

On the Riemann-Lie Algebras and Riemann-Poisson Lie Groups

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Abstract. A Riemann-Lie algebra is a Lie algebra \mathcal{G} such that its dual \mathcal{G}^* carries a Riemannian metric compatible (in the sense introduced by the author in C. R. Acad. Sci. Paris, t. 333, Série I, (2001) 763–768) with the canonical linear Poisson structure of \mathcal{G}^* . The notion of Riemann-Lie algebra has its origins in the study, by the author, of Riemann-Poisson manifolds (see Differential Geometry and its Applications, Vol. 20, Issue 3(2004), 279–291).

In this paper, we show that, for a Lie group G , its Lie algebra \mathcal{G} carries a structure of Riemann-Lie algebra iff G carries a flat left-invariant Riemannian metric. We use this characterization to construct examples of Riemann-Poisson Lie groups (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure).

1. Introduction

Riemann-Lie algebras first arose in the study by the author of Riemann-Poisson manifolds (see [2]). A Riemann-Poisson manifold is a Poisson manifold (P, π) endowed with a Riemannian metric \langle, \rangle such that the couple (π, \langle, \rangle) is compatible in the sense introduced by the author in [1]. The notion of Riemann-Lie algebra appeared when we looked for the Riemannian metrics compatible with the canonical Poisson structure on the dual of a Lie algebra. We pointed out (see [2]) that the dual space \mathcal{G}^* of a Lie algebra \mathcal{G} carries a Riemannian metric compatible with the linear Poisson structure iff the Lie algebra \mathcal{G} carries a structure which we called Riemann-Lie algebra. Moreover, the isotropy Lie algebra at a point on a Riemann-Poisson manifold is a Riemann-Lie algebra. In particular, the dual Lie algebra of a Riemann-Poisson Lie group is a Riemann-Lie algebra (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure). In this paper, we will show that a Lie algebra \mathcal{G} carries a structure of Riemann-Lie algebra iff \mathcal{G} is a semi-direct

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product of two abelian Lie algebras. Hence, according to a well-known result of Milnor [5], we deduce that, for a Lie group G , its Lie algebra carries a structure of Riemann-Lie algebra iff G carries a flat left-invariant Riemannian metric. We apply this geometrical characterization to construct examples of Riemann-Poisson Lie groups. In particular, we give many examples of bialgebras $(\mathcal{G}, [\cdot, \cdot], \mathcal{G}^*, [\cdot, \cdot]^*)$ such that both $(\mathcal{G}, [\cdot, \cdot])$ and $(\mathcal{G}^*, [\cdot, \cdot]^*)$ are Riemann-Lie algebras.

2. Definitions and main results

Notations. Let G be a connected Lie group and $(\mathcal{G}, [\cdot, \cdot])$ its Lie algebra. For any $u \in \mathcal{G}$, we denote by u^l (resp. u^r) the left-invariant (resp. right-invariant) vector field of G corresponding to u . We denote by θ the right-invariant Maurer-Cartan form on G given by

$$\theta(u^r) = -u, \quad u \in \mathcal{G}. \quad (1)$$

Let $\langle \cdot, \cdot \rangle$ be a scalar product on \mathcal{G} i.e. a bilinear, symmetric, non-degenerate and positive definite form on \mathcal{G} .

Let us enumerate some mathematical objects which are naturally associated with $\langle \cdot, \cdot \rangle$:

1. an isomorphism $\# : \mathcal{G}^* \longrightarrow \mathcal{G}$;
2. a scalar product $\langle \cdot, \cdot \rangle^*$ on the dual space \mathcal{G}^* by

$$\langle \alpha, \beta \rangle^* = \langle \#(\alpha), \#(\beta) \rangle \quad \alpha, \beta \in \mathcal{G}^*; \quad (2)$$

3. a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle^l$ on G by

$$\langle u^l, v^l \rangle^l = \langle u, v \rangle \quad u, v \in \mathcal{G}; \quad (3)$$

4. a left-invariant contravariant Riemannian metric $\langle \cdot, \cdot \rangle^{*l}$ on G by

$$\langle \alpha, \beta \rangle_g^{*l} = \langle T_e^* L_g(\alpha), T_e^* L_g(\beta) \rangle^* \quad (4)$$

where $\alpha, \beta \in \Omega^1(G)$ and L_g is the left translation of G by g .

The infinitesimal Levi-Civita connection associated with $(\mathcal{G}, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is the bilinear map $A : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ given by

$$2\langle A_u v, w \rangle = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle, \quad u, v, w \in \mathcal{G}. \quad (5)$$

Note that A is the unique bilinear map from $\mathcal{G} \times \mathcal{G}$ to \mathcal{G} which verifies:

1. $A_u v - A_v u = [u, v]$;
2. for any $u \in \mathcal{G}$, $A_u : \mathcal{G} \longrightarrow \mathcal{G}$ is skew-adjoint i.e.

$$\langle A_u v, w \rangle + \langle v, A_u w \rangle = 0, \quad v, w \in \mathcal{G}.$$

The terminology used here is motivated by the fact that the Levi-Civita connection ∇ associated with $(G, \langle \cdot, \cdot \rangle^l)$ is given by

$$\nabla_{u^l} v^l = (A_u v)^l \quad u, v \in \mathcal{G}. \quad (6)$$

We will introduce now a Lie subalgebra of \mathcal{G} which will play a crucial role in this paper.

For any $u \in \mathcal{G}$, we denote by $ad_u : \mathcal{G} \rightarrow \mathcal{G}$ the endomorphism given by $ad_u(v) = [u, v]$, and by $ad_u^t : \mathcal{G} \rightarrow \mathcal{G}$ its adjoint given by

$$\langle ad_u^t(v), w \rangle = \langle v, ad_u(w) \rangle \quad v, w \in \mathcal{G}.$$

The space

$$S_{\langle, \rangle} = \{u \in \mathcal{G}; ad_u + ad_u^t = 0\} \quad (7)$$

is obviously a subalgebra of \mathcal{G} . We call $S_{\langle, \rangle}$ the orthogonal subalgebra associated with $(\mathcal{G}, [,], \langle, \rangle)$.

Remark 2.1. The scalar product \langle, \rangle is bi-invariant if and only if $\mathcal{G} = S_{\langle, \rangle}$. In this case \mathcal{G} is the product of an abelian Lie algebra and a semi-simple and compact Lie algebra (see [5]). The general case where \langle, \rangle is not positive definite has been studied by A. Medina and P. Revoy in [4] and they called a Lie algebra \mathcal{G} with an inner product \langle, \rangle such that $\mathcal{G} = S_{\langle, \rangle}$ an orthogonal Lie algebra which justifies the terminology used here.

Let $(\mathcal{G}, [,], \langle, \rangle)$ be a Lie algebra endowed with a scalar product.

The triple $(\mathcal{G}, [,], \langle, \rangle)$ is called a Riemann-Lie algebra if

$$[A_u v, w] + [u, A_w v] = 0 \quad (8)$$

for all $u, v, w \in \mathcal{G}$ and where A is the infinitesimal Levi-Civita connection associated to $(\mathcal{G}, [,], \langle, \rangle)$.

From the relation $A_u v - A_v u = [u, v]$ and the Jacobi identity, we deduce that (8) is equivalent to

$$[u, [v, w]] = [A_u v, w] + [v, A_u w] \quad (9)$$

for any $u, v, w \in \mathcal{G}$. We refer the reader to [2] for the origins of this definition.

Briefly, if $(\mathcal{G}, [,], \langle, \rangle)$ is a Lie algebra endowed with a scalar product. The scalar product \langle, \rangle defines naturally a contravariant Riemannian metric on the Poisson manifold \mathcal{G}^* which we denote also by \langle, \rangle . If we denote by π^l the canonical Poisson tensor on \mathcal{G}^* , $(\mathcal{G}^*, \pi^l, \langle, \rangle)$ is a Riemann-Poisson manifold iff the triple $(\mathcal{G}, [,], \langle, \rangle)$ is a Riemann-Lie algebra.

Characterization of Riemann-Lie algebras. With the notations and the definitions above, we can state now the main result of this paper.

Theorem 2.2. Let G be a Lie group, $(\mathcal{G}, [,], \langle, \rangle)$ its Lie algebra and \langle, \rangle a scalar product on \mathcal{G} . Then, the following assertions are equivalent:

- 1) $(\mathcal{G}, [,], \langle, \rangle)$ is a Riemann-Lie algebra.
- 2) $(\mathcal{G}^*, \pi^l, \langle, \rangle)$ is a Riemann-Poisson manifold (π^l is the canonical Poisson tensor on \mathcal{G}^* and \langle, \rangle is considered as a contravariant metric on \mathcal{G}^*).
- 3) The 2-form $d\theta \in \Omega^2(G, \mathcal{G})$ is parallel with respect the Levi-Civita connection ∇ i.e. $\nabla d\theta = 0$.
- 4) (G, \langle, \rangle^l) is a flat Riemannian manifold.
- 5) The orthogonal subalgebra $S_{\langle, \rangle}$ of $(\mathcal{G}, [,], \langle, \rangle)$ is abelian and \mathcal{G} split as an orthogonal direct sum $S_{\langle, \rangle} \oplus \mathcal{U}$ where \mathcal{U} is a commutative ideal.

The equivalence “1) \Leftrightarrow 2)” of this theorem was proven in [2] and the equivalence “4) \Leftrightarrow 5)” was proven by Milnor in [5]. We will prove the equivalence “1) \Leftrightarrow 3)” and the equivalence “1) \Leftrightarrow 5)”.

The equivalence “1) \Leftrightarrow 3)” is a direct consequence of the following formula which it is easy to verify:

$$\nabla d\theta(u^l, v^l, w^l)_g = Ad_g([u, [v, w]] - [A_u v, w] - [v, A_u w]), \quad u, v, w \in \mathcal{G}, g \in G. \tag{10}$$

If G is compact and connected, the condition $\nabla d\theta = 0$ implies that $d\theta$ is harmonic and, according to the Hodge Theorem must vanishes since it is exact. Now, the vanishing of $d\theta$ is equivalent to G being abelian and hence we get the following lemma which will be used in the proof of the equivalence “1) \Leftrightarrow 5)” in Section 3.

Lemma 2.3. *Let G be a compact, connected and non abelian Lie group. Then the Lie algebra of G does not admit any structure of Riemann-Lie algebra.*

A proof of the equivalence “1) \Leftrightarrow 5)” will be given in Section 3.

Examples of Riemann-Poisson Lie groups. This subsection is devoted to the construction, using Theorem 2.2, of some examples of Riemann-Poisson Lie groups. A Riemann-Poisson Lie group is a Poisson Lie group with a left-invariant Riemannian metric compatible with the Poisson structure (see [2]).

We refer the reader to [6] for background material on Poisson Lie groups.

Let G be a Poisson Lie group with Lie algebra \mathcal{G} and π the Poisson tensor on G . Pulling π back to the identity element e of G by left translations, we get a map $\pi_l : G \rightarrow \mathcal{G} \wedge \mathcal{G}$ defined by $\pi_l(g) = (L_{g^{-1}})_* \pi(g)$ where $(L_g)_*$ denotes the tangent map of the left translation of G by g . Let

$$d_e \pi : \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{G}$$

be the intrinsic derivative of π at e given by

$$v \mapsto L_X \pi(e),$$

where X can be any vector field on G with $X(e) = v$.

The dual map of $d_e \pi$

$$[\ , \]_e : \mathcal{G}^* \wedge \mathcal{G}^* \rightarrow \mathcal{G}^*$$

is exactly the Lie bracket on \mathcal{G}^* obtained by linearizing the Poisson structure at e . The Lie algebra $(\mathcal{G}^*, [\ , \]_e)$ is called the dual Lie algebra associated with the Poisson Lie group (G, π) .

We consider now a scalar product $\langle \ , \ \rangle^*$ on \mathcal{G}^* . We denote by $\langle \ , \ \rangle^{*l}$ the left-invariant contravariant Riemannian metric on G given by (4).

We have shown (cf. [2] Lemma 5.1) that $(G, \pi, \langle \ , \ \rangle^{*l})$ is a Riemann-Poisson Lie group iff, for any $\alpha, \beta, \gamma \in \mathcal{G}^*$ and for any $g \in G$,

$$[Ad_g^*(A_\alpha^* \gamma + ad_{\pi_l(g)(\alpha)}^* \gamma), Ad_g^*(\beta)]_e + [Ad_g^*(\alpha), Ad_g^*(A_\beta^* \gamma + ad_{\pi_l(g)(\beta)}^* \gamma)]_e = 0, \tag{11}$$

where $A^* : \mathcal{G}^* \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is the infinitesimal Levi-Civita connection associated to $(\mathcal{G}^*, [\ , \]_e, \langle \ , \ \rangle^*)$ and where $\pi_l(g)$ also denotes the linear map from \mathcal{G}^* to \mathcal{G} induced by $\pi_l(g) \in \mathcal{G} \wedge \mathcal{G}$.

This complicated equation can be simplified enormously in the case where the Poisson tensor arises from a solution of the classical Yang-Baxter equation. However, we need to work more in order to give this simplification.

Let G be a Lie group and let $r \in \mathcal{G} \wedge \mathcal{G}$. We will also denote by $r : \mathcal{G}^* \longrightarrow \mathcal{G}$ the linear map induced by r . Define a bivector π on G by

$$\pi(g) = (L_g)_*r - (R_g)_*r \quad g \in G.$$

(G, π) is a Poisson Lie group if and only if the element $[r, r] \in \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$ defined by

$$[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)]) \quad (12)$$

is *ad*-invariant. In particular, when r satisfies the Yang-Baxter equation

$$[r, r] = 0, \quad (Y - B)$$

it defines a Poisson Lie group structure on G and, in this case, the bracket of the dual Lie algebra \mathcal{G}^* is given by

$$[\alpha, \beta]_e = ad_{r(\beta)}^* \alpha - ad_{r(\alpha)}^* \beta, \quad \alpha, \beta \in \mathcal{G}^*. \quad (13)$$

We will denote by $[\]_r$ this bracket.

We will give now another description of the solutions of the Yang-Baxter equation which will be useful latter.

We observe that to give $r \in \mathcal{G} \wedge \mathcal{G}$ is equivalent to give a vectorial subspace $S_r \subset \mathcal{G}$ and a non-degenerate 2-form $\omega_r \in \wedge^2 S_r^*$.

Indeed, for $r \in \mathcal{G} \wedge \mathcal{G}$, we put $S_r = Im r$ and $\omega_r(u, v) = r(r^{-1}(u), r^{-1}(v))$ where $u, v \in S_r$ and $r^{-1}(u)$ is any antecedent of u by r .

Conversely, let (S, ω) be a vectorial subspace of \mathcal{G} with a non-degenerate 2-form. The 2-form ω defines an isomorphism $\omega^b : S \longrightarrow S^*$ by $\omega^b(u) = \omega(u, \cdot)$, we denote by $\# : S^* \longrightarrow S$ its inverse and we put

$$r = \# \circ i^*$$

where $i^* : \mathcal{G}^* \longrightarrow S^*$ is the dual of the inclusion $i : S \hookrightarrow \mathcal{G}$.

With this observation in mind, the following proposition gives another description of the solutions of the Yang-Baxter equation.

Proposition 2.4. *Let $r \in \mathcal{G} \wedge \mathcal{G}$ and (S_r, ω_r) its associated subspace. The following assertions are equivalent:*

- 1) $[r, r] = 0$.
- 2) $r([\alpha, \beta]_r) = [r(\alpha), r(\beta)]$. ($[\]_r$ is the bracket given by (13)).
- 3) S_r is a subalgebra of \mathcal{G} and

$$\delta\omega_r(u, v, w) := \omega_r(u, [v, w]) + \omega_r(v, [w, u]) + \omega_r(w, [u, v]) = 0$$

for any $u, v, w \in S_r$.

Proof. The proposition follows from the following formulas:

$$\gamma(r([\alpha, \beta]_r) - [r(\alpha), r(\beta)]) = -[r, r](\alpha, \beta, \gamma), \quad \alpha, \beta, \gamma \in \mathcal{G}^* \quad (14)$$

and, if S is a subalgebra,

$$[r, r](\alpha, \beta, \gamma) = -\delta\omega_r(r(\alpha), r(\beta), r(\gamma)). \quad (15) \quad \blacksquare$$

This proposition shows that to give a solution of the Yang-Baxter equation is equivalent to give a symplectic subalgebra of \mathcal{G} . We recall that a symplectic algebra (see [3]) is a Lie algebra S endowed with a non-degenerate 2-form ω such that $\delta\omega = 0$.

Remark 2.5. Let G be a Lie group, \mathcal{G} its Lie algebra and S an even dimensional abelian subalgebra of \mathcal{G} . Any non-degenerate 2-form ω on S verifies the assertion 3) in Proposition 2.1 and hence (S, ω) defines a solution of the Yang-Baxter equation and then a structure of Poisson Lie group on G .

The following proposition will be crucial in the simplification of the equation (11).

Proposition 2.6. Let $(\mathcal{G}, [\ , \], \langle \cdot, \cdot \rangle)$ be a Lie algebra endowed with a scalar product, $r \in \mathcal{G} \wedge \mathcal{G}$ a solution of $(Y - B)$ and (S_r, ω_r) its associated symplectic Lie algebra. Then, $S_r \subset S_{\langle \cdot, \cdot \rangle}$ iff the infinitesimal Levi-Civita connection A^* associated with $(\mathcal{G}^*, [\ , \], \langle \cdot, \cdot \rangle^*)$ is given by

$$A_\alpha^* \beta = -ad_{r(\alpha)}^* \beta, \quad \alpha, \beta \in \mathcal{G}^*. \quad (16)$$

Moreover, if $S_r \subset S_{\langle \cdot, \cdot \rangle}$, the curvature of A^* vanishes and hence $(\mathcal{G}^*, [\ , \], \langle \cdot, \cdot \rangle^*)$ is a Riemann-Lie algebra.

Proof. A^* is the unique bilinear map from $\mathcal{G}^* \times \mathcal{G}^*$ to \mathcal{G}^* such that:

- 1) $A_\alpha^* \beta - A_\beta^* \alpha = [\alpha, \beta]_r$ for any $\alpha, \beta \in \mathcal{G}^*$;
- 2) the endomorphism $A_\alpha^* : \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is skew-adjoint with respect to $\langle \cdot, \cdot \rangle^*$.

The bilinear map $(\alpha, \beta) \mapsto -ad_{r(\alpha)}^* \beta$ verifies 1) obviously and verifies 2) iff $S_r \subset S_{\langle \cdot, \cdot \rangle}$.

If $A_\alpha^* \beta = -ad_{r(\alpha)}^* \beta$, the curvature of A^* is given by

$$R(\alpha, \beta)\gamma = A_{[\alpha, \beta]_r}^* \gamma - (A_\alpha^* A_\beta^* \gamma - A_\beta^* A_\alpha^* \gamma) = ad_{r([\alpha, \beta]_r) - [r(\alpha), r(\beta)]}^* \gamma = 0$$

from Proposition 2.4 2). We conclude by Theorem 2.2. \blacksquare

Proposition 2.7. Let $(\mathcal{G}, [\ , \], \langle \cdot, \cdot \rangle)$ be a Lie algebra with a scalar product. Let $r \in \mathcal{G} \wedge \mathcal{G}$ be a solution of $(Y - B)$ such that S_r is a subalgebra of the orthogonal subalgebra $S_{\langle \cdot, \cdot \rangle}$. Then S_r is abelian.

Proof. S_r is unimodular and symplectic and then solvable (see [3]). Also S_r carries a bi-invariant scalar product so S_r must be abelian (see [5]). ■

We can now simplify the equation (11) and give the construction of Riemann-Poisson Lie groups announced before.

Let G be a Lie group, $(\mathcal{G}, [\ , \])$ its Lie algebra and \langle , \rangle a scalar product on \mathcal{G} . We assume that the orthogonal subalgebra $S_{\langle , \rangle}$ contains an abelian even dimensional subalgebra S endowed with a non-degenerate 2-form ω .

As in Remark 2.5, (S, ω) defines a solution r of $(Y - B)$ and then a Poisson Lie tensor π on G . It is easy to see that, for any $g \in G$,

$$\pi^l(g) = r - Ad_g(r).$$

This implies that (11) can be rewritten

$$\begin{aligned} [Ad_g^*(A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*(A_\beta^*\gamma + ad_{r(\beta)}^*\gamma)]_r = \\ [Ad_g^*(ad_{Ad_g(r)(\alpha)}^*\gamma), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*(ad_{Ad_g(r)(\beta)}^*\gamma)]_r. \end{aligned}$$

Now, since $S \subset S_{\langle , \rangle}$, we have by Proposition 2.6

$$A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma = 0$$

for any $\alpha, \gamma \in \mathcal{G}^*$. On other hand, it is easy to get the formula

$$Ad_g^*[ad_{r(\alpha)}^*\beta] = ad_{(Ad_{g^{-1}r})(Ad_g^*\alpha)}^*(Ad_g^*\beta), \quad g \in G, \alpha, \beta \in \mathcal{G}^*.$$

Finally, $(G, \pi, \langle , \rangle^{*l})$ is a Riemann-Poisson Lie group iff

$$[ad_{r(\alpha)}^*\gamma, \beta]_r + [\alpha, ad_{r(\beta)}^*\gamma]_r = 0, \quad \alpha, \beta, \gamma \in \mathcal{G}^*.$$

But, also since $A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma = 0$, this condition is equivalent to $(\mathcal{G}^*, [\ , \]_r, \langle , \rangle^*)$ is a Riemann-Lie algebra which is true by Proposition 2.6. So, we have shown:

Theorem 2.8. *Let G be a Lie group, $(\mathcal{G}, [\ , \])$ its Lie algebra and \langle , \rangle a scalar product on \mathcal{G} . Let S be an even dimensional abelian subalgebra of the orthogonal subalgebra $S_{\langle , \rangle}$ and ω a non-degenerate 2-form on S . Then, the solution of the Yang-Baxter equation associated with (S, ω) defines a structure of Poisson Lie group (G, π) and $(G, \pi, \langle , \rangle^{*l})$ is a Riemann-Poisson Lie group.*

Let us enumerate some important cases where this theorem can be used.

1) Let G be a compact Lie group and \mathcal{G} its Lie algebra. For any bi-invariant scalar product \langle , \rangle on the Lie algebra \mathcal{G} , $S_{\langle , \rangle} = \mathcal{G}$. By Theorem 2.8, we can associate to any even dimensional abelian subalgebra of \mathcal{G} a Riemann-Poisson Lie group structure on G .

2) Let $(\mathcal{G}, [\ , \], \langle , \rangle)$ be a Riemann-Lie algebra. By Theorem 2.2, the orthogonal subalgebra $S_{\langle , \rangle}$ is abelian and any even dimensional subalgebra of $S_{\langle , \rangle}$ gives rise to a structure of a Riemann-Poisson Lie group on any Lie group whose the Lie algebra is \mathcal{G} . Moreover, we get a structure of bialgebra $(\mathcal{G}, [\ , \], \mathcal{G}^*, [\ , \]_r)$ where both \mathcal{G} and \mathcal{G}^* are Riemann-Lie algebras.

Finally, we observe that the Riemann-Lie groups constructed above inherit the properties of Riemann-Poisson manifolds (see [2]). Namely, the symplectic leaves of these Poisson Lie groups are Kählerian and their Poisson structures are unimodular.

3. Proof of the equivalence “1) \Leftrightarrow 5)” in Theorem 2.2

In this section we will give a proof of the equivalence “1) \Leftrightarrow 5)” in Theorem 2.2. The proof is a sequence of lemmas. Namely, we will show that, for a Riemann-Lie algebra $(\mathcal{G}, [,], \langle , \rangle)$, the orthogonal subalgebra $S_{\langle , \rangle}$ is abelian. Moreover, $S_{\langle , \rangle}$ is the \langle , \rangle -orthogonal of the ideal $[\mathcal{G}, \mathcal{G}]$. This result will be the key of the proof.

We begin by a characterization of Riemann-Lie subalgebras.

Proposition 3.1. *Let $(\mathcal{G}, [,], \langle , \rangle)$ be a Riemann-Lie algebra and \mathcal{H} a subalgebra of \mathcal{G} . For any $u, v \in \mathcal{H}$, we put $A_u v = A_u^0 v + A_u^1 v$, where $A_u^0 v \in \mathcal{H}$ and $A_u^1 v \in \mathcal{H}^\perp$. Then, $(\mathcal{H}, [,], \langle , \rangle)$ is a Riemann-Lie algebra if and only if, for any $u, v, w \in \mathcal{H}$, $[A_u^1 v, w] + [v, A_u^1 w] \in \mathcal{H}^\perp$.*

Proof. We have, from (9), that for any $u, v, w \in \mathcal{H}$

$$[u, [v, w]] = [A_u^0 v, w] + [v, A_u^0 w] + [A_u^1 v, w] + [v, A_u^1 w].$$

Now $A^0 : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ is the infinitesimal Levi-Civita connection associated with the restriction of \langle , \rangle to \mathcal{H} and the proposition follows. \blacksquare

We will introduce now some objects which will be useful latter.

Let $(\mathcal{G}, [,], \langle , \rangle)$ a Lie algebra endowed with a scalar product.

From (5), we deduce that the infinitesimal Levi-Civita connection A associated to \langle , \rangle is given by

$$A_u v = \frac{1}{2}[u, v] - \frac{1}{2}(ad_u^t v + ad_v^t u) \quad u, v \in \mathcal{G}. \quad (17)$$

On other hand, the orthogonal with respect to \langle , \rangle of the ideal $[\mathcal{G}, \mathcal{G}]$ is given by

$$[\mathcal{G}, \mathcal{G}]^\perp = \bigcap_{u \in \mathcal{G}} \ker ad_u^t. \quad (18)$$

Let us introduce, for any $u \in \mathcal{G}$, the endomorphism

$$D_u = ad_u - A_u. \quad (19)$$

We have, by a straightforward calculation, the relations

$$\begin{aligned} D_u(v) &= \frac{1}{2}[u, v] + \frac{1}{2}(ad_u^t v + ad_v^t u), \\ D_u^t(v) &= \frac{1}{2}[u, v] + \frac{1}{2}(ad_u^t v - ad_v^t u). \end{aligned}$$

From these relations, we remark that, for any $u, v \in \mathcal{G}$, $D_u^t(v) = -D_v^t(u)$ and then

$$\forall u \in \mathcal{G}, D_u^t(u) = 0. \quad (20)$$

We remark also that

$$D_u^t = D_u \quad \Leftrightarrow \quad \forall v \in \mathcal{G}, ad_v^t u = 0.$$

So, by (18), we get

$$[\mathcal{G}, \mathcal{G}]^\perp = \{u \in \mathcal{G}; D_u^t = D_u\}. \quad (21)$$

Now, we prove a sequence of results which will give a proof of the equivalence “1) \Leftrightarrow 5)” in Theorem 2.2.

Proposition 3.2. *Let $(\mathcal{G}, [,], \langle , \rangle)$ be a Riemann-Lie algebra. Then $Z(\mathcal{G})^\perp$ ($Z(\mathcal{G})$ is the center of \mathcal{G}) is an ideal of \mathcal{G} which contains the ideal $[\mathcal{G}, \mathcal{G}]$. In particular,*

$$\mathcal{G} = Z(\mathcal{G}) \oplus Z(\mathcal{G})^\perp.$$

Proof. For any $u \in Z(\mathcal{G})$ and $v \in \mathcal{G}$, from (17) and the fact that A_u is skew-adjoint, $A_u v = -\frac{1}{2} \text{ad}_v^t u \in Z(\mathcal{G})^\perp$. By (8), for any $w \in \mathcal{G}$

$$[A_u v, w] = [A_w v, u] = 0,$$

so $A_u v \in Z(\mathcal{G})$ and then $A_u v = -\frac{1}{2} \text{ad}_v^t u = 0$ which shows that $u \in [\mathcal{G}, \mathcal{G}]^\perp$. So $Z(\mathcal{G}) \subset [\mathcal{G}, \mathcal{G}]^\perp$ and the proposition follows. ■

From this proposition and the fact that for a nilpotent Lie algebra \mathcal{G} $Z(\mathcal{G}) \cap [\mathcal{G}, \mathcal{G}] \neq \{0\}$, we get the following lemma.

Lemma 3.3. *A nilpotent Lie algebra \mathcal{G} carries a structure of Riemann-Lie algebra if and only if \mathcal{G} is abelian.*

We can now get the following crucial result.

Lemma 3.4. *Let $(\mathcal{G}, [,], \langle , \rangle)$ be a Riemann-Lie algebra. Then the orthogonal Lie subalgebra $S_{\langle , \rangle}$ is abelian.*

Proof. By (17), $A_u v = \frac{1}{2}[u, v]$ for any $u, v \in S_{\langle , \rangle}$. So, by Proposition 3.1, $S_{\langle , \rangle}$ is a Riemann-subalgebra. By (9), we have, for any $u, v, w \in S_{\langle , \rangle}$,

$$[u, [v, w]] = [A_u v, w] + [v, A_u w] = \frac{1}{2}[[u, v], w] + \frac{1}{2}[v, [u, w]] = \frac{1}{2}[u, [v, w]]$$

and then $[S_{\langle , \rangle}, [S_{\langle , \rangle}, S_{\langle , \rangle}]] = 0$ i.e. $S_{\langle , \rangle}$ is a nilpotent Lie algebra and then abelian by Lemma 3.3. ■

Lemma 3.5. *Let $(\mathcal{G}, [,], \langle , \rangle)$ be a Riemann-Lie algebra. Then*

$$[\mathcal{G}, \mathcal{G}]^\perp = \{u \in \mathcal{G}; D_u = 0\}.$$

Proof. Firstly, we notice that, by (21), $[\mathcal{G}, \mathcal{G}]^\perp \supset \{u \in \mathcal{G}; D_u = 0\}$. On other hand, remark that the relation (8) can be rewritten

$$[D_u(v), w] + [v, D_u(w)] = 0$$

for any $u, v, w \in \mathcal{G}$. So, we can deduce immediately that $[\ker D_u, \text{Im} D_u] = 0$ for any $u \in \mathcal{G}$.

Now we observe that, for any $u \in [\mathcal{G}, \mathcal{G}]^\perp$, the endomorphism D_u is auto-adjoint and then diagonalizable on \mathbb{R} . Let $u \in [\mathcal{G}, \mathcal{G}]^\perp$, $\lambda \in \mathbb{R}$ be an eigenvalue of D_u and $v \in \mathcal{G}$ an eigenvector associated with λ . We have

$$\langle D_u(v), v \rangle = \lambda \langle v, v \rangle \stackrel{(\alpha)}{=} -\langle A_v u, v \rangle \stackrel{(\beta)}{=} -\langle [v, u], v \rangle \stackrel{(\gamma)}{=} 0.$$

So $\lambda = 0$ and we obtain that D_u vanishes identically. Hence the lemma follows.

The equality (α) is a consequence of the definition of D_u , and the equality (β) follows from the definition of A . We observe that $v \in \text{Im} D_u$ and $u \in \ker D_u$ since $D_u(u) = D_u^t(u) = 0$ (see (20)) and the equality (γ) follows from the remark above. ■

Lemma 3.6. *Let $(\mathcal{G}, [\ , \], \langle \cdot, \cdot \rangle)$ be a Riemann-Lie algebra. Then*

$$S_{\langle \cdot, \cdot \rangle} = [\mathcal{G}, \mathcal{G}]^\perp.$$

Proof. From Lemma 3.5, for any $u \in [\mathcal{G}, \mathcal{G}]^\perp$, $A_u = ad_u$ and then ad_u is skew-adjoint. So $[\mathcal{G}, \mathcal{G}]^\perp \subset S_{\langle \cdot, \cdot \rangle}$. To prove the second inclusion, we need to work harder than the first one.

Firstly, remark that one can suppose that $Z(\mathcal{G}) = \{0\}$. Indeed, $\mathcal{G} = Z(\mathcal{G}) \oplus Z(\mathcal{G})^\perp$ (see Proposition 3.2), $Z(\mathcal{G})^\perp$ is a Riemann-Lie algebra (see Proposition 3.1), $[\mathcal{G}, \mathcal{G}] = [Z(\mathcal{G})^\perp, Z(\mathcal{G})^\perp]$ and $S_{\langle \cdot, \cdot \rangle} = Z(\mathcal{G}) \oplus S'_{\langle \cdot, \cdot \rangle}$ where $S'_{\langle \cdot, \cdot \rangle}$ is the orthogonal subalgebra associated to $(Z(\mathcal{G})^\perp, \langle \cdot, \cdot \rangle)$.

We suppose now that $(\mathcal{G}, [\ , \], \langle \cdot, \cdot \rangle)$ is a Riemann-Lie algebra such that $Z(\mathcal{G}) = \{0\}$ and we want to prove the inclusion $[\mathcal{G}, \mathcal{G}]^\perp \supset S_{\langle \cdot, \cdot \rangle}$. Notice that it suffices to show that, for any $u \in S_{\langle \cdot, \cdot \rangle}$, $A_u = ad_u$.

The proof requires some preparation. Let us introduce the subalgebra K given by

$$K = \bigcap_{u \in S_{\langle \cdot, \cdot \rangle}} \ker ad_u.$$

Firstly, we notice that K contains $S_{\langle \cdot, \cdot \rangle}$ because $S_{\langle \cdot, \cdot \rangle}$ is abelian (see Lemma 3.4).

On other hand, we remark that, for any $u \in S_{\langle \cdot, \cdot \rangle}$, the endomorphism A_u leaves invariant K and K^\perp . Indeed, for any $v \in K$ and any $w \in S_{\langle \cdot, \cdot \rangle}$, we have

$$[w, A_u v] \stackrel{(\alpha)}{=} [w, A_v u] \stackrel{(\beta)}{=} -[A_w u, v] \stackrel{(\gamma)}{=} 0$$

and then $A_u v \in K$, this shows that A_u leaves invariant K . Furthermore, A_u being skew-adjoint, we have $A_u(K^\perp) \subset K^\perp$.

The equality (α) follows from the relation $A_u v = A_v u + [u, v] = A_u v$, the equality (β) follows from (8) and (γ) follows from the relation $A_w u = \frac{1}{2}[w, u] = 0$.

With this observation in mind, we consider the representation $\rho : S_{\langle \cdot, \cdot \rangle} \longrightarrow so(K^\perp)$ given by

$$\rho(u) = ad_{u|_{K^\perp}} \quad u \in S_{\langle \cdot, \cdot \rangle}.$$

It is clear that

$$\bigcap_{u \in S_{\langle \cdot, \cdot \rangle}} \ker \rho(u) = \{0\}. \tag{*}$$

This relation and the fact that $S_{\langle \cdot, \cdot \rangle}$ is abelian imply that $\dim K^\perp$ is even and that there is an orthonormal basis $(e_1, f_1, \dots, e_p, f_p)$ of K^\perp such that

$$\forall i \in \{1, \dots, p\}, \forall u \in S_{\langle \cdot, \cdot \rangle}, \quad ad_u e_i = \lambda^i(u) f_i \quad \text{and} \quad ad_u f_i = -\lambda^i(u) e_i, \tag{**}$$

where $\lambda^i \in S_{\langle \cdot, \cdot \rangle}^*$.

Now, for any $u \in S_{\langle \cdot, \cdot \rangle}$, since A_u leaves K^\perp invariant, we can write

$$\begin{aligned} A_u e_i &= \sum_{j=1}^p (\langle A_u e_i, e_j \rangle e_j + \langle A_u e_i, f_j \rangle f_j), \\ A_u f_i &= \sum_{j=1}^p (\langle A_u f_i, e_j \rangle e_j + \langle A_u f_i, f_j \rangle f_j). \end{aligned}$$

From (9), we have for any $v \in S_{\langle, \rangle}$ and for any $i \in \{1, \dots, p\}$,

$$\begin{aligned} [u, [v, e_i]] &= [A_u v, e_i] + [v, A_u e_i], \\ [u, [v, f_i]] &= [A_u v, f_i] + [v, A_u f_i]. \end{aligned}$$

Using the equality $A_u v = 0$ and (**) and substituting we get

$$\begin{aligned} -\lambda^i(u)\lambda^i(v)e_i &= \sum_{j=1}^p \lambda^j(v)\langle A_u e_i, e_j \rangle f_j - \sum_{j=1}^p \lambda^j(v)\langle A_u e_i, f_j \rangle e_j, \\ -\lambda^i(u)\lambda^i(v)f_i &= \sum_{j=1}^p \lambda^j(v)\langle A_u f_i, e_j \rangle f_j - \sum_{j=1}^p \lambda^j(v)\langle A_u f_i, f_j \rangle e_j. \end{aligned}$$

Now, it is clear from (*) that, for any $i \in \{1, \dots, p\}$, there exists $v \in S_{\langle, \rangle}$ such that $\lambda^i(v) \neq 0$. Using this fact and the relations above, we get

$$A_u e_i = \lambda^i(u)f_i \quad \text{and} \quad A_u f_i = -\lambda^i(u)e_i.$$

So we have shown that, for any $u \in S_{\langle, \rangle}$,

$$A_u|_{K^\perp} = ad_u|_{K^\perp}.$$

Now, for any $u \in S_{\langle, \rangle}$ and for any $k \in K$, $ad_u(k) = 0$. So, to complete the proof of the lemma, we will show that, for any $u \in S_{\langle, \rangle}$ and for any $k \in K$, $A_u k = 0$. This will be done by showing that $A_u k \in Z(\mathcal{G})$ and conclude by using the assumption $Z(\mathcal{G}) = \{0\}$.

Indeed, for any $h \in K$, by (8)

$$[A_u k, h] = [A_h k, u].$$

Since $A_u(K) \subset K$ and since K is a subalgebra, $[A_u k, h] \in K$. Now, $K \subset \ker ad_u$ and ad_u is skew-adjoint so $[A_h k, u] \in \text{Im} ad_u \subset K^\perp$. So $[A_u k, h] = 0$. On other hand, for any $f \in K^\perp$, we have, also from (8),

$$[A_u k, f] = [A_k u, f] = [A_f u, k] = 0$$

since $A_f u = [f, u] + A_u f = [f, u] + [u, f] = 0$.

We deduce that $A_u k \in Z(\mathcal{G})$ and then $A_u k = 0$. The proof of the lemma is complete. \blacksquare

Lemma 3.7. *Let $(\mathcal{G}, [,], \langle, \rangle)$ be a Riemann-Lie algebra such that $Z(\mathcal{G}) = 0$. Then*

$$\mathcal{G} \neq [\mathcal{G}, \mathcal{G}].$$

Proof. Let $(\mathcal{G}, [,], \langle, \rangle)$ be a Riemann-Lie algebra such that $Z(\mathcal{G}) = 0$. We will show that the assumption $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$ implies that the Killing form of \mathcal{G} is strictly negative definite and then \mathcal{G} is semi-simple and compact which is in contradiction with lemma 2.3.

Let $u \in \mathcal{G}$ fixed. Since A_u is skew-adjoint, there is an orthonormal basis $(a_1, b_1, \dots, a_r, b_r, c_1, \dots, c_l)$ of \mathcal{G} and $(\mu_1, \dots, \mu_r) \in \mathbb{R}^r$ such that, for any $i \in \{1, \dots, r\}$ and any $j \in \{1, \dots, l\}$,

$$A_u a_i = \mu_i b_i, \quad A_u b_i = -\mu_i a_i \quad \text{and} \quad A_u c_j = 0.$$

Moreover, $\mu_i > 0$ for any $i \in \{1, \dots, r\}$.

By applying (9), we can deduce, for any $i, j \in \{1, \dots, r\}$ and for any $k, h \in \{1, \dots, l\}$, the relations:

$$\begin{aligned} [u, [a_i, a_j]] &= \mu_i [b_i, a_j] + \mu_j [a_i, b_j], & [u, [b_i, b_j]] &= -\mu_j [b_i, a_j] - \mu_i [a_i, b_j], \\ [u, [a_i, b_j]] &= -\mu_j [a_i, a_j] + \mu_i [b_i, b_j], & [u, [b_i, a_j]] &= -\mu_i [a_i, a_j] + \mu_j [b_i, b_j], \\ [u, [c_k, a_j]] &= \mu_j [c_k, b_j], & [u, [c_k, b_j]] &= -\mu_j [c_k, a_j], & [u, [c_k, c_h]] &= 0. \end{aligned}$$

From these relations we deduce

$$\begin{aligned} ad_u \circ ad_u([a_i, a_j]) &= -(\mu_i^2 + \mu_j^2)[a_i, a_j] + 2\mu_i \mu_j [b_i, b_j], \\ ad_u \circ ad_u([b_i, b_j]) &= 2\mu_i \mu_j [a_i, a_j] - (\mu_i^2 + \mu_j^2)[b_i, b_j], \\ ad_u \circ ad_u([b_i, a_j]) &= -(\mu_i^2 + \mu_j^2)[b_i, a_j] - 2\mu_i \mu_j [a_i, b_j], \\ ad_u \circ ad_u([a_i, b_j]) &= -2\mu_i \mu_j [b_i, a_j] - (\mu_i^2 + \mu_j^2)[a_i, b_j], \\ ad_u \circ ad_u([c_k, a_j]) &= -\mu_j^2 [c_k, a_j], \\ ad_u \circ ad_u([c_k, b_j]) &= -\mu_j^2 [c_k, b_j], \\ ad_u \circ ad_u([c_k, c_h]) &= 0. \end{aligned}$$

By an obvious transformation we obtain

$$\begin{aligned} ad_u \circ ad_u([a_i, a_j] + [b_i, b_j]) &= -(\mu_i - \mu_j)^2 ([a_i, a_j] + [b_i, b_j]), \\ ad_u \circ ad_u([a_i, a_j] - [b_i, b_j]) &= -(\mu_i + \mu_j)^2 ([a_i, a_j] - [b_i, b_j]), \\ ad_u \circ ad_u([b_i, a_j] + [a_i, b_j]) &= -(\mu_i + \mu_j)^2 ([b_i, a_j] + [a_i, b_j]), \\ ad_u \circ ad_u([b_i, a_j] - [a_i, b_j]) &= -(\mu_i - \mu_j)^2 ([b_i, a_j] - [a_i, b_j]), \\ ad_u \circ ad_u([c_k, a_j]) &= -\mu_j^2 [c_k, a_j], \\ ad_u \circ ad_u([c_k, b_j]) &= -\mu_j^2 [c_k, b_j], \\ ad_u \circ ad_u([c_k, c_h]) &= 0. \end{aligned}$$

Suppose now $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$. Then the family of vectors

$$\{[a_i, a_j] + [b_i, b_j], [a_i, a_j] - [b_i, b_j], [b_i, a_j] + [a_i, b_j],$$

$$[b_i, a_j] - [a_i, b_j], [c_k, a_i], [c_k, b_j], [c_k, c_h]; \quad i, j \in \{1, \dots, r\}, h, k \in \{1, \dots, l\}\}$$

spans \mathcal{G} and then $ad_u \circ ad_u$ is diagonalizable and all its eigenvalues are non positive.

Now it's easy to deduce that $ad_u \circ ad_u = 0$ if and only if $ad_u = 0$. Since $Z(\mathcal{G}) = 0$ we have shown that, for any $u \in \mathcal{G} \setminus \{0\}$, $Tr(ad_u \circ ad_u) < 0$ and then the Killing form of \mathcal{G} is strictly negative definite and then \mathcal{G} is semi-simple compact. We can conclude with Lemma 2.3. \blacksquare

Proof of the equivalence "1) \Leftrightarrow 5)" in Theorem 2.2.

It is an obvious and straightforward calculation to show that 5) \Rightarrow 1).

Conversely, let $(\mathcal{G}, [,], \langle , \rangle)$ be a Riemann-Lie algebra. By Proposition 3.2, we can suppose that $Z(\mathcal{G}) = \{0\}$.

We have, from Lemma 3.7 and Lemma 3.6, $\mathcal{G} \neq [\mathcal{G}, \mathcal{G}]$ which implies $S_{\langle, \rangle} \neq 0$ and $\mathcal{G} = S_{\langle, \rangle} \dot{\oplus} [\mathcal{G}, \mathcal{G}]$. Moreover, $[\mathcal{G}, \mathcal{G}]$ is a Riemann-Lie algebra (see Proposition 3.1) and we can repeat the argument above to deduce that eventually \mathcal{G} is solvable which implies that $[\mathcal{G}, \mathcal{G}]$ is nilpotent and then abelian by Lemma 3.3 and the implication follows.

Remark 3.8. The pseudo-Riemann-Lie algebras are completely different from the Riemann-Lie algebras. Indeed, the 3-dimensional Heisenberg Lie algebra which is nilpotent carries a Lorentzian Lie algebra structure. On other hand, the non trivial 2-dimensional Lie algebra carries a Lorentzian inner product whose curvature vanishes and does not carry any structure of a pseudo-Riemann-Lie algebra.

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