_1, \dots, x_n, \xi^1, \dots, \xi^r) \mapsto \left(x_1, \dots, x_n, \sum\nolimits_{i=1}^r g^{1i}\xi_i, \dots, \sum\nolimits_{i=1}^r g^{ri}\xi_i\right),$$
$$E = \frac{1}{2}\sum\nolimits_{i,j} g_{ij}\mu_i\mu_j,$$
$$X_E = \sum\nolimits_{i=1}^n \{E, x_i\}\partial_{x_i} + \sum\nolimits_{j=1}^r \{E, \mu_j\}\partial_{\mu_j}.$$

According to (7), we must show that, for $i = 1, \dots, n$ and $j = 1, \dots, r$,

$$\{E, x_i\} = \sum_k \mu_k b^{ki} \quad \text{and} \quad \{E, \mu_j\} = -\sum_{s,t} \mu_s\mu_t\Gamma^j_{st}, \tag{18}$$

where Γ^j_{st} are the Christoffel symbols given by (12), i.e.,

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^r\sum_{u=1}^n g^{kl}\left(b^{iu}\partial_{x_u}(g_{jl}) + b^{ju}\partial_{x_u}(g_{il}) - b^{lu}\partial_{x_u}(g_{ij})\right) + \frac{1}{2}\sum_{l=1}^r \times \sum_{u=1}^r g^{kl}\left(C^u_{ij}g_{ul} + C^u_{li}g_{uj} + C^u_{lj}g_{ui}\right).$$

1. The first relation in (18) is a straightforward computation. Indeed,

$$\begin{aligned}
\{E, x_i\} &= \frac{1}{2}\sum_{k,l} g_{kl}\{\mu_k\mu_l, x_i\} = \frac{1}{2}\sum_{k,l} g_{kl}(\mu_k\{\mu_l, x_i\} + \mu_l\{\mu_k, x_i\})\\
&= \sum_{k,l} g_{kl}\mu_k\{\mu_l, x_i\} = \sum_{k,l} g_{kl}\mu_k\{\sum_j g^{lj}\xi_j, x_i\}\\
&= \sum_{k,l,j} g_{kl}g^{lj}\mu_k\{\xi_j, x_i\} = \sum_{k,l,j} g_{kl}g^{lj}\mu_k b^{ji}\\
&= \sum_{k,j}\left(\sum_l g_{kl}g^{lj}\right)\mu_k b^{ji} = \sum_k \mu_k b^{ki}.
\end{aligned}$$

2. We must work much more to establish the second relation in (18).Note first that

$$\begin{aligned}
\sum\nolimits_{s,t}\mu_s\mu_t\Gamma^j_{st} = \frac{1}{2}&\\
&\times \sum_{s,t,u,l} g^{jl}\left(b^{su}\partial_{x_u}(g_{tl}) + b^{tu}\partial_{x_u}(g_{sl}) - b^{lu}\partial_{x_u}(g_{st})\right)\mu_s\mu_t\\
&+\frac{1}{2}\sum_{s,t,u,l} g^{jl}\left(C^u_{st}g_{ul} + C^u_{ls}g_{ut} + C^u_{lt}g_{us}\right)\mu_s\mu_t \overset{(a)}{=}\\
&\times \sum_{s,t,u,l} g^{jl}\left(b^{su}\partial_{x_u}(g_{tl}) - \frac{1}{2}b^{lu}\partial_{x_u}(g_{st})\right)\mu_s\mu_t\\
&+\sum_{s,t,u,l} g^{jl}g_{ut}C^u_{ls}\mu_s\mu_t.
\end{aligned} \tag{19}$$

We have used in (a) the fact that $C^{st}_u = -C^{ts}_u$. Now

$$\begin{aligned}
2\{E, \mu_j\} &= \sum_{s,t}\{g_{st}\mu_s\mu_t, \mu_j\}\\
&= \sum_{s,t}\left(g_{st}\{\mu_s\mu_t, \mu_j\} + \mu_s\mu_t\{g_{st}, \mu_j\}\right)\\
&= \sum_{s,t}\left(g_{st}\mu_s\{\mu_t, \mu_j\} + g_{st}\mu_t\{\mu_s, \mu_j\}\right)\\
&\quad + \sum_{s,t,l}\mu_s\mu_t g^{jl}\{g_{st}, \xi_l\}\\
&= 2\sum_{s,t} g_{st}\mu_s\{\mu_t, \mu_j\} - \sum_{s,t,l,u} g^{jl}b^{lu}\partial_{x_u}(g_{st})\mu_s\mu_t.
\end{aligned} \tag{20}$$

By comparing (19) and (20), one can see that the desired relation is equivalent to

$$\begin{aligned}
\sum_{s,t} g_{st}\mu_s\{\mu_t, \mu_j\} &= \sum_{s,t,u,l} g^{jl}\left(-b^{su}\partial_{x_u}(g_{tl}) + b^{lu}\partial_{x_u}(g_{st})\right)\mu_s\mu_t\\
&\quad - \sum_{s,t,u,l} g^{jl}g_{ut}C^u_{ls}\mu_s\mu_t.
\end{aligned} \tag{21}$$

Let us establish this relation. Note first that

$$\begin{aligned}
\{\mu_i, \mu_j\} &= \sum_{k,l}\{g^{il}\xi_l, g^{jk}\xi_k\}\\
&= \sum_{k,l}\left(g^{il}g^{jk}\{\xi_l, \xi_k\} + g^{il}\xi_k\{\xi_l, g^{jk}\} + g^{jk}\xi_l\{g^{il}, \xi_k\}\right)\\
&= \sum_{k,l,u} g^{il}g^{jk}C^u_{lk}\xi_u + \sum_{k,l,u} g^{il}\xi_k b^{lu}\partial_{x_u}(g^{jk})\\
&\quad - \sum_{k,l,u} g^{jk}\xi_l b^{ku}\partial_{x_u}(g^{il})\\
&= \sum_{k,l,u} g^{il}g^{jk}C^u_{lk}\xi_u + \sum_{k,l,u} b^{lk}\left(g^{il}\partial_{x_k}(g^{ju}) - g^{jl}\partial_{x_k}(g^{iu})\right)\xi_u.
\end{aligned}$$

Hence

$$\sum_{s,t} g_{st}\mu_s\{\mu_t,\mu_j\} = \sum_{s,t,k,l,u} g_{st}g^{tl}g^{jk}C^u_{lk}\mu_s\xi_u + \sum_{s,t,k,l,u} b^{lk}\left(g^{tl}\partial_{x_k}(g^{ju}) - g^{jl}\partial_{x_k}(g^{tu})\right)g_{st}\mu_s\xi_u.$$

Now

$$\sum_{s,t,k,l,u} g_{st}g^{tl}g^{jk}C^u_{lk}\mu_s\xi_u = \sum_{s,k,u} g^{jk}C^u_{sk}\mu_s\xi_u = \sum_{s,t,k,u} g^{jk}C^u_{sk}\mu_s g_{ut}\mu_t = -\sum_{s,t,u,l} g^{jl}g_{ut}C^u_{ls}\mu_s\mu_t.$$

$$\begin{aligned}\sum_{s,t,k,l,u} b^{lk}g^{tl}\partial_{x_k}(g^{ju})g_{st}\mu_s\xi_u &= \sum_{s,k,u} b^{sk}\partial_{x_k}(g^{ju})\mu_s\xi_u\\ &= \sum_{s,t,k,u} b^{sk}g_{ut}\partial_{x_k}(g^{ju})\mu_s\mu_t\\ &= \sum_{s,t,k,u} b^{sk}\partial_{x_k}(g_{ut}g^{ju})\mu_s\mu_t - \sum_{s,t,k,u} b^{sk}g^{ju}\partial_{x_k}(g_{ut})\mu_s\mu_t\\ &= -\sum_{s,t,u,l} b^{su}g^{jl}\partial_{x_u}(g_{lt})\mu_s\mu_t.\end{aligned}$$

$$\begin{aligned}\sum_{s,t,k,l,u} b^{lk}g^{jl}\partial_{x_k}(g^{tu})g_{st}\mu_s\xi_u &= -\sum_{s,t,k,l,u} b^{lk}g^{jl}\partial_{x_k}(g_{st})g^{tu}\mu_s\xi_u\\ &= -\sum_{s,t,k,l,u,h} b^{lk}g^{jl}\partial_{x_k}(g_{st})g^{tu}g_{uh}\mu_s\mu_h\\ &= -\sum_{s,t,k,l} b^{lk}g^{jl}\partial_{x_k}(g_{st})\mu_s\mu_t\\ &= -\sum_{s,t,u,l} b^{lu}g^{jl}\partial_{x_u}(g_{st})\mu_s\mu_t.\end{aligned}$$

Thus we get (21) and the theorem follows. □

The flow of the Hamiltonian vector field X_E is called the *geodesic flow* of $\langle\ ,\ \rangle$.

Remark 3.5. Let $p: A \to M$ be a Riemannian Lie algebroid. Then:

1. For any leaf L, the geodesic vector field X_E is tangent to A_L and to $\mathcal{G}_x$ for any $x \in L$. This follows from the fact that geodesics are A-paths.
2. From Proposition 3.2, one can deduce that, for any leaf L, the geodesic vector field X_E is tangent to $\mathcal{G}_L^\perp$.

Corollary 3.6. *Let $p: A \to M$ be Riemannian Lie algebroid. Then*

1. *If L is a compact leaf then the geodesic flow is complete in restriction to A_L.*
2. *If M is compact then the geodesic flow is complete and for any leaf L the induced Riemannian metric $\langle\ ,\ \rangle_L$ is complete.*

We will now construct an analogous of the Sasaki metric on A and study the divergence of the geodesic flow with respect to this metric. Actually, the Sasaki metric is not defined on A but only on A_L where L is a leaf of the characteristic foliation.

Let $p: A \to M$ be a Riemannian Lie algebroid with anchor $\#$. Fix a leaf L, consider $p_L: A_L \to L$ and put $\mathcal{V}A_L = \mathrm{Ker}dp_L$.
For any $a \in A_L$, we consider the subspace $\mathcal{H}^\perp A_L$ of T_aA_L consisting of the tangent vectors V_a such that there exists an horizontal A-path $\alpha: [0,1] \to \mathcal{G}_L^\perp$ satisfying $p(\alpha(0)) = p(a)$ and $V_a = \frac{d}{dt|_{t=0}}\tau^t_\alpha(a)$, where τ_α is the parallel transport along α. We have

$$TA_L = \mathcal{V}A_L \oplus \mathcal{H}^\perp A_L. \tag{22}$$

Indeed, we define $K: TA_L \to A_L$ as follows. Fix $a \in A_L$ and $Z \in T_aA_L$ and choose $\beta: [0,1] \to A_L$ such that $\beta(0) = a$ and $\dot\beta(0) = Z$. There exists an unique horizontal A-path $\alpha: [0,1] \to \mathcal{G}_L^\perp$ with the base path $p \circ \beta(t)$. Put

$$K(Z) = (\mathcal{D}^\alpha\beta)(0).$$

It is easy to check that K is well-defined, $\mathrm{Ker}K = \mathcal{H}^\perp A_L$ and, for any $Z \in \mathcal{V}A_L$, $K(Z) = Z$. Then the relation (22) follows.
Let $(x_1,\ldots,x_l)$ be a system of local coordinates on an open set U in L and $(a_1,\ldots,a_r)$ is a basis of local sections (over U) of A_L. This defines a system of coordinates $(x_1,\ldots,x_l,\mu_1,\ldots,\mu_r)$ on A_L and if $Z = \sum_j b_j\partial_{x_j} + \sum_j Z^j\partial_{\mu_j}$ then

$$K(Z) = \sum_l\left(Z^l + \sum_{i,j}\alpha_i\mu_j\Gamma^l_{ij}\right)a_l, \tag{23}$$

where $dp_L(Z) = \#(\sum_i\alpha_i a_i)$ and $\sum_i\alpha_i a_i \in \mathcal{G}_L^\perp$.

Remark 3.7. In general, the geodesic vector field does not lies in $\mathrm{Ker}K$. Indeed, one can check easily that for any $a \in A_L$

$$K(X_E(a)) = -\mathcal{D}_{a^v}a.$$

We define the Sasaki metric on A_L by

$$g_L(Z_a,Z_a) = \langle d_ap(Z_a), d_ap(Z_a)\rangle_L + \langle K(Z_a), K(Z_a)\rangle.$$

The projection $p_L: A_L \to L$ becomes a Riemannian submersion. We consider now the Liouville vector field $\overrightarrow{r}$ on A_L which is the vector field generating the flow $\phi_t(a) = e^ta$. By direct computation one can get

$$[\overrightarrow{r}, X_E] = X_E. \tag{24}$$

From this relation, one deduce that X_E preserves the Riemannian volume on A_L associated to g_L if and only if X_E preserves the Riemannian volume of the restriction of g_L to the spheres bundle $UA_L = \{a \in A_L; \langle a,a\rangle = 1\}$. Let us compute the divergence of the geodesic vector field with respect to g_L.

Theorem 3.8. *The divergence the geodesic vector field X_E with respect to the Sasaki metric g_L is given by*

$$\mathrm{div}(X_E)(a) = \mathrm{Tr}ad_{a^v} + \langle a^h, N\rangle, \tag{25}$$

where $ad_{a^v}: \mathcal{G}_{p(a)} \to \mathcal{G}_{p(a)}$, $b \to [a^v, b]$ and $N = \sum_i T_{b_i}b_i$ where $(b_1,\ldots,b_s)$ is any orthonormal basis of $\mathcal{G}_{p(a)}$ and T is the O'Neill tensor.

Proof. Denote by l the dimension of L and choose a system of local coordinates $(x_1,\ldots,x_l)$ in some open set U of L. Choose $(a_1,\ldots,a_l)$ an orthonormal basis of sections of $\mathcal{G}_L^\perp \to U$ and $(b_1,\ldots,b_{r-l})$ an orthonormal basis of sections of $\mathcal{G}_L \to U$. We get a system of coordinates (x,μ) in A_L. Put, for any $i = 1,\ldots,l$,

$$\#(a_i) = \sum_j p^{ij}\partial_{x_j} \quad \text{and} \quad Z^i = \sum_j p^{ij}\partial_{x_j} - \sum_l (\sum_j \mu_j \Gamma^l_{ij})\partial_{\mu_l}.$$

By using (23), one can check easily that $K(Z^i) = 0$ and $K(\partial_{\mu_i}) = a_i$ for $i = 1, \dots, l$ and $K(\partial_{\mu_i}) = b_i$ for $i = l+1, \dots, l-r$. Moreover $(Z^1, \dots, Z^l, \partial_{\mu_1}, \dots, \partial_{\mu_r})$ is an orthonormal frame of g_L and hence

$$\mathrm{div}(X_E) = \sum_i g_L([Z^i, X_E], Z^i) + \sum_j g_L([\partial_{\mu_j}, X_E], \partial_{\mu_j}).$$

Recall that

$$X_E = \sum_{i,k=1}^{l} p^{ki}\mu_k \partial_{x_i} - \sum_{j,s,t} \mu_s\mu_t \Gamma^j_{st}\partial_{\mu_j}.$$

So, for $1 \leqslant j \leqslant l$,

$$[\partial_{\mu_j}, X_E] = \sum_i p^{ji}\partial_{x_i} - \sum_{i,t} \mu_t(\Gamma^i_{jt} + \Gamma^i_{tj})\partial_{\mu_i},$$

$$g_L([\partial_{\mu_j}, X_E], \partial_{\mu_j}) = \langle K([\partial_{\mu_j}, X_E]), K(\partial_{\mu_j})\rangle \overset{(23)}{=} -\sum_t \mu_t \Gamma^j_{tj} = 0,$$

since $\Gamma^j_{tj} = \langle \mathcal{D}_{a_t}a_j, a_j\rangle = -\langle a_j, \mathcal{D}_{a_t}a_j\rangle$.

For $j \geqslant l+1$

$$[\partial_{\mu_j}, X_E] = -\sum_{i,t} \mu_t(\Gamma^i_{jt} + \Gamma^i_{tj})\partial_{\mu_i},$$

$$g_L([\partial_{\mu_j}, X_E], \partial_{\mu_j}) = -\sum_t \mu_t \Gamma^j_{jt}.$$

Hence

$$\begin{aligned}\sum_j g_L([\partial_{\mu_j}, X_E], \partial_{\mu_j}) &= -\sum_{j\geqslant l+1}\sum_t \mu_t\Gamma^j_{jt} \\ &= -\sum_{j\geqslant l+1}\sum_{t=1}^{l} \mu_t\langle \mathcal{D}_{b_j}a_t, b_j\rangle - \sum_{j\geqslant l+1} \\ &\quad \times \sum_{t\geqslant l+1} \mu_t \langle \mathcal{D}_{b_j}b_t, b_j\rangle \\ &= \langle a^h, \sum_{j\geqslant l+1} \mathcal{D}_{b_j}b_j\rangle - \sum_{j\geqslant l+1}\langle \mathcal{D}_{b_j}a^v, b_j\rangle \\ &= \langle a^h, \sum_{j\geqslant l+1} T_{b_j}b_j\rangle - \sum_{j\geqslant l+1}\langle [b_j, a^v], b_j\rangle \\ &= \langle a^h, N\rangle + \mathrm{Tr}ad_{a^v}.\end{aligned}$$

On the other hand, one can see easily that

$$X_E = \sum_{k=1}^{l}\mu_k Z^k - \sum_{j=1}^{r}\sum_{s\geqslant l+1,t} \mu_s\mu_t\Gamma^j_{st}\partial_{\mu_j} = \sum_{k=1}^{l}\mu_k Z^k + V.$$

Note that V is vertical and since, for any $i = 1, \dots, l$, Z^i is basic (with respect to the Riemannian submersion $p_L : A_L \to L$) then $[Z^i, V]$ is vertical. Note also that, for any $i, k = 1, \dots, l$, $dp_L([Z^i, Z^k]) = \#([a_i, a_k])$. Hence

$$\begin{aligned}\sum_i g_L([Z^i, X_E], Z^i) &= \sum_{i,k} g_L([Z^i, \mu_k Z^k], Z^i) \\ &= \sum_i Z^i(\mu_i) + \sum_{i,k}\mu_k\langle \#([a_i, a_k]), \#a_i\rangle_L \\ &= \sum_i Z^i(\mu_i) + \sum_{i,k}\mu_k\langle [a_i, a_k]^h, a_i\rangle \\ &= \sum_i Z^i(\mu_i) + \sum_{i,k}\mu_k\langle [a_i, a_k], a_i\rangle \\ &= -\sum_{i,k}\mu_k\Gamma^i_{ik} + \sum_{i,k}\mu_k\Gamma^i_{ik} = 0.\end{aligned}$$

Finally, we get the desired formula. □

The following proposition gives an interesting interpretation of $\mathrm{div}X_E$, namely $\mathrm{div}X_E$ is a modular cocycle.

Proposition 3.9. *Let $p : A \to M$ be a transitive Riemannian Lie algebroid such that both A and TM are orientable. Denote by $\lambda \in \Gamma(\wedge^{\mathrm{top}}A)$ and $v \in \Gamma(\wedge^{\mathrm{top}}T^*M)$, respectively, the Riemannian volume associated to $\langle\, ,\, \rangle$ and the Riemannian volume associated to $\langle\, ,\, \rangle_M$ then*

$$\mathcal{D}^A(\lambda \otimes v) = \mathrm{div}(X_E)(\lambda \otimes v),$$

where $\mathcal{D}^A$ is the canonical representation of A. Thus $\mathrm{div}(X_E)$ *is a modular cocycle.*

Proof. Choose a local orthonormal basis $(a_1, \dots, a_n)$ of sections of $\mathcal{G}_L^{\perp}$ and a local orthonormal basis $(b_1, \dots, b_{r-n})$ of sections of $\mathcal{G}_L$. Recall that

$$\mathcal{D}^A_a(\lambda \otimes v) = [a, \lambda] \otimes v + \lambda \otimes \mathcal{L}_{\#(a)}v.$$

Now

$$\begin{aligned}[a, \lambda] &= \left(\sum_{i=1}^{n}\langle [a, a_i], a_i\rangle + \sum_{i=1}^{r-n}\langle [a, b_i], b_i\rangle\right)\lambda, \\ &= \left(\sum_{i=1}^{n}\langle [a^h, a_i], a_i\rangle + \sum_{i=1}^{r-n}\langle [a, b_i], b_i\rangle\right)\lambda,\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{\#(a)}v &= \left(\sum_{i=1}^{n}\langle [\#(a_i), \#(a)], \#(a_i)\rangle_M\right)v \\ &= \left(\sum_{i=1}^{n}\langle [a_i, a^h], a_i\rangle\right)v.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{r-n}\langle [a, b_i], b_i\rangle &= \sum_{i=1}^{r-n}\langle [a^v, b_i], b_i\rangle + \sum_{i=1}^{r-n}\langle [a^h, b_i], b_i\rangle \\ &= Trad_{a^v} + \sum_{i=1}^{r-n}\langle a^h, \mathcal{D}_{b_i}b_i\rangle \overset{(17)}{=} Trad_{a^v} \\ &\quad + \langle a^h, \sum_{i=1}^{r-n} T_{b_i}b_i\rangle,\end{aligned}$$

which completes the proof. □

Remark 3.10.

1. If $A = TM$ then $\mathrm{div}(X_E) = 0$ and one recover the classical Liouville Theorem.
2. If A is a Lie algebra then $\mathrm{div}(X_E) = 0$ if and only if A is unimodular.
3. If A is a transitive unimodular Lie algebroid then there exists a Riemannian metric on A such that $\mathrm{div}(X_E) = 0$.

We will now establish the first and the second variation formulas in the context of Riemannian Lie algebroids.

Let $p : A \to M$ be a Riemannian Lie algebroid with anchor $\#$. For any A-path $\alpha : [0, 1] \to A$, the energy and the length of α are given, respectively, by

$$\mathbf{E}(\alpha)=\frac{1}{2}\int_0^1\langle\alpha(t),\alpha(t)\rangle dt \quad \text{and} \quad \mathcal{L}(\alpha)=\int_0^1\sqrt{\langle\alpha(t),\alpha(t)\rangle}dt.$$

For any m, q lying in the same leaf of the characteristic foliation, we denote by Ω_{mq} the set of A-path α such that $p(\alpha(0))=m$ and $p(\alpha(1))=q$.

Proposition 3.11 (First variation formulas). *Let $p: A\to M$ be a Riemannian Lie algebroid. Then:*

1. *For any variation of A-paths $\alpha:[0,1]\times[0,1]\to A$ and for any β a transverse variation to α, one has*
$$\frac{d}{d\epsilon}\mathbf{E}(\alpha)=\langle\beta(\epsilon,1),\alpha(\epsilon,1)\rangle-\langle\beta(\epsilon,0),\alpha(\epsilon,0)\rangle-\int_0^1\langle\beta,\mathcal{D}_t\alpha\rangle dt$$
$$-\int_0^1\langle\Delta(\alpha,\beta),\alpha\rangle dt.$$
2. *The h-critical points of $\mathbf{E}:\Omega_{mq}\to\mathbb{R}$, namely the A-paths α_0 such that*
$$\frac{d}{d\epsilon}\mathbf{E}(\alpha)_{|\epsilon=0}=0$$

for any A-homotopy α in Ω_{mq} starting at α_0, are geodesics.

3. *For any variation of A-paths α such that α_0 is parameterized with arc-length,*
$$\frac{d}{d\epsilon}\mathbf{E}(\alpha)_{|\epsilon=0}=\frac{d}{d\epsilon}\mathcal{L}(\alpha)_{|\epsilon=0}.$$
4. *An A-path $\alpha_0\in\Omega_{mq}$ is h-critical for $\mathcal{L}$, namely*
$$\frac{d}{d\epsilon}\mathcal{L}(\alpha)_{|\epsilon=0}=0$$
for any A-homotopy in Ω_{mq} starting at α_0, if and only if there exists a change of parameter μ such that the A-path $\tilde{\alpha}_0=\mu'\alpha_0(\mu)$ is a geodesic.

Proof.

1. Let us compute $\frac{d}{d\epsilon}\mathbf{E}(\alpha)$. We have
$$\begin{aligned}\frac{d}{d\epsilon}\mathbf{E}(\alpha_\epsilon)&=\frac{1}{2}\frac{d}{d\epsilon}\int_0^1\langle\alpha,\alpha\rangle dt=\frac{1}{2}\int_0^1\frac{d}{d\epsilon}\langle\alpha,\alpha\rangle dt=\int_0^1\langle\mathcal{D}_\epsilon\alpha,\alpha\rangle dt\\&=\int_0^1\langle\mathcal{D}_t\beta,\alpha\rangle dt-\int_0^1\langle\Delta(\alpha,\beta),\alpha\rangle dt \quad (\text{Proposition2.10})\\&=\int_0^1\partial_t(\langle\beta,\alpha\rangle)dt-\int_0^1(\langle\beta,\mathcal{D}_t\alpha\rangle)dt-\int_0^1\langle\Delta(\alpha,\beta),\alpha\rangle dt\\&=\langle\beta(\epsilon,1),\alpha(\epsilon,1)\rangle-\langle\beta(\epsilon,0),\alpha(\epsilon,0)\rangle-\int_0^1\langle\beta,\mathcal{D}_t\alpha\rangle dt\\&\quad-\int_0^1\langle\Delta(\alpha,\beta),\alpha\rangle dt.\end{aligned}$$
Analogously one can get
$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}(\alpha)&=\int_0^1|\alpha|^{-1/2}\partial_t(\langle\beta,\alpha\rangle)dt\\&\quad-\int_0^1|\alpha|^{-1/2}(\langle\beta,\mathcal{D}_t\alpha\rangle)dt\\&\quad-\int_0^1|\alpha|^{-1/2}\langle\Delta(\alpha,\beta),\alpha\rangle dt.\end{aligned}\tag{26}$$
2. Let α_0 be geodesic and let α be an A-homotopy with fixed end-point starting at α_0. Then there exists a transverse variation β to α such that $\beta(\epsilon,0)=\beta(\epsilon,1)=0$ and $\Delta(\alpha,\beta)=0$. Hence from 1, we get
$$\frac{d}{d\epsilon_{|\epsilon=0}}\mathbf{E}(\alpha)=0.$$
Conversely, suppose that α_0 is an A-path which is a h-critical point of $\mathbf{E}:\Omega_{mq}\to\mathbb{R}$. Consider the α_0-section $\beta_0(t)=f(t)\mathcal{D}_t\alpha_0$ where $f:[0,1]\to\mathbb{R}$ is a smooth function such that $f(0)=f(1)=0$. According to Lemma 2.12, there exists an A-homotopy α with fixed end-points and starting at α_0 and such the corresponding transverse variation β satisfies $\beta(0,t)=\beta_0(t)$. By applying the formula in 1., we get
$$0=\int_0^1 f(t)\langle\mathcal{D}_t\alpha_0,\mathcal{D}_t\alpha_0\rangle dt$$
and hence $\mathcal{D}_t\alpha_0=0$ which means that α_0 is a geodesic.
3. This is a consequence of (26) and $|\alpha_0|=1$.
4. Immediate from 2. and 3. □

Proposition 3.12. Second variation formulas). *Let $p: A\to M$ be a Riemannian Lie algebroid. Then the following assertions hold.*

1. *For any variation of A-paths α such that α_0 is a geodesic and for any β a transverse variation to α such that $\Delta(\alpha,\beta)=0$, one has*
$$\frac{d^2}{d\epsilon^2}\mathbf{E}(\alpha)_{|\epsilon=0}=\langle\mathcal{D}_\epsilon\beta(0,1),\alpha(0,1)\rangle-\langle\mathcal{D}_\epsilon\beta(0,0),\alpha(0,0)\rangle$$
$$+\int_0^1\langle\mathcal{D}_t\beta_0,\mathcal{D}_t\beta_0\rangle dt+\int_0^1\langle\beta_0,R(\alpha_0,\beta_0)\alpha_0\rangle dt.$$
2. *Let α be an A-homotopy of A-paths such that α_0 is a geodesic and let β be the corresponding transverse variation. One has*
$$\frac{d^2}{d\epsilon^2}\mathbf{E}(\alpha)_{|\epsilon=0}=\int_0^1\langle\mathcal{D}_t\beta_0,\mathcal{D}_t\beta_0\rangle dt+\int_0^1\langle\beta_0,R(\alpha_0,\beta_0)\alpha_0\rangle dt.$$
3. *Let α be a variation of A-paths such that α_0 is a geodesic parameterized by arc length and let β a transverse variation to α such that $\Delta(\alpha,\beta)=0$. One has*
$$\frac{d^2}{d\epsilon^2}\mathcal{L}(\alpha)_{|\epsilon=0}=\langle\mathcal{D}_\epsilon\beta(0,1),a(0,1)\rangle-\langle\mathcal{D}_\epsilon\beta(0,0),\alpha(0,0)\rangle$$
$$+\int_0^1\langle\mathcal{D}_t\beta_0,\mathcal{D}_t\beta_0\rangle dt+\int_0^1\langle\beta_0,R(\alpha_0,\beta_0)\alpha_0\rangle dt$$
$$-\int_0^1\langle\alpha_0,\mathcal{D}_t\beta_0\rangle dt.$$
4. *Let α be an A-homotopy of A-paths such that α_0 is a geodesic parameterized by arc length and let β be the corresponding transverse variation. One has*
$$\frac{d^2}{d\epsilon^2}\mathcal{L}(\alpha)_{|\epsilon=0}=\int_0^1\langle\mathcal{D}_t\beta_0,\mathcal{D}_t\beta_0\rangle dt+\int_0^1\langle\beta_0,R(\alpha_0,\beta_0)\alpha_0\rangle dt$$
$$-\int_0^1\langle\alpha_0,\mathcal{D}_t\beta_0\rangle dt.$$

Proof.

1. From the first variation formula, we have

$$\frac{d}{d\epsilon}\mathbf{E}(\alpha) = \langle\beta(\epsilon,1),\alpha(\epsilon,1)\rangle - \langle\beta(\epsilon,0),\alpha(\epsilon,0)\rangle - \int_0^1\langle\beta,\mathcal{D}_t\alpha\rangle dt.$$

Then

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\mathbf{E}(\alpha) &= \langle\mathcal{D}_\epsilon\beta(\epsilon,1),\alpha(\epsilon,1)\rangle + \langle\beta(\epsilon,1),\mathcal{D}_\epsilon\alpha(\epsilon,1)\rangle\\ &\quad - \langle\mathcal{D}_\epsilon\beta(\epsilon,0),\alpha(\epsilon,0)\rangle - \langle\beta(\epsilon,0),\mathcal{D}_\epsilon\alpha(\epsilon,0)\rangle\\ &\quad - \int_0^1\langle\mathcal{D}_\epsilon\beta,\mathcal{D}_t\alpha\rangle dt - \int_0^1\langle\beta,\mathcal{D}_\epsilon\mathcal{D}_t\alpha\rangle dt.\end{aligned}$$

$$\begin{aligned}\int_0^1\langle\beta,\mathcal{D}_\epsilon\mathcal{D}_t\alpha\rangle dt &= \int_0^1\langle\beta,\mathcal{D}_t\mathcal{D}_\epsilon\alpha\rangle dt\\ &\quad + \int_0^1\langle\beta,R(\beta,\alpha)\alpha\rangle dt \quad \text{(Proposition2.10)}\\ &= \int_0^1\partial_t(\langle\beta,\mathcal{D}_\epsilon\alpha\rangle)dt - \int_0^1\langle\mathcal{D}_t\beta,\mathcal{D}_\epsilon\alpha\rangle dt\\ &\quad + \int_0^1\langle\beta,R(\beta,\alpha)\alpha\rangle dt\\ &= \langle\beta(\epsilon,1),\mathcal{D}_\epsilon\alpha(\epsilon,1)\rangle - \langle\beta(\epsilon,0),\mathcal{D}_\epsilon\alpha(\epsilon,0)\rangle\\ &\quad - \int_0^1\langle\mathcal{D}_t\beta,\mathcal{D}_t\beta\rangle dt + \int_0^1\langle\beta,R(\beta,\alpha)\alpha\rangle dt\end{aligned}$$

Hence

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\mathbf{E}(\alpha) &= \langle\mathcal{D}_\epsilon\beta(\epsilon,1),\alpha(\epsilon,1)\rangle - \langle\mathcal{D}_\epsilon\beta(\epsilon,0),\alpha(\epsilon,0)\rangle\\ &\quad - \int_0^1\langle\mathcal{D}_\epsilon\beta,\mathcal{D}_t\alpha\rangle dt + \int_0^1\langle\mathcal{D}_t\beta,\mathcal{D}_t\beta\rangle dt\\ &\quad + \int_0^1\langle\beta,R(\alpha,\beta)\alpha\rangle dt.\end{aligned}$$

2. In this situation, we have $\mathcal{D}_\epsilon\beta(\epsilon,1) = \mathcal{D}_\epsilon\beta(\epsilon,0) = \mathcal{D}_t\alpha = 0$ and the formula follows:3 and 4 are left to the reader. □

As an application of Proposition 3.11 2, we give now a description of the geodesics of a left invariant Riemannian metric on a Lie group using the geodesics of its Lie algebra considered as a Riemannian Lie algebroid.

Let G be a Lie group and $\mathcal{G} = T_eG$ its Lie algebra. For any $u\in\mathcal{G}$, we denote by u^+ the associated left invariant vector field on G. Suppose that G is endowed with a left invariant Riemannian metric g and put $\langle\ ,\ \rangle = g_e$. If we think $\mathcal{G}$ as a Lie algebroid, $(\mathcal{G},\langle\ ,\ \rangle)$ is a Riemannian Lie algebroid and we will explain how one can construct the geodesics of (G,g) from the geodesics of $(\mathcal{G},\langle\ ,\ \rangle)$. Choose a basis $(e_1,\ldots,e_n)$ of $\mathcal{G}$ and put $g_{ij} = \langle e_i,e_j\rangle$. Recall that the geodesics of $(\mathcal{G},\langle\ ,\ \rangle)$ are the integral curves of the geodesic vector field X_E given in the linear coordinates $(x_1,\ldots,x_n)$ associated to $(e_1,\ldots,e_n)$ by

$$X_E = -\sum_{s,t,j}x_sx_t\Gamma_{st}^j\partial_{x_j},$$

where Γ_{st}^j are given by

$$\Gamma_{st}^j = \frac{1}{2}\sum_{l,u}g^{lj}\left(g_{ul}C_{st}^u + g_{ut}C_{ls}^u + g_{us}C_{lt}^u\right).$$

Here (g^{ij}) is the inverse matrix of (g_{ij}) and C_{ij}^k are given by $[e_i,e_j] = \sum_u C_{ij}^u e_u$.

Proposition 3.13. *Let $h\in G$ and $v\in T_hG$. Then the geodesic $\gamma:\mathbb{R}\to G$ of (G,g) satisfying $\gamma(0)=h$ and $\dot\gamma(0)=v$ is the integral curve passing through h of the time-depending family of left invariant vector fields $(\alpha^+(t))_{t\in\mathbb{R}}$ where $\alpha:\mathbb{R}\to\mathcal{G}$ is the geodesic of $(\mathcal{G},\langle\ ,\ \rangle)$ satisfying $\alpha(0) = \left(L_{h^{-1}}\right)_*(v)$.*

Proof. Note first that by invariance the integral curves of $(\alpha^+(t))_{t\in\mathbb{R}}$ are complete. Note also that both (G,g) and $(\mathcal{G},\langle\ ,\ \rangle)$ are geodesically complete. Let $\gamma:\mathbb{R}\to G$ be the integral curve of $(\alpha^+(t))_{t\in\mathbb{R}}$ satisfying $\gamma(0)=h$. We have

$$\dot\gamma(0) = \alpha^+(0) = (L_h)_*(\alpha(0)) = (L_h\circ L_{h^{-1}})_*(v) = v.$$

We will show that for any $t_1,t_2\in\mathbb{R}$, the restriction of γ to $[t_1,t_2]$ is a critical point of the energy functional $\mathbf{E}_g:\Omega\to\mathbb{R}$ where Ω is the space of smooth curves $\mu:[t_1,t_2]\to G$ such that $\mu(t_1)=\gamma(t_1)$ and $\mu(t_2)=\gamma(t_2)$.

Let $\tilde\gamma:[0,1]\times[t_1,t_2]\to G$ be an homotopy with end-fixed points such that $\tilde\gamma(0,.)=\gamma$. It is well-known (see [7]) that the variation $\tilde\alpha:[0,1]\times[t_1,t_2]\to\mathcal{G}$ given by

$$\tilde\alpha(\epsilon,t) = \left(L_{\tilde\gamma(\epsilon,t)^{-1}}\right)_*\left(\frac{\partial\tilde\gamma}{\partial t}(\epsilon,t)\right)$$

is a $\mathcal{G}$-homotopy. Moreover, $\tilde\alpha(0,.)=\alpha$ and, by invariance, $\mathbf{E}_g(\tilde\gamma) = \mathbf{E}_{\langle\ ,\ \rangle}(\tilde\alpha)$. By applying Proposition 3.11, we get $\frac{d}{d\epsilon}\mathbf{E}_{\langle\ ,\ \rangle}(\tilde\alpha)_{|\epsilon=0}=0$. Thus, $\frac{d}{d\epsilon}\mathbf{E}_g(\tilde\gamma)_{|\epsilon=0}=0$ and, by applying the classical result on geodesics of Riemannian metric we deduce that γ is a geodesic. □

Remark 3.14. If the Riemannian metric g is bi-invariant then $\Gamma_{ij}^k = \frac{1}{2}C_{ij}^k$ and hence X_E vanishes identically. We deduce from Proposition 3.5 that the geodesic of (G,g) passing through $h\in G$ and with initial velocity $v\in T_hG$ is the integral curve (passing through h) of the left invariant vector field $(L_{h^{-1}*}(v))^+$.

Let us define now Jacobi sections along a geodesic.

Definition 3.15. Let A be a Riemannian Lie algebroid and $\alpha:[0,1]\to A$ a geodesic. A Jacobi α-section is an α-section β which satisfies

$$\beta'' - R(\alpha,\beta)\alpha = 0,$$

where β' is the derivative of β along α and so on.

Proposition 3.16. *Let $\alpha:[0,1]\to A$ be a geodesic in a Riemannian Lie algebroid A. Then for any $a,b\in A_{p(\alpha(0))}$ there exists one and only one Jacobi α-section such that $\beta(0)=a$ and $\beta'(0)=b$. If $\beta(0)=0$ and $\beta'(0)=k\alpha(0)$ then $\beta(t)=kt\alpha(t)$ for any t. If $\beta(0)$ and $\beta'(0)$ are orthogonal to $\alpha(0)$, then $\beta(t)$ is orthogonal to $\alpha(t)$ for any t. In particular the vector space of Jacobi α-sections has dimension $2r$ and the subspace of Jacobi α-sections which are normal to α has dimension $2(r-1)$.*

Proof. Take an orthonormal basis $(a_1,\ldots,a_r)$ of $A_{p(\alpha(0))}$ such that $a_1 = k\alpha(0)$. The parallel transport along α of the vectors a_i gives a basis of orthonormal α-sections $(s_1,\ldots,s_r)$ with $s_1 = k\alpha$. Every Jacobi α-section β is a linear combination of s_i, say $\beta = \sum_i y_is_i$, whose coefficients satisfy the differential system

$$y_i'' - \sum_{j=2}^{r} \langle R(\alpha, s_j)\alpha, s_i \rangle \, y_j = 0.$$

For given initial conditions $\beta(0) = a$ and $\beta\prime(0) = b$, the existence and uniqueness of β come from standard results about linear differential systems.

If $\beta(0) = 0$ and $\beta'(0) = k\alpha(0)$ then $\beta(t) = kt\alpha(t)$ since $\beta''(t) = 0$.

The condition $\beta(0)$ and $\beta'(0)$ to be orthogonal to α means that $y_1(0) = 0$ and $y_1'(0) = 0$. In that case $y_1(t) = 0$ for any t, since $y''(t) = 0$. □

Proposition 3.17. *Let $\alpha_0 : [0, 1] \to A$ be a geodesic, and α be a variation of α_0 such that all A-paths $\alpha(\epsilon, .)$ are geodesics. Then, for any transverse variation β of α such that $\Delta(\alpha, \beta) = 0$, β_0 is a Jacobi α_0-section. Conversely, every Jacobi α_0-section can be obtained in this way.*

Proof. We have

$$\beta_0''(t) = \mathcal{D}_t\mathcal{D}_t\beta(0, t).$$

Performing the two exchanges of t and ϵ, we get from Proposition 2.10

$$\beta_0''(t) = \mathcal{D}_t\mathcal{D}_\epsilon\alpha(0, t) = \mathcal{D}_\epsilon\mathcal{D}_t\alpha(0, t) + R(\alpha_0, \beta_0)\alpha_0.$$

Since the A-paths α_ϵ are geodesics, the first term vanishes and we get

$$\beta_0'' = R(\alpha_0, \beta_0)\alpha_0.$$

Conversely, take a Jacobi α_0-section b and the geodesic c such that $c(0) = b(0)$. Take parallel sections s_0 and s_1 along c such that $s_0(0) = \alpha_0(0)$ and $s_1(0) = b'(0)$. Set

$$s(\epsilon) = s_0(\epsilon) + \epsilon s_1(\epsilon) \quad \text{and} \quad \alpha(\epsilon, t) = \phi_t(s(\epsilon)),$$

where ϕ_t is the geodesic flow. Consider the transverse variation β to α such that $\beta(\epsilon, 0) = c(\epsilon)$ and $\Delta(\alpha, \beta) = 0$. We will show that $\beta(0, .)$ and b coincide. Remark first that these two α_0-sections satisfy the same differential equation namely

$$y'' - R(\alpha_0, y)\alpha_0 = 0.$$

Since $b(0) = \beta(0, 0) = c(0)$, let us show that $\mathcal{D}_t\beta(0, 0) = b'(0)$. Since $\mathcal{D}_t\beta = \mathcal{D}_\epsilon\alpha$, we have $\mathcal{D}_\epsilon\alpha(0, 0)$ is the value at 0 of the derivative of the curve $\alpha(\epsilon, 0)$ along the A-path $\beta(\epsilon, 0)$. Or $\alpha(\epsilon, 0) = s(\epsilon)$ and $\beta(\epsilon, 0) = c(\epsilon)$ and we get $\mathcal{D}_\epsilon\alpha(0, 0) = s_1(0) = b'(0)$. □

As the classical case, the Jacobi sections can be used to compute the derivative of the exponential which can be defined as follows. Let $p : A \to M$ be a Riemannian Lie algebroid. Fix a point $m \in M$ and denote by L the leaf containing m. We define the exponential

$$exp_m : \mathbb{U} \subset A_m \to L$$

where $\mathbb{U}_m = \{\alpha \in A_m, \phi_1(\alpha) \text{ is defined}\}$ and $exp_m(\alpha) = p \circ \phi_1(\alpha)$ (ϕ is the geodesic flow).

Proposition 3.18. *We have*

$$d_a exp_m(u) = \#(\beta(1))$$

where β is the Jacobi section along $t \mapsto \phi_t(a)$ with initial condition $\beta(0) = 0$ and $\beta'(0) = u$.

Proof. We have

$$d_a exp_m(u) = \frac{d}{d\epsilon_{|\epsilon=0}} p(\phi_1(a + \epsilon u)).$$

We consider the variation of geodesics $\alpha(\epsilon, t) = \phi_t(a + \epsilon u)$ with fixed initial point. We consider the transverse variation β such that $\beta(\epsilon, 0) = 0$ and $\Delta(\alpha, \beta) = 0$. We have that β_0 is a Jacobi α_0-section such that $\beta_0(0) = 0$ and $\#(\beta_0(1)) = \frac{d}{d\epsilon_{|\epsilon=0}} p(\phi_1(a + \epsilon u))$ by construction. □

As the classical case, we define the sectional curvature of two linearly independent vectors $a, b \in A_m$ by

$$K(a, b) = -\frac{\langle R(a, b)a, b \rangle}{\langle a, a \rangle \langle b, b \rangle - \langle a, b \rangle^2}.$$

Proposition 3.19. *Let $p : A \to M$ be a Riemannian Lie algebroid. If the sectional curvature is everywhere nonpositive then exp_m is a submersion for every $m \in M$.*

Proof. Fix $a \in A_m$ and let $\mathcal{J}_0^\alpha$ be the space of Jacobi sections β along $\alpha(t) = \phi_t(a)$ such that $\beta(0) = 0$ (ϕ is the geodesic flow). We define the linear application

$$\xi : \mathcal{J}_0^\alpha \to A_{p(\phi_1(\alpha_0))}$$

by $\xi(\beta) = \beta(1)$. We will show that ξ is injective and hence an isomorphism since $\dim\mathcal{J}_0^\alpha = \dim A_{p(\phi_1(\alpha_0))}$. Suppose that $\beta \in \mathcal{J}_0^\alpha$ satisfies $\beta(1) = 0$. The function $f : [0, 1] \to \mathbb{R}$ given by $f(t) = \langle \beta(t), \beta(t) \rangle$ satisfies

$$\begin{aligned} f'(t) &= 2\langle \beta'(t), \beta(t) \rangle, \\ f''(t) &= 2\langle \beta'(t), \beta'(t) \rangle + 2\langle \beta''(t), \beta(t) \rangle \\ &= 2\langle \beta'(t), \beta'(t) \rangle + 2\langle R(\alpha(t), \beta(t))\alpha(t), \beta(t) \rangle. \end{aligned}$$

Hence $f'' \geqslant 0$ and since $f(0) = f(1) = 0$ we deduce that f vanishes identically and then $\beta = 0$. This shows that ξ is injective and hence an isomorphism. From Proposition 3.18, one can identify $\mathrm{Ker} d_a exp_m$ with $\xi^{-1}(\mathcal{G}_{p(\phi_1(a))})$ and the proposition follows. □

4. O'Neill's formulas for curvature

Let $p : A \to M$ be a Riemannian Lie algebroid. The different curvatures (sectional curvature, Ricci curvature and scalar curvature) can be defined as the classical case (when $A = TM$). For any leaf L, the short exact sequence

$$0 \to \mathcal{G}_L \to A_L \to TL$$

is formally identical to a Riemannian submersion and hence all formulas on curvature given by O'Neill are valid in this context. We denote by K, $\hat{K}$ and $\widetilde{K}$, respectively, the sectional curvature of the Riemannian metrics $\langle \, , \, \rangle$, the restriction of $\langle \, , \, \rangle$ to $\mathcal{G}_L$ and the induced metric on L. The following proposition is a reformulation of Corollary 9.29 pp. 241 in [1].

Proposition 4.1. *Let $\alpha, \beta, s_1, s_2 \in \Gamma(A_L)$ such that α, β are vertical, s_1, s_2 are horizontal and $|\alpha \wedge \beta| = 1$, $|s_1| = |\alpha| = 1$, $|s_1 \wedge s_2| = 1$. Then*

$$\begin{aligned} K(\alpha, \beta) &= \widehat{K}(\alpha, \beta) + |T_\alpha\beta|^2 - \langle T_\alpha\alpha, T_\beta\beta \rangle, \\ K(s_1, \alpha) &= \langle (\mathcal{D}_{s_1} T)_\alpha\alpha, s_1 \rangle - |T_\alpha s_1| + |H_{s_1}\alpha|^2, \\ K(s_1, s_2) &= \widetilde{K}(s_1, s_2) - 3|H_{s_1}s_2|^2. \end{aligned}$$

The last formula says that the leaves carry "more curvature" than the Lie algebroid and by applying Mayer theorem (see for instance [1]) we get:

Proposition 4.2. *Let $A \to M$ be a complete Riemannian algebroid and let L be a leaf of the characteristic foliation such that for any linearly independent horizontal sections s_1, s_2 over L, $K(s_1, s_2) \geqslant k$. Then $diam L \leqslant \frac{\pi}{\sqrt{k}}$ and hence L is compact.*

There is another case when one can apply Mayer theorem. Consider a Riemannian Lie algebroid $p: A \to M$ such that the O'Neill tensor T vanishes and fix a leaf L and denote by r and $\tilde{r}$ respectively the Ricci curvature of the Riemannian metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_L$. The formula 9.36c pp.244 in [1] applies in our context and gives

$$r(s_1, s_2) = \tilde{r}(\#(s_1), \#(s_2)) - 2\sum_{i=1}^{l}\langle H_{s_1}a_i, H_{s_2}a_i\rangle$$

where $(a_1, \dots, a_l)$ is any orthonormal basis of $\mathcal{G}_L^{\perp}$. By applying Mayer theorem (see for instance [1]) we get:

Proposition 4.3. *Let $A \to M$ be a complete Riemannian algebroid such that $T = 0$ and let L be a leaf of the characteristic foliation such that there exists a constant k such that the restriction of r to $\mathcal{G}_L^{\perp}$ satisfies*

$$r \geqslant (n-1)k^{-2}\langle\ ,\ \rangle.$$

Then $diam L \leqslant \frac{\pi}{\sqrt{k}}$ and hence L is compact.

5. Integrability of Riemannian Lie algebroids

In this section, we show that a Riemannian Lie algebroid such that the O'Neill tensor H vanishes is integrable and we give a large class of Riemannian Lie algebroids which satisfy this condition.

A groupoid is a small category $\mathcal{C}$ in which all the arrows are invertible. We shall write M for the set of objects of $\mathcal{C}$, while the set of arrows of $\mathcal{C}$ will be denoted by $\mathcal{C}$. We shall often identify M with the subset of units of $\mathcal{C}$. The structure maps of $\mathcal{C}$ will be denoted as follows: $\mathbf{s}, \mathbf{t}: \mathcal{C} \to M$ will stand for the source map, respectively the target map, $m: \mathcal{C}^2 = \{(g,h); \mathbf{s}(g) = \mathbf{t}(h)\} \to \mathcal{C}$ the multiplication map ($m(g,h) = gh$), $i: \mathcal{C} \to \mathcal{C}_1$ ($i(g) = g^{-1}$) for the inverse map and $u: M \to \mathcal{C}$ ($u(x) = 1_x$) for the unit map. Given $g \in \mathcal{C}$, the right multiplication by g is only defined on the $\mathbf{s}$-fiber at $\mathbf{t}(g)$, and induces a bijection

$$R_g: \mathbf{s}^{-1}(\mathbf{t}(g)) \to \mathbf{s}^{-1}(\mathbf{s}(g)).$$

A Lie groupoid is a groupoid $\mathcal{C}$, equipped with the structure of smooth manifold both on the $\mathcal{C}$ and on the M such that all the structure maps are smooth and $\mathbf{s}$ and $\mathbf{t}$ are submersions.

The construction of a Lie algebra of a given Lie group extends to Lie groupoids. Explicitly, if $\mathcal{C}$ is a Lie groupoid, the vector bundle $T^{\mathbf{s}}\mathcal{C} = \mathrm{Ker}(d\mathbf{s})$ over $\mathcal{C}$ of $\mathbf{s}$-vertical tangent vectors pulls back along $i: M \to \mathcal{C}$ to a vector bundle A over M. This vector bundle has the structure of a Lie algebroid. Its anchor $\#: A \to TM$ is induced by the differential of the target map, $d\mathbf{t}: T\mathcal{C} \to TM$. The sections of A over M can be identified by the space of right invariant $\mathbf{s}$-vertical vector fields which induce a Lie bracket on the space of sections of A. With this construction in mind, one can see that a Riemannian structure on A is equivalent to the data of a Riemannian metric on any $\mathbf{s}$-fiber such that, for any $g \in \mathcal{C}$, $R_g: \mathbf{s}^{-1}(\mathbf{t}(g)) \to \mathbf{s}^{-1}(\mathbf{s}(g))$ is an isometry. In this case, for any $x \in M$, $\mathbf{t}: \mathbf{s}^{-1}(x) \to L_x$ is a Riemannian submersion where the leaf L_x is endowed with the metric defined in 3.1.

A Lie algebroid A is called integrable if it is isomorphic to the Lie algebroid associated to a Lie groupoid. In [7], Crainic and Fernandes give a final solution to the problem of integrability of Lie algebroids. They show that the obstruction to integrability can be controlled by two computable quantities.

The following proposition is a direct application of Crainic–Fernandes results on integrability.

Proposition 5.1. *Let $p: A \to M$ be a Riemannian Lie algebroid such that $H = 0$. Then A is integrable.*

Proof. For any leaf L, the vanishing of H implies, according to (13), that the space of sections of $\mathcal{G}_L^{\perp} \to L$ is a Lie subalgebra of $\Gamma(A_L)$ and hence there is a splitting $\sigma: TL \to A_L$ of the anchor, which is compatible with the Lie bracket. By applying Corollary 5.2 in [7], we get the result. $\square$

There is a large class of Lie algebroids for which one can apply this result. Let (M, π) be a Poisson manifold. The cotangent bundle T^*M carries a structure of a Lie algebroid where the anchor is the contraction by π, $\pi_{\#}: T^*M \to TM$ and the Lie bracket is given by the Koszul bracket

$$[\alpha, \beta] = \mathcal{L}_{\pi_{\#}(\alpha)}\beta - \mathcal{L}_{\pi_{\#}(\beta)}\alpha - d\pi(\alpha, \beta)$$

where $\alpha, \beta \in \Omega^1(M)$. Let $\langle\ ,\ \rangle$ be a Riemannian structure in T^*M. In [3], the author studied the triple $(M, \pi, \langle\ ,\ \rangle)$ such that π is parallel with respect the Levi-Civita T^*M-connection $\mathcal{D}$. A triple $(M, \pi, \langle\ ,\ \rangle)$ satisfying $\mathcal{D}\pi = 0$ is called Riemann–Poisson manifold. The condition $\mathcal{D}\pi = 0$ implies that $\mathrm{Ker}\pi_{\#}$ is invariant by parallel transport and hence $\mathcal{D}$ is strongly compatible with the Lie algebroid structure of T^*M. By using Proposition 3.3, we deduce that $H = 0$. So we get the following result.

Corollary 5.2. *Let $(M, \pi, \langle\ ,\ \rangle)$ be a Riemann–Poisson manifold. Then the Lie algebroid structure of T^*M associated to π is integrable.*

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