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On flat pseudo-Euclidean nilpotent Lie algebras

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ABSTRACT

A flat pseudo-Euclidean Lie algebra is a real Lie algebra with a non degenerate symmetric bilinear form and a left symmetric product whose the commutator is the Lie bracket and such that the left multiplications are skew-symmetric. We show that the center of a flat pseudo-Euclidean nilpotent Lie algebra of signature $(2, n-2)$ must be degenerate and all flat pseudo-Euclidean nilpotent Lie algebras of signature $(2, n-2)$ can be obtained by using the double extension process from flat Lorentzian nilpotent Lie algebras. We show also that the center of a flat pseudo-Euclidean 2-step nilpotent Lie algebra is degenerate and all these Lie algebras are obtained by using a sequence of double extension from an abelian Lie algebra. In particular, we determine all flat pseudo-Euclidean 2-step nilpotent Lie algebras of signature $(2, n-2)$. The paper contains also some examples in low dimension.

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1. Introduction

A *flat pseudo-Euclidean Lie algebra* is a real Lie algebra with a non degenerate symmetric bilinear form and a left symmetric product whose the commutator is the Lie bracket and such that the left multiplications are skew-symmetric. In geometrical terms, a flat pseudo-Euclidean Lie algebra is the Lie algebra of a Lie group with a left-invariant pseudo-Riemannian metric with vanishing curvature. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean Lie algebra of dimension n. If the metric $\langle\ ,\ \rangle$ is definite positive (resp. of signature $(1, n-1)$), then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called Euclidean (resp. Lorentzian). Flat pseudo-Euclidean Lie algebras have been studied mostly in the Euclidean and the Lorentzian cases. Let us enumerate some important results on flat pseudo-Euclidean Lie algebras:

1. In [6], Milnor showed that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a flat Euclidean Lie algebra if and only if $\mathfrak{g}$ splits orthogonally as $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{u}$ is an abelian ideal, $\mathfrak{b}$ is an abelian subalgebra, and ad_b is skew-symmetric for any $b \in \mathfrak{b}$. According to this theorem, a nilpotent (non-abelian) Lie algebra can not admit a flat Euclidean metric.
2. In [3], Aubert and Medina showed that all flat Lorentzian nilpotent Lie algebras are obtained by the double extension process from Euclidean abelian Lie algebras.
3. Guédiri showed in [5] that a flat Lorentzian 2-step nilpotent Lie algebra is a trivial central extension of the 3-dimensional Heisenberg Lie algebra $\mathcal{H}_3$. Recall that the Heisenberg Lie algebra $\mathcal{H}_{2k+1}$, is defined as the vector space $\mathcal{H}_{2k+1} = \mathrm{span}\{z, x_1, \ldots, x_k, y_1, \ldots, y_k\}$ such that all brackets are zeros except $[x_i, y_i] = z$ for $1 \le i \le k$.
4. In [2], M. Ait Ben Haddou and the authors showed that all flat Lorentzian Lie algebras with degenerate center can be obtained by double extension process from flat Euclidean Lie algebras. In [4], the authors showed that all flat nonunimodular Lorentzian Lie algebras can be obtained by double extension process from flat Euclidean Lie algebras.

The study of flat pseudo-Euclidean Lie algebras of signature other than $(0, n)$ and $(1, n-1)$ is an open problem. In this paper, we study a part of this problem, more precisely, we study flat pseudo-Euclidean nilpotent Lie algebras of signature $(2, n-2)$ and flat pseudo-Euclidean 2-step nilpotent Lie algebras of any signature. There are our main results:

1. In Theorem 3.1, we show that the center of a flat pseudo-Euclidean nilpotent Lie algebra of signature $(2, n-2)$ must be degenerate. From this theorem and Theorem 4.1 we deduce that all flat pseudo-Euclidean nilpotent Lie algebra of signature $(2, n-2)$ are obtained by the double extension process.
2. We give some general properties of flat pseudo-Euclidean 2-step nilpotent Lie algebras and we show that their center is degenerate. We show also that we can construct

all this Lie algebras by applying a sequence of double extension starting from a pseudo-Euclidean abelian Lie algebra.

3. We give all 2-step nilpotent Lie algebras which can admit flat pseudo-Euclidean metrics of signature $(2, n-2)$ (Theorem 6.1 and Theorem 6.2). We will see that a class of 2-step nilpotent Lie algebras which can admit a flat pseudo-Euclidean metrics of signature $(2, n-2)$ is very rich, contrary to the Euclidean and the Lorentzian cases. As example, we show that any 6-dimensional 2-step nilpotent Lie algebra which is not a trivial central extension of a 5-dimensional Heisenberg Lie algebra, admits such metric.

The paper is organized as follows. In section 2, we give some generalities on flat pseudo-Euclidean Lie algebras. In section 3 and section 4, we study flat pseudo-Euclidean metrics of signature $(2, n-2)$ on nilpotent Lie algebras. In section 5, we study flat pseudo-Euclidean 2-step nilpotent Lie algebra of any signature. In section 6, we give all flat pseudo-Euclidean 2-step nilpotent Lie algebras of signature $(2, n-2)$. We end the paper by giving some examples.

2. Preliminaries

In this section, we give some general results on nilpotent Lie algebras and on flat pseudo-Euclidean nilpotent Lie algebras which will be crucial in the proofs of our main results.

Let us start with two useful lemmas. Recall that a pseudo-Euclidean vector space is a real finite dimensional vector space endowed with a non degenerate bilinear symmetric form.

Lemma 2.1. *Let $(V, \langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space and A a skew-symmetric endomorphism satisfying $A^2 = 0$ and $\dim \mathrm{Im} A \leq 1$. Then $A = 0$.*

Proof. Suppose that $A \neq 0$. Then $\mathrm{Im}A$ is a totally isotropic vector space of dimension 1. This implies that $\ker A$ is an hyperplan which contains $\mathrm{Im}A$. Let e be a generator of $\mathrm{Im}A$ and choose an isotropic vector $\bar{e} \notin \ker A$ such that $\langle e, \bar{e}\rangle = 1$. We have $V = \ker A \oplus \mathbb{R}\bar{e}$ and $A(\bar{e}) = \alpha e$. Then $\alpha = \langle A(\bar{e}), \bar{e}\rangle = 0$ which gives a contradiction and completes the proof. □

Lemma 2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\mathfrak{a}$ and $\mathfrak{h}$, respectively, a Lie subalgebra of codimension one and an ideal of codimension two. Then $[\mathfrak{g}, \mathfrak{g}]$ is contained in $\mathfrak{a}$ and in $\mathfrak{h}$.*

Proof. We have $\mathfrak{g}/\mathfrak{h}$ is a 2-dimensional nilpotent Lie algebra and hence must be abelian. This implies that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$. On the other hand, write $\mathfrak{g} = \mathfrak{a} \oplus \mathbb{R}y$. For any $x \in \mathfrak{a}$, we have

$$[x, y] = a(x)y + u_1, \text{ where } u_1 \in \mathfrak{a}.$$

Since $\mathfrak{a}$ is a Lie subalgebra then, for any $n \in \mathbb{N}^*$, $\mathrm{ad}_x^n(y) = a(x)^n y + u_n$ with $u_n \in \mathfrak{a}$. Since ad_x is nilpotent then $a(x) = 0$ and the result follows. □

We pursue with some general properties of flat pseudo-Euclidean Lie algebras. A pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a finite dimensional real Lie algebra $\mathfrak{g}$ endowed with a non degenerate symmetric bilinear form $\langle\ ,\ \rangle$. We define a product $(u, v) \mapsto u.v$ on $\mathfrak{g}$ called Levi-Civita product by Koszul's formula

$$2\langle u.v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle, \tag{2.1}$$

for any $u, v, w \in \mathfrak{g}$. We denote by $\mathrm{L}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ and $\mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$, respectively, the left multiplication and the right multiplication by u given by $\mathrm{L}_u v = u.v$ and $\mathrm{R}_u v = v.u$. For any $u \in \mathfrak{g}$, L_u is skew-symmetric with respect to $\langle\ ,\ \rangle$ and $\mathrm{ad}_u = \mathrm{L}_u - \mathrm{R}_u$, where $\mathrm{ad}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by $\mathrm{ad}_u v = [u, v]$. We call $(\mathfrak{g}, \langle\ ,\ \rangle)$ *flat pseudo-Euclidean Lie algebra* if the Levi-Civita product is left symmetric, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}(u, v, w) = \mathrm{ass}(v, u, w), \tag{2.2}$$

where $\mathrm{ass}(u, v, w) = (u.v).w - u.(v.w)$.

Remark 1. Let G be a Lie group, and μ a left-invariant pseudo-Riemannian metric on G. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\langle\ ,\ \rangle = \mu_e$. Then the curvature of (G, μ) vanishes if and only if $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean Lie algebra.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean Lie algebra. The condition (2.2) is also equivalent to one of the following relations:

$$\mathrm{L}_{[u,v]} = [\mathrm{L}_u, \mathrm{L}_v], \tag{2.3}$$

$$\mathrm{R}_{u.v} - \mathrm{R}_v \circ \mathrm{R}_u = [\mathrm{L}_u, \mathrm{R}_v], \tag{2.4}$$

for any $u, v \in \mathfrak{g}$. We denote by $Z(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{ad}_u = 0\}$ the center of $\mathfrak{g}$. For any $u, v \in Z(\mathfrak{g})$ and $a, b \in \mathfrak{g}$, one can deduce easily from (2.1)-(2.4) that

$$u.v = 0,\ \mathrm{L}_u = \mathrm{R}_u,\ \mathrm{L}_u \circ \mathrm{L}_v = 0 \quad \text{and} \quad u.(a.b) = a.(u.b). \tag{2.5}$$

Proposition 2.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean nilpotent non abelian Lie algebra. If $Z(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{L}_u = \mathrm{R}_u = 0\}$ then $Z(\mathfrak{g})$ is degenerate.*

Proof. One can see easily that the orthogonal of the derived ideal of $\mathfrak{g}$ is given by

$$[\mathfrak{g}, \mathfrak{g}]^\perp = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\}. \tag{2.6}$$

Then $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]^\perp$ and hence $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})^\perp$. Since $\mathfrak{g}$ is nilpotent non abelian then $\{0\} \neq [\mathfrak{g}, \mathfrak{g}] \cap Z(\mathfrak{g}) \subset Z(\mathfrak{g})^\perp \cap Z(\mathfrak{g})$. This shows that $Z(\mathfrak{g})$ is degenerate. □

Proposition 2.2. *Let* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a flat pseudo-Euclidean nilpotent Lie algebra. Then:*

1. *If* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *is Euclidean then* $\mathfrak{g}$ *is abelian.*
2. *If* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *is non abelian Lorentzian then* $Z(\mathfrak{g})$ *is degenerate.*

Proof. 1. According to (2.5), for any $u \in Z(\mathfrak{g})$, L_u is a nilpotent skew-symmetric endomorphism and hence must vanishes. This gives the result, by virtue of Proposition 2.1.
2. This is a consequence of (2.5), Lemma 2.1 and Proposition 2.1. □

Put $N(\mathfrak{g}) = \bigcap_{u\in Z(\mathfrak{g})} \ker \mathrm{L}_u$, $\mathfrak{g}_0 := N(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ and $\mathfrak{h}_0 := N(\mathfrak{g})^\perp$. These vector spaces and the following lemma which states their main properties will play a central role in this paper, namely, in the proof of Theorem 3.1.

Lemma 2.3. *Let* $(\mathfrak{g}, \langle\ ,\ \rangle)$ *be a flat pseudo-Euclidean nilpotent non abelian Lie algebra of signature* $(2, n-2)$ *with* $n \geq 4$. *Then:*

1. $N(\mathfrak{g})$, $\mathfrak{g}_0$ *and* $\mathfrak{h}_0$ *are left ideals for the Levi-Civita product,* $\mathfrak{h}_0 \subset \mathfrak{g}_0$, *and* $\mathfrak{h}_0$ *is totally isotropic with* $\dim \mathfrak{h}_0 \leq 2$.
2. *If* $Z(\mathfrak{g})$ *is non degenerate then the restriction of* $\langle\ ,\ \rangle$ *to* $Z(\mathfrak{g})$ *is positive definite,* $\dim \mathfrak{h}_0 = 2$ *and* $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) = 1$. *Moreover, if* z_0 *is a generator of* $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ *with* $\langle z_0, z_0\rangle = 1$ *then for any* $u, v \in \mathfrak{g}$,
$$[u, v] = [u, v]_1 - 2\langle \mathrm{L}_{z_0} u, v\rangle z_0, \tag{2.7}$$
where $[u, v]_1 \in Z(\mathfrak{g})^\perp$.

Proof. 1. Note first that, for any $u \in \mathfrak{g}$, $(\ker \mathrm{L}_u)^\perp = \mathrm{Im}\mathrm{L}_u$ and hence $\mathfrak{h}_0 = \sum_{u\in Z(\mathfrak{g})} \mathrm{Im}\mathrm{L}_u$. From (2.5), we have clearly that $Z(\mathfrak{g}) \subset N(\mathfrak{g})$ and, for any $u, v \in Z(\mathfrak{g})$, $\mathrm{Im}\mathrm{L}_u \subset \ker \mathrm{L}_v$. Thus $\mathfrak{h}_0 \subset \mathfrak{g}_0$. This implies that $\mathfrak{h}_0$ is totally isotropic and since the signature is $(2, n-2)$ one must have $\dim \mathfrak{h}_0 \leq 2$. One can deduce easily from the third relation in (2.5) that $N(\mathfrak{g})$ is a left ideal. This implies, since the left multiplication are skew-symmetric that $\mathfrak{h}_0$ and $\mathfrak{g}_0$ are also left ideals.
2. Suppose now that $Z(\mathfrak{g})$ is non degenerate. If $\dim \mathfrak{h}_0 \leq 1$ then, according to Lemma 2.1, $\mathrm{L}_u = 0$ for any $u \in Z(\mathfrak{g})$ and hence, by virtue of Proposition 2.1, $Z(\mathfrak{g})$ is degenerate. So we must have $\dim \mathfrak{h}_0 = 2$ and the restriction of $\langle\ ,\ \rangle$ to $Z(\mathfrak{g})^\perp$ is of signature $(2, \dim Z(\mathfrak{g})^\perp - 2)$ which implies that the restriction of $\langle\ ,\ \rangle$ to $Z(\mathfrak{g})$ is definite positive. On the other hand, according to what above we can choose two vectors $(\bar{e}_1, \bar{e}_2)$ of $Z(\mathfrak{g})^\perp$ such that $Z(\mathfrak{g})^\perp = \mathfrak{g}_0 \oplus \mathrm{Span}\{\bar{e}_1, \bar{e}_2\}$. So,
$$[\mathfrak{g}, \mathfrak{g}] = [Z(\mathfrak{g})^\perp, Z(\mathfrak{g})^\perp] = \mathbb{R}[\bar{e}_1, \bar{e}_2] + [\bar{e}_1, \mathfrak{g}_0] + [\bar{e}_2, \mathfrak{g}_0] + [\mathfrak{g}_0, \mathfrak{g}_0].$$
We have that $\mathfrak{g}_0$ is a left ideal for the Levi-Civita product and for any $a \in \mathfrak{g}_0$, $b \in \mathfrak{g}$ and $u \in Z(\mathfrak{g})$,

$$\langle a.b, u\rangle = -\langle b, a.u\rangle = -\langle b, u.a\rangle = 0$$

and hence $\mathfrak{g}_0.\mathfrak{g} \subset Z(\mathfrak{g})^\perp$. This implies that $[\bar{e}_1, \mathfrak{g}_0] + [\bar{e}_2, \mathfrak{g}_0] + [\mathfrak{g}_0, \mathfrak{g}_0] \subset Z(\mathfrak{g})^\perp$. Moreover, $[\bar{e}_1, \bar{e}_2] = z + v_0$, where $z \in Z(\mathfrak{g})$, $z \neq 0$ since $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] \neq 0$ and $v_0 \in Z(\mathfrak{g})^\perp$. So $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}z \oplus F$ where F is a vector subspace of $Z(\mathfrak{g})^\perp$. From this relation, we can deduce that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}z$ and (2.7) follows immediately. □

3. The center of a flat pseudo-Euclidean nilpotent Lie algebra of signature $(2, n-2)$ is degenerate

The purpose of this section is to prove the following theorem.

Theorem 3.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean nilpotent non abelian Lie algebra of signature $(2, n-2)$ with $n \geq 4$. Then $Z(\mathfrak{g})$ is degenerate.*

Proof. We proceed by contradiction and we suppose that $Z(\mathfrak{g})$ is non degenerate, i.e., $\mathfrak{g} = Z(\mathfrak{g}) \oplus Z(\mathfrak{g})^\perp$. As in Lemma 2.3, we consider $\mathfrak{g}_0 = \{v \in Z(\mathfrak{g})^\perp / \mathrm{L}_u v = 0,\ \forall u \in Z(\mathfrak{g})\}$ and $\mathfrak{h}_0$ its orthogonal in $Z(\mathfrak{g})^\perp$. We have both $\mathfrak{h}_0$ and $\mathfrak{g}_0$ are left ideals for the Levi-Civita product, $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and $\mathfrak{h}_0$ is totally isotropic of dimension 2. Moreover, if z_0 is a generator of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ such that $\langle z_0, z_0\rangle = 1$ then, for any $u, v \in \mathfrak{g}$,

$$[u, v] = [u, v]_1 - 2\langle \mathrm{L}_{z_0} u, v\rangle z_0, \tag{3.1}$$

where $[u, v]_1 \in Z(\mathfrak{g})^\perp$. This relation shows that $\mathrm{L}_{z_0} \neq 0$ and since $\mathrm{L}_{z_0}^2 = 0$ and $\mathrm{Im}\mathrm{L}_{z_0} \subset \mathfrak{h}_0$, by virtue of Lemma 2.1, $\mathrm{Im}\mathrm{L}_{z_0} = \mathfrak{h}_0$ and $\ker \mathrm{L}_{z_0} = Z(\mathfrak{g}) \oplus \mathfrak{g}_0$. Moreover, from (3.1), one can check easily that $[\ ,\ ]_1$ satisfies Jacobi identity and $(Z(\mathfrak{g})^\perp, [\ ,\ ]_1)$ becomes a nilpotent Lie algebra. We denote by $\circ$ the Levi-Civita product of $(Z(\mathfrak{g})^\perp, [\ ,\ ]_1, \langle\ ,\ \rangle)$ and we have obviously, for any $u, v \in Z(\mathfrak{g})^\perp$,

$$u.v = u \circ v - \langle \mathrm{L}_{z_0} u, v\rangle z_0. \tag{3.2}$$

Let $C(\mathfrak{g})$ denote the center of $(Z(\mathfrak{g})^\perp, [\ ,\ ]_1)$. We have $C(\mathfrak{g}) \neq 0$ and $C(\mathfrak{g}) \cap \mathfrak{g}_0 = \{0\}$. Indeed, if $u \in C(\mathfrak{g}) \cap \mathfrak{g}_0$, then for any $v \in Z(\mathfrak{g})^\perp$,

$$[u, v] = [u, v]_1 - 2\langle \mathrm{L}_{z_0} u, v\rangle z_0 = 0,$$

hence $u \in Z(\mathfrak{g})$ and then $u = 0$. This implies that $1 \leq \dim C(\mathfrak{g}) \leq 2$ and for any $u \in C(\mathfrak{g}) \setminus \{0\}$, $z_0.u \neq 0$.

Let z be a non-null vector in $C(\mathfrak{g})$ then $z_0.z$ is a non-null vector in $\mathfrak{h}_0$. From (2.3) we get $\mathrm{L}_z \circ \mathrm{L}_{z_0} = \mathrm{L}_{z_0} \circ \mathrm{L}_z$ and by using (2.4) we have

$$\mathrm{R}_{z.z_0} = \mathrm{R}_{z_0} \circ \mathrm{R}_z = \mathrm{L}_{z_0} \circ \mathrm{R}_z.$$

For any $u \in Z(\mathfrak{g})^\perp$, we have from (3.2) and the fact that $z \in C(\mathfrak{g})$,

$$\mathrm{L}_z u = z \circ u - \langle z_0.z, u\rangle z_0 \quad \text{and} \quad \mathrm{R}_z u = u \circ z + \langle z_0.z, u\rangle z_0 = z \circ u + \langle z_0.z, u\rangle z_0.$$

Thus $\mathrm{L}_z u = \mathrm{R}_z u - 2\langle z_0.z, u\rangle z_0$. This relation is also true for $u \in Z(\mathfrak{g})$ since $z_0.u = 0$ and hence $\mathrm{L}_z = \mathrm{R}_z + A_z$, where $\mathrm{A}_z = -2\langle z.z_0, .\rangle z_0$. Since $\mathrm{L}_{z_0} \circ A_z = 0$, we deduce that

$$\mathrm{R}_{z.z_0} = \mathrm{L}_{z_0} \circ \mathrm{R}_z = \mathrm{L}_{z_0} \circ (\mathrm{L}_z - A_z) = \mathrm{L}_{z_0} \circ \mathrm{L}_z = \mathrm{L}_z \circ \mathrm{L}_{z_0}. \tag{3.3}$$

This relation implies that $\mathrm{R}_{z.z_0}$ is symmetric and $\mathfrak{g}_0 \oplus Z(\mathfrak{g}) \subset \ker \mathrm{R}_{z.z_0}$. From (3.2), we have $z.z = 0$, and hence $\mathfrak{g}_0 \oplus \mathbb{R}z \oplus Z(\mathfrak{g}) \subset \ker \mathrm{R}_{z.z_0}$. From the symmetry of $\mathrm{R}_{z.z_0}$ we deduce that $\mathrm{Im}\mathrm{R}_{z.z_0} = (\ker \mathrm{R}_{z.z_0})^\perp$ and finally $\mathrm{Im}\mathrm{R}_{z.z_0} \subset (\mathfrak{g}_0 \oplus \mathbb{R}z \oplus Z(\mathfrak{g}))^\perp = \mathbb{R}z.z_0$. So we can write, for any $u \in \mathfrak{g}$,

$$\mathrm{R}_{z.z_0}(u) = a_1(u) z.z_0 = \alpha \langle z.z_0, u\rangle z.z_0, \tag{3.4}$$

where $a_1 \in \mathfrak{g}^*$ and $\alpha \in \mathbb{R}$. We will show now that $\mathrm{R}_{z.z_0} = 0$.

Put $e_1 = z_0.z$. Since the orthogonal of z in $Z(\mathfrak{g})^\perp$ is different from the orthogonal of e_1 in $Z(\mathfrak{g})^\perp$, we can choose $\bar{z} \in Z(\mathfrak{g})^\perp$ such that $\langle z, \bar{z}\rangle = 0$ and $\langle e_1, \bar{z}\rangle = 1$. We put $e_2 = -z_0.\bar{z}$. We have $\langle e_2, z\rangle = 1$, $Z(\mathfrak{g})^\perp = \mathfrak{g}_0 \oplus \mathrm{span}\{z, \bar{z}\}$ and (e_1, e_2) is a basis of $\mathfrak{h}_0$. Now $\mathfrak{h}_0$ is a 2-dimensional subalgebra of a nilpotent Lie algebra then it must be abelian and since $\mathfrak{h}_0 \subset \ker \mathrm{R}_{e_1}$ we deduce that $e_1.e_1 = e_1.e_2 = e_2.e_1 = 0$. Moreover, $\mathfrak{h}_0$ is a left ideal and we can write, for any $u \in \mathfrak{g}$,

$$u.e_1 = a_1(u) e_1 \quad \text{and} \quad u.e_2 = a_2(u) e_1 + b_2(u) e_2.$$

From the relation $u.(z_0.z) = z_0.(u.z)$ shown in (2.5), we deduce that $a_1(u) z_0.z = z_0.(u.z)$, $a_1(u)z - u.z \in \ker \mathrm{L}_{z_0} = \mathfrak{h}_0^\perp$ and hence

$$0 = a_1(u)\langle z, e_2\rangle - \langle u.z, e_2\rangle = a_1(u)\langle z, e_2\rangle + \langle z, u.e_2\rangle = a_1(u) + b_2(u).$$

Thus $b_2 = -a_1$. Using the fact that the curvature vanishes, we get

$$\begin{aligned}
[u, v].e_2 &= u.(v.e_2) - v.(u.e_2) \\
&= u.(a_2(v)e_1 - a_1(v)e_2) - v.(a_2(u)e_1 - a_1(u)e_2) \\
&= 2(a_2(v)a_1(u) - a_1(v)a_2(u))e_1.
\end{aligned}$$

Thus

$$a_2([u, v]) = 2(a_2(v)a_1(u) - a_1(v)a_2(u)).$$

By taking $u = z$ and $v = \bar{z}$ in this relation and since $a_2(z_0) = 0$, $a_1(z) = 0$ and, by virtue of (3.1), $[z, \bar{z}] = -2z_0$, we get $a_2(z)a_1(\bar{z}) = 0$. Now

$$a_1(\bar{z})e_1 = \mathrm{R}_{e_1}(\bar{z}) \overset{(3.3)}{=} \mathrm{L}_z \circ \mathrm{L}_{z_0}(\bar{z}) = -z.e_2 = -a_2(z)e_1.$$

This relation and $a_2(z)a_1(\bar{z}) = 0$ imply that $\mathrm{R}_{e_1}(\bar{z}) = 0$. But $\mathfrak{g}_0 \oplus \mathbb{R}z \oplus Z(\mathfrak{g}) \subset \ker \mathrm{R}_{e_1}$ so finally $\mathrm{R}_{e_1} = 0$. To complete, we will show that $e_1 \in Z(\mathfrak{g})$, i.e., $\mathrm{L}_{e_1} = \mathrm{ad}_{e_1} = 0$ and we will get a contradiction.

Note first that L_{e_1} is nilpotent, $\mathrm{L}_{e_1}(\mathfrak{h}_0) = 0$ and $\mathrm{L}_{e_1}(\mathfrak{g}_0) \subset \mathfrak{g}_0$. So L_{e_1} induces on the Euclidean vector space $\mathfrak{g}_0/\mathfrak{h}_0$ a skew-symmetric nilpotent endomorphism which must then vanish. So $\mathrm{L}_{e_1}(\mathfrak{g}_0) \subset \mathfrak{h}_0$. On the other hand, by virtue of (3.1), $e_1.z = [e_1, z] = 0$. So for any $x \in \mathfrak{g}_0$, $e_1.x = [e_1, x] = a(x)e_1 + b(x)e_2$. This implies that $b(x) = \langle e_1.x, z\rangle = -\langle x, e_1.z\rangle = 0$. But ad_x is nilpotent so $a(x) = 0$ and we deduce that $\mathrm{L}_{e_1}(\mathfrak{g}_0) = 0$. So far, we have shown that $\mathfrak{g}_0 \oplus \mathbb{R}z \oplus Z(\mathfrak{g}) \subset \ker \mathrm{L}_{e_1}$ and hence its image has a dimension less or equal to 1. Moreover, $\mathrm{ImL}_{e_1} \subset \mathfrak{h}_0$ and hence $\mathrm{L}_{e_1}^2 = 0$ and we can conclude by using Lemma 2.1. □

4. Flat pseudo-Euclidean nilpotent Lie algebras of signature $(2, n-2)$ are obtained by the double extension process

In this section, based on Theorem 3.1, we will show that any flat pseudo-Euclidean nilpotent Lie algebra of signature $(2, n-2)$ can be obtained by the double extension process from a Lorentzian or an Euclidean flat nilpotent Lie algebra. To do so we need first to recall the double extension process introduced by Aubert and Medina [3]. Note that Propositions 3.1 and 3.2 in the paper [3] are essential in this process.

Let $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ be a pseudo-Riemannian flat Lie algebra, $\xi, D : B \longrightarrow B$ two endomorphisms of B, $b_0 \in B$ and $\mu \in \mathbb{R}$ such that:

1. ξ is a 1-cocycle of $(B, [\ ,\ ]_0)$ with respect to the representation $\mathrm{L} : B \longrightarrow \mathrm{End}(B)$ defined by the left multiplication associated to the Levi-Civita product, i.e., for any $a, b \in B$,

$$\xi([a, b]) = \mathrm{L}_a\xi(b) - \mathrm{L}_b\xi(a), \tag{4.1}$$

2. $D - \xi$ is skew-symmetric with respect to $\langle\ ,\ \rangle_0$,

$$[D, \xi] = \xi^2 - \mu\xi - \mathrm{R}_{b_0}, \tag{4.2}$$

and for any $a, b \in B$

$$a.\xi(b) - \xi(a.b) = D(a).b + a.D(b) - D(a.b). \tag{4.3}$$

We call (ξ, D, μ, b_0) satisfying the two conditions above *admissible*.

Given (ξ, D, μ, b_0) admissible, we endow the vector space $\mathfrak{g} = \mathbb{R}e \oplus B \oplus \mathbb{R}\bar{e}$ with the inner product $\langle\ ,\ \rangle$ which extends $\langle\ ,\ \rangle_0$, for which $\mathrm{span}\{e, \bar{e}\}$ and B are orthogonal, $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$ and $\langle e, \bar{e}\rangle = 1$. We define also on $\mathfrak{g}$ the bracket

$$[\bar{e}, e] = \mu e,\ [\bar{e}, a] = D(a) - \langle b_0, a\rangle_0 e \quad \text{and} \quad [a, b] = [a, b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 e, \tag{4.4}$$

where $a, b \in B$ and ξ^* is the adjoint of ξ with respect to $\langle\ ,\ \rangle_0$. Then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean Lie algebra called *double extension* of $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to (ξ, D, μ, b_0). Using this method, Aubert and Medina characterize a flat Lorentzian nilpotent Lie algebras. They show that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a flat Lorentzian nilpotent Lie algebra if and only if $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a double extension of an Euclidean abelian Lie algebra according to $\mu = 0$, $D = \xi$ and b_0 where $D^2 = 0$.

Theorem 4.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean nilpotent non abelian Lie algebra of signature $(2, n-2)$ with $n \geq 4$. Then, for any $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, $\mathrm{L}_e = \mathrm{R}_e = 0$. Moreover, $Z(\mathfrak{g}) + Z(\mathfrak{g})^\perp$ is a two-sided ideal with respect to the Levi-Civita product.*

Proof. Recall that $[\mathfrak{g}, \mathfrak{g}]^\perp = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\}$, put $\mathfrak{a} = Z(\mathfrak{g}) + Z(\mathfrak{g})^\perp$ and consider $N(\mathfrak{g}) = \{v \in \mathfrak{g}/\ \mathrm{L}_u v = 0,\ \forall u \in Z(\mathfrak{g})\}$ and $\mathfrak{h}_0$ its orthogonal. We have seen in Lemma 2.3 that both $N(\mathfrak{g})$ and $\mathfrak{h}_0$ are left ideals and $\mathfrak{h}_0$ is totally isotropic. We have seen that if $\dim \mathfrak{h}_0 \leq 1$ then $N(\mathfrak{g}) = \mathfrak{g}$ and hence any vector $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ satisfies the conditions required. Suppose that $\dim \mathfrak{h}_0 = 2$. We claim that $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset \mathfrak{h}_0$. This is a consequence of the fact that $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset Z(\mathfrak{g}) \subset N(\mathfrak{g})$ and the fact that $N(\mathfrak{g})/\mathfrak{h}_0$ is Euclidean. We distinguish two cases:

1. $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = \mathfrak{h}_0$ and hence $\mathfrak{a} = N(\mathfrak{g})$. We have that $\mathfrak{g}.N(\mathfrak{g}) \subset N(\mathfrak{g})$ and for any $u \in N(\mathfrak{g})$, $w \in \mathfrak{g}$ and $v \in \mathfrak{h}_0$, $v.u = u.v = 0$ and hence $\langle u.w, v\rangle = 0$. This implies that $N(\mathfrak{g})$ is an ideal for the Lie bracket and, according to Lemma 2.2, $[\mathfrak{g}, \mathfrak{g}] \subset N(\mathfrak{g})$. We deduce that $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset [\mathfrak{g}, \mathfrak{g}]^\perp$ and hence for any $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, L_e is both skew-symmetric and symmetric and hence $\mathrm{L}_e = \mathrm{R}_e = 0$.
2. $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$. Since $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset \mathfrak{h}_0$, we have $N(\mathfrak{g}) \subset \mathfrak{a}$ and $\mathfrak{a} = N(\mathfrak{g}) \oplus \mathbb{R}y$. We have $\mathfrak{g}.N(\mathfrak{g}) \subset \mathfrak{a}$ and for any $u \in N(\mathfrak{g})$, $w \in \mathfrak{g}$ and $v \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, $v.u = u.v = 0$ and hence $\langle u.w, v\rangle = 0$. Thus $N(\mathfrak{g}).\mathfrak{g} \subset \mathfrak{a}$. Moreover, for any $v \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, $\langle y.y, v\rangle = 0$ and then $y.y \in \mathfrak{a}$. In particular, $\mathfrak{a}.\mathfrak{a} \subset \mathfrak{a}$ and hence $\mathfrak{a}$ is a subalgebra. According to Lemma 2.2, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$ and hence $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset [\mathfrak{g}, \mathfrak{g}]^\perp$. This implies that for any $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, $\mathrm{L}_e = \mathrm{R}_e = 0$ and $\mathfrak{a}$ is a two-sided ideal. □

Theorem 4.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean nilpotent non abelian Lie algebra of signature $(2, n-2)$ with $n \geq 4$. Then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a double extension of a flat Lorentzian nilpotent Lie algebra, according to $\mu = 0$, D, ξ and b_0 where D is a nilpotent endomorphism.*

Proof. Let e be a non-null vector in $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ and put $\mathrm{I} = \mathbb{R}e$. According to Theorem 4.1, I is a totally isotropic two-sided ideal with respect to the Levi-Civita product. Moreover, $\mathrm{I}^\perp$ is also a two sided ideal. Then, according to [3], $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a double extension of flat Lorentzian Lie algebra $(B, \langle\ ,\ \rangle_B)$ with $\mu = 0$. From (4.4) and the fact that

$\mathfrak{g}$ is nilpotent we deduce that D is a nilpotent endomorphism, and B is a nilpotent Lie algebra. □

Remark 2. According to [3], flat Lorentzian nilpotent Lie algebra are double extension of abelian Euclidean Lie algebras. Then flat pseudo-Euclidean nilpotent Lie algebras of signature $(2, n-2)$ are obtained by applying twice the double extension process, starting from abelian Euclidean Lie algebras.

Example 1. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a 4-dimensional flat pseudo-Euclidean nilpotent Lie algebras of signature $(2,2)$. According to Theorem 4.2, $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a double extension of a 2-dimensional abelian Lorentzian Lie algebra $(B, \langle\ ,\ \rangle_B)$ with $D^2 = 0$. The conditions (4.1)-(4.3) are equivalent to $[D, \xi] = \xi^2$ and $D - \xi$ is skew-symmetric, which implies that $D = \xi$. Then there exists a basis $\{e_1, e_2\}$ of B such that the matrix of D in this basis has the form

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \text{ where } \alpha \in \mathbb{R}.$$

Let $\langle\ ,\ \rangle_B$ be any Lorentzian metric in B. Then according to (4.4), $\mathfrak{g} = \operatorname{span}\{\bar{e}, e, e_1, e_2\}$ with the non vanishing Lie brackets

$$[\bar{e}, e_1] = \beta e,\ [\bar{e}, e_2] = \alpha e_1 + \gamma e,\ [e_1, e_2] = \delta e, \text{where } \alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R},$$

and the metric in $\mathfrak{g}$ is an extension orthogonal of $\langle\ ,\ \rangle_B$ such that $\langle \bar{e}, \bar{e}\rangle = \langle e, e\rangle = 0$ and $\langle \bar{e}, e\rangle = 1$. It is easy to show that $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:

- $\mathbb{R}^4$: The 4-dimensional abelian Lie algebra (if $\alpha = \beta = \gamma = \delta = 0$).
- $\mathcal{H}_3 \oplus \mathbb{R}$: The trivial central extension of $\mathcal{H}_3$ (if $\alpha = 0$ and $(\beta, \gamma) \neq (0, 0)$ or $\alpha \neq 0$ and $\beta = \delta = 0$).
- The 4-dimensional filiform Lie algebra: $[\bar{e}, e_1] = e,\ [\bar{e}, e_2] = e_1$ (if $\alpha \neq 0$ and $(\beta, \delta) \neq (0, 0)$).

5. Flat pseudo-Euclidean 2-step nilpotent Lie algebras

A 2-step nilpotent Lie algebra is a non-abelian Lie algebra $\mathfrak{g}$ which satisfies $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$. Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra. In [5], the author showed that if the metric $\langle\ ,\ \rangle$ is Lorentzian, then $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$, where $\mathcal{H}_3$ is a 3-dimensional Heisenberg Lie algebra. Let us studies some properties of $(\mathfrak{g}, \langle\ ,\ \rangle)$ in other signatures.

We consider $N(\mathfrak{g}) = \bigcap_{u \in Z(\mathfrak{g})} \ker \mathrm{L}_u$, and $\mathfrak{h}_0 := N(\mathfrak{g})^{\perp}$. According to Lemma 2.3, $\mathfrak{h}_0 \subset N(\mathfrak{g})$. If $N(\mathfrak{g}) \neq \mathfrak{g}$, then $N(\mathfrak{g})$ is degenerate. On the other hand, for any $z \in Z(\mathfrak{g})$, $a \in N(\mathfrak{g})$ and $u \in \mathfrak{g}$ we have

$$\langle u.a, z\rangle = -\langle a, z.u\rangle = \langle z.a, u\rangle = -\langle a.u, z\rangle = 0.$$

This implies that $\mathfrak{g}.N(\mathfrak{g}) \subset Z(\mathfrak{g})^\perp$ and $N(\mathfrak{g}).\mathfrak{g} \subset Z(\mathfrak{g})^\perp$. Thus

$$[\mathfrak{g}, N(\mathfrak{g})] \subset Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp. \tag{5.1}$$

Proposition 5.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra. Then*

1. *$Z(\mathfrak{g})$ is degenerate.*
2. *For any $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, $\mathrm{L}_e = \mathrm{R}_e = 0$.*
3. *For any $x, y \in Z(\mathfrak{g})^\perp$, $\langle [x, y], [x, y]\rangle = 0$.*

Proof. 1. Suppose that $Z(\mathfrak{g})$ is non degenerate, i.e., $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = \{0\}$.
- If $\mathfrak{g} = N(\mathfrak{g})$ then according to (5.1), $[\mathfrak{g}, \mathfrak{g}] = 0$ which is impossible.
- If $\mathfrak{g} \neq N(\mathfrak{g})$ then $[\mathfrak{g}, N(\mathfrak{g})] = 0$ and hence $N(\mathfrak{g}) = Z(\mathfrak{g})$ which is impossible since $N(\mathfrak{g})$ is degenerate.

2. Let $e \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Since $Z(\mathfrak{g})^\perp \subset [\mathfrak{g}, \mathfrak{g}]^\perp$, then according to (2.6), $\mathrm{L}_e = \mathrm{R}_e$ is both symmetric and skew-symmetric and hence must vanish.
3. According to (2.1), we have for any $x, y \in Z(\mathfrak{g})^\perp$ $x.y = \frac{1}{2}[x, y]$. Using (2.3), we have $[x, y].x = x.(y.x) - y.(x.x)$, then $[x, y].x = 0$. In particular $\langle [x, y].x, y\rangle = 0$. Since L_x is skew-symmetric, thus $\langle [x, y], [x, y]\rangle = 0$. □

Proposition 5.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra. Then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is obtained by a sequence of double extension, starting from an abelian pseudo-Euclidean Lie algebra.*

Proof. Let e be a non-null vector in $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Since $\mathrm{L}_e = \mathrm{R}_e = 0$, then $\mathrm{I} = \mathbb{R}e$ is a totally isotropic two sided ideal, and $\mathrm{I}^\perp$ is also a two sided ideal. Thus, $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a double extension of a pseudo-Euclidean Lie algebra $(B_1, \langle\ ,\ \rangle_1)$. According to (4.4), B_1 is either abelian or 2-step nilpotent. If B_1 is 2-step nilpotent, then it's also a double extension of $(B_2, \langle\ ,\ \rangle_2)$. Since a 2-step nilpotent Lie algebra can not admit a flat Euclidean metric, then there exists $k \in \mathbb{N}^*$ such that B_k is abelian. □

Proposition 5.3. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra of signature $(p, p+q)$. If $\dim(Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp) = p$ then $Z(\mathfrak{g})^\perp$ is abelian.*

Proof. Let $\{e_1, \ldots, e_p\}$ be a basis of $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$, then we can write $Z(\mathfrak{g}) = Z_1 \oplus \mathrm{span}\{e_1, \ldots, e_p\}$ where $(Z_1, \langle\ ,\ \rangle_{/Z_1 \times Z_1})$ is euclidean. In $Z_1^\perp$ we can choose a totally isotropic subspace $\mathrm{span}\{\bar{e}_1, \ldots, \bar{e}_p\}$ such that, $\langle e_i, \bar{e}_j\rangle = 0$ for $i \neq j$, and $\langle e_i, \bar{e}_i\rangle = 1$. Let B_1 be the orthogonal of $Z_1 \oplus \mathrm{span}\{e_1, \ldots, e_p\} \oplus \mathrm{span}\{\bar{e}_1, \ldots, \bar{e}_p\}$. Thus we get a decomposition

$$\mathfrak{g} = Z_1 \oplus \operatorname{span}\{e_1, \dots, e_p\} \oplus B_1 \oplus \operatorname{span}\{\bar{e}_1, \dots, \bar{e}_p\}. \tag{5.2}$$

We have $Z(\mathfrak{g})^\perp = B_1 \oplus \operatorname{span}\{e_1, \dots, e_p\}$, and $(B_1, \langle\ ,\ \rangle_{/B_1 \times B_1})$ is euclidean. Let $\{b_1, \dots, b_r\}$ be an orthonormal basis of B_1. Since $\langle [b_i, b_j], [b_i, b_j]\rangle = 0$ and Z_1 is euclidean, then $[b_i, b_j] \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ for all $i, j \in \{1, \dots, r\}$. Thus, it suffices to show that $\langle [b_i, b_j], \bar{e}_k\rangle = 0$ for all $i, j \in \{1, \dots, r\}$ and $k \in \{1, \dots, p\}$. Let $z \in Z(\mathfrak{g})$, $k \in \{1, \dots, p\}$ and $i, j \in \{1, \dots, r\}$. We have $z.\bar{e}_k \in Z(\mathfrak{g})^\perp$ and $\langle z.\bar{e}_k, z.\bar{e}_k\rangle = 0$. Since B_1 is euclidean, then $z.\bar{e}_k \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Thus

$$\langle [\bar{e}_k, b_i], z\rangle = 2\langle \bar{e}_k.b_i, z\rangle = -2\langle z.\bar{e}_k, b_i\rangle = 0,$$

which implies that $[\bar{e}_k, b_i] \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ and $\bar{e}_k.b_i \in Z(\mathfrak{g})^\perp$. We have $\langle \bar{e}_k.b_i, b_j\rangle = -\frac{1}{2}\langle [b_i, b_j], \bar{e}_k\rangle$, then $\bar{e}_k.b_i = e_0 - \frac{1}{2}\sum_{j=1}^{r}\langle [b_i, b_j], \bar{e}_k\rangle b_j$, where $e_0 \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Using the flatness of the metric, we have $[\bar{e}_k, b_i].b_i = \bar{e}_k.(b_i.b_i) - b_i.(\bar{e}_k.b_i)$. Since $[\bar{e}_k, b_i].b_i = 0$, and $b_i.b_i = 0$, thus $b_i.(\bar{e}_k.b_i) = 0$. From the facts that $b_i.e_0 = 0$ and $b_i.b_j = \frac{1}{2}[b_i, b_j]$, we deduce that

$$\begin{aligned} b_i.(\bar{e}_k.b_i) &= -\frac{1}{2}\sum_{j=1}^{r}\langle [b_i, b_j], \bar{e}_k\rangle b_i.b_j \\ &= -\frac{1}{4}\sum_{j=1}^{r}\langle [b_i, b_j], \bar{e}_k\rangle [b_i, b_j], \end{aligned}$$

which implies that $\sum_{j=1}^{r}\langle [b_i, b_j], \bar{e}_k\rangle^2 = 0$, and completes the proof. □

Suppose that $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$. Then the decomposition (5.2) becomes

$$\mathfrak{g} = Z_1 \oplus \mathbb{R}e \oplus B_1 \oplus \mathbb{R}\bar{e}, \tag{5.3}$$

and the restriction of $\langle\ ,\ \rangle$ to Z_1 and B_1 is nondegenerate.

Proposition 5.4. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra such that $\dim(Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp) = 1$. With notations as in (5.3), if the restriction of the metric $\langle\ ,\ \rangle$ to B_1 is positive or negative definite, then $\dim B_1 = 1$, and $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$, where $\mathcal{H}_3$ is the 3-dimensional Heisenberg Lie algebra.*

Proof. Let $z \in Z(\mathfrak{g})$, and $b \in B_1$. We have $z.\bar{e} \in B_1$ and $z.b \in Z(\mathfrak{g})^\perp$. Since $\langle\ ,\ \rangle_{/B_1 \times B_1}$ is positive definite or negative definite and $\langle z.\bar{e}, z.\bar{e}\rangle = 0$, then $z.\bar{e} = 0$. Thus $\langle z.b, \bar{e}\rangle = 0$, which implies that $z.b \in B_1$. Using the same argument, then we can conclude that $z.b = 0$, and $\mathrm{L}_z = 0$ for any $z \in Z(\mathfrak{g})$. Let $x, y \in B_1$. We have for any $z \in Z(\mathfrak{g})$

$$\langle [x, y], z\rangle = 2\langle x.y, z\rangle = 0,$$

thus $[x,y]=\alpha e$, where $\alpha\in\mathbb{R}$. Using the flatness of the metric, then we get $[\bar{e},x].x=\bar{e}.(x.x)-x.(\bar{e}.x)$, thus $x.(\bar{e}.x)=0$. Let $\{b_1,\ldots,b_r\}$ be an orthonormal basis of B_1. Then

$$\bar{e}.x=\beta e\mp\frac{1}{2}\sum_{i=1}^{r}\langle[x,b_i],\bar{e}\rangle b_i$$

where $\beta\in\mathbb{R}$. Thus

$$x.(\bar{e}.x)=\mp\frac{1}{4}\sum_{i=1}^{r}\langle[x,b_i],\bar{e}\rangle[x,b_i]=0,$$

which implies that B_1 is abelian. On the other hand, we have for any $z\in Z(\mathfrak{g})$,

$$0=\langle z.\bar{e},x\rangle=-\frac{1}{2}\langle[\bar{e},x],z\rangle,$$

thus $[\bar{e},x]\in Z(\mathfrak{g})\cap Z(\mathfrak{g})^{\perp}$. Put $[\bar{e},b_i]=\alpha_i e$, where $\alpha_i\in\mathbb{R}^*$ for any $i\in\{1,\ldots,r\}$. In fact, if $\alpha_i=0$ then $b_i\in Z(\mathfrak{g})$, which contradicts the fact that $Z(\mathfrak{g})\cap B_1=\{0\}$. Suppose that $\dim B_1>1$. For any $i\in\{2,\ldots,r\}$, we put $b_i'=b_i-\frac{\alpha_i}{\alpha_1}b_1$, thus $[\bar{e},b_i']=0$ and $b_i'\in Z(\mathfrak{g})$ which is a Contradiction. Then $\dim B_1=1$ and the only non vanishing brackets in $\mathfrak{g}$ is $[\bar{e},b_1]=\alpha_1 e$, thus $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$. □

6. Flat pseudo-Euclidean 2-step nilpotent Lie algebras of signature $(2,n-2)$

Let us start by an example which play an important role in this section. Let L_6^4 be a 6-dimensional Lie algebra defined by the non vanishing Lie brackets, giving in the basis $\{x_1,\ldots,x_6\}$ by

$$[x_1,x_2]=x_5\ ,\ [x_1,x_3]=[x_2,x_4]=x_6.$$

This Lie algebra appear in the classification of 2-step nilpotent Lie algebras of dimension 6, as for example in [1, pp. 3], or in [7, pp. 97], where it is denoted by $L_{6,3}$.

It is clear that this Lie algebra admits no flat Euclidean or Lorentzian metrics. However, L_6^4 admits a flat pseudo-Euclidean metrics of signature $(2,n-2)$. In fact, let $\langle\ ,\ \rangle_0$ be a pseudo-Euclidean metric of signature $(2,4)$ defined in the basis $\{x_1,\ldots,x_6\}$ by the matrix

$$\langle\ ,\ \rangle_0=\begin{pmatrix}0&0&0&0&a&1\\0&0&b&c&0&0\\0&b&0&0&0&0\\0&c&0&d&0&0\\a&0&0&0&\frac{1}{3d}&0\\1&0&0&0&0&0\end{pmatrix}$$

where $a, c \in \mathbb{R}$, $b \in \mathbb{R}^*$ and $d > 0$. Straightforward calculations using (2.1) shows that, the only non vanishing Levi-Civita products are

$$x_1.x_1 = -\frac{1}{b}x_2 - \left(\frac{a}{b} + \frac{c^2}{b^2 d}\right)x_3 + \frac{c}{bd}x_4,\ x_1.x_2 = \frac{c}{2bd}x_3 - \frac{1}{2d}x_4 + \frac{1}{2}x_5 + \frac{a}{2}x_6,$$
$$x_1.x_3 = x_6,\ x_1.x_4 = \frac{1}{2b}x_3,\ x_1.x_5 = x_5.x_1 = -\frac{1}{6bd}x_3,\ x_2.x_4 = \frac{1}{2}x_6,$$
$$x_2.x_5 = x_5.x_2 = \frac{1}{6d}x_6,$$
$$x_2.x_1 = \frac{c}{2bd}x_3 - \frac{1}{2d}x_4 - \frac{1}{2}x_5 + \frac{a}{2}x_6,\ x_4.x_1 = \frac{1}{2b}x_3,\ x_4.x_2 = -\frac{1}{2}x_6.$$

One can verify that for any $x, y \in \mathrm{L}_6^4$, we have $\mathrm{L}_{[x,y]} = [\mathrm{L}_x, \mathrm{L}_y]$, which shows that $(\mathrm{L}_6^4, \langle\ ,\ \rangle_0)$ is flat. The following Theorem shows that this example, is the only non trivial one such that $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Euclidean 2-step nilpotent Lie algebra of signature $(2, n-2)$. According to Theorem 3.1, the dimension of $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ is 1 or 2.

Theorem 6.1. *A 2-step nilpotent Lie algebra $\mathfrak{g}$ admits a flat pseudo-Euclidean metric of signature $(2, n-2)$ such that $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$ if and only if $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$ or $\mathfrak{g}$ is a trivial central extension of L_6^4. Furthermore, in the second case, the restriction of the metric to L_6^4 is giving by $\langle\ ,\ \rangle_0$.*

Proof. If $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$, then we can split $\mathfrak{g}$ as

$$\mathfrak{g} = Z_1 \oplus \mathbb{R}e \oplus B_1 \oplus \mathbb{R}\bar{e},$$

where $Z(\mathfrak{g}) = Z_1 \oplus \mathbb{R}e$, $Z(\mathfrak{g})^\perp = \mathbb{R}e \oplus B_1$, $\mathrm{span}\{e, \bar{e}\} = (Z_1 \oplus B_1)^\perp$, $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$ and $\langle e, \bar{e}\rangle = 1$. We have two cases:

First case: $\langle\ ,\ \rangle/_{B_1 \times B_1}$ is positive or negative definite. Then according to Proposition 5.4, $\dim B_1 = 1$ and $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$.

Second case: $\langle\ ,\ \rangle/_{B_1 \times B_1}$ is Lorentzian. Then $\dim B_1 \geq 2$. For any $z, z' \in Z(\mathfrak{g})$, we have $\langle z.\bar{e}, z'.\bar{e}\rangle = 0$, then $\mathrm{R}_{\bar{e}}(Z(\mathfrak{g}))$ is a totally isotropic subspace. Since $\mathrm{R}_{\bar{e}}(Z(\mathfrak{g})) \subset B_1$ and $(B_1, \langle\ ,\ \rangle/_{B_1 \times B_1})$ is Lorentzian, then there exists an isotropic vector $b_0 \in B_1$ and a covector $\lambda \in Z(\mathfrak{g})^*$ such that $z.\bar{e} = \lambda(z) b_0$ for any $z \in Z(\mathfrak{g})$.

Let $x, y \in Z(\mathfrak{g})^\perp$. Recall that $x.y = \frac{1}{2}[x, y]$ and $\langle [x, y], [x, y]\rangle = 0$. Since Z_1 is Euclidean then $[x, y] \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Choose a basis $\{b_0, \bar{b}, b_1, \ldots, b_r\}$ of B_1 such that $\{b_1, \ldots, b_r\}$ is orthonormal, $\mathrm{span}\{b_0, \bar{b}\}$ and $\mathrm{span}\{b_1, \ldots, b_r\}$ are orthogonal, $\bar{b}$ is isotropic and $\langle b_0, \bar{b}\rangle = 1$. Then for any $i \in \{0, 1, \ldots, r\}$, we have from (2.1)

$$\langle [\bar{e}, b_i].\bar{e}, b_i\rangle = -\frac{1}{2}\langle [\bar{e}, b_i], [\bar{e}, b_i]\rangle.$$

On the other hand, we have $\langle [\bar{e}, b_i].\bar{e}, bi\rangle = \langle \lambda\left([\bar{e}, b_i]\right) b_0, b_i\rangle = 0$, then $[\bar{e}, b_i] \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$.

We can write from the condition of flatness, for any $x, y, z \in \mathfrak{g}$

$$[x, y].z = x.(y.z) - y.(x.z). \tag{6.1}$$

If we take $x = \bar{e}$ and $y = z = b_0$, we get $b_0.(\bar{e}.b_0) = 0$. Let $i \in \{0, 1, \dots, r\}$, since $\bar{e}.b_0 \in Z(\mathfrak{g})^\perp$ and $\langle \bar{e}.b_0, b_i\rangle = -\frac{1}{2}\langle [b_0, b_i], \bar{e}\rangle$, thus

$$\bar{e}.b_0 = \alpha e + \beta b_0 - \frac{1}{2}\sum_{i=1}^{r}\langle [b_0, b_i], \bar{e}\rangle,$$

where $\alpha, \beta \in \mathbb{R}$. It follows that $b_0.(\bar{e}.b_0) = -\frac{1}{4}\sum_{i=1}^{r}\langle [b_0, b_i], \bar{e}\rangle [b_0, b_i]$, thus $\sum_{i=1}^{r}\langle [b_0, b_i], \bar{e}\rangle^2 = 0$ which implies that $[b_0, b_i] = 0$ for any $i \in \{0, 1, \dots, r\}$.

If we take in (6.1), $x = \bar{e}$, $y = b_0$ and $z = \bar{b}$ we get $b_0.(\bar{e}.\bar{b}) = 0$. Using the fact that $b_0.u = 0$ for any $u \in Z(\mathfrak{g})$, we deduce that $b_0.(\bar{e}.\bar{b}) = -\frac{1}{4}\langle [\bar{b}, b_0], \bar{e}\rangle [\bar{b}, b_0]$, thus $[\bar{b}, b_0] = 0$. Similarly, for any $i \in \{1, \dots, r\}$, if we take in (6.1), $x = \bar{e}$ and $y = z = e_i$ we get

$$0 = b_i.(\bar{e}.b_i) = -\frac{1}{4}\sum_{j=1}^{r}\langle [b_i, b_j], \bar{e}\rangle [b_i, b_j],$$

thus $[b_i, b_j] = 0$ for any $i, j \in \{1, \dots, r\}$. It follows that $\mathrm{span}\{b_0, b_1, \dots, b_r\}$ is abelian and $[b_0, \bar{b}] = 0$. We put

$$[\bar{e}, b_i] = \alpha_i e,\ [\bar{e}, \bar{b}] = \alpha e + z_0,\ [\bar{b}, b_i] = \beta_i e,$$

where $\alpha_i, \beta_i, \alpha \in \mathbb{R}$, $z_0 \in Z_1$ and $i = 0, 1, \dots, r$. If we take in (6.1), $x = \bar{e}$ and $y = z = \bar{b}$ we get $z_0.\bar{b} = -\bar{b}(\bar{e}.\bar{b})$, then $\frac{3}{2}z_0.\bar{b} - \frac{1}{2}\sum_{i=1}^{r}\beta_i^2 e = 0$, which implies that

$$3\langle z_0, z_0\rangle = \sum_{i=1}^{r}\beta_i^2. \tag{6.2}$$

We have $\dim B_1 \geq 3$. In fact, if $\dim B_1 = 2$ then $B_1 = \mathrm{span}\{b_0, \bar{b}\}$ and (6.2) implies that $z_0 = 0$. Then the Lie brackets are reduced to $[\bar{e}, b_i] = \alpha_i e$ and $[\bar{e}, \bar{b}] = \alpha e$, and as in the proof of Proposition 5.4 we can deduce that $\dim B_1 = 1$, which is a contradiction. The same argument shows that $z_0 \neq 0$. Then there exists $i \in \{1, \dots, r\}$ such that $\beta_i \neq 0$. To simplify, we can suppose that $\beta_1 \neq 0$, and we have also $\alpha_0 \neq 0$ because $b_0 \notin Z(\mathfrak{g})$.

Let us show that $\dim B_1 = 3$. In fact, if $\dim B_1 \geq 4$, then we put for any $i \geq 4$,

$$b_i' = b_i - \frac{\beta_i}{\beta_1}b_1 - \left(\frac{\alpha_i\beta_1 - \alpha_1\beta_i}{\alpha_0\beta_1}\right)b_0,$$

and we can verify easily that $[b_i', x] = 0$ for any $x \in \mathfrak{g}$. Thus $b_i' \in Z(\mathfrak{g})$ which contradicts the fact that $Z(\mathfrak{g}) \cap B_1 = \{0\}$. We put $x_1 = \bar{e}$, $x_2 = \bar{b}$, $x_3 = \frac{b_0}{\alpha_0}$, $x_4 = \frac{1}{\beta_1} b_1 - \frac{\alpha_1}{\beta_1 \alpha_0} b_0$, $x_5 = \alpha e + z_0$ and $x_6 = e$. Then the only non vanishing brackets on $\mathfrak{g}$ are

$$[x_1, x_2] = x_5, \ [x_1, x_3] = [x_2, x_4] = x_6.$$

It follows that $\mathfrak{g}$ is a trivial central extension of L_6^4. Furthermore, with the condition (6.2), one can verify that the restriction of the metric to L_6^4 is given by $\langle\ ,\ \rangle_0$. Conversely, if $\mathfrak{g}$ splits orthogonally into $\mathfrak{g} = Z_1 \oplus \mathrm{L}_6^4$, where $Z_1 \subset Z(\mathfrak{g})$ and the restriction of the metric to L_6^4 is $\langle\ ,\ \rangle_0$, and the restriction to Z_1 is Euclidean, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a flat pseudo-Euclidean 2-step nilpotent Lie algebra of signature $(2, n-2)$ and $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$. □

Corollary 6.1. *The Heisenberg Lie algebra $\mathcal{H}_{2k+1}$ admits a flat pseudo-Euclidean metric of signature $(2, n-2)$ if and only if $k = 1$.*

Proof. Since $Z(\mathcal{H}_{2k+1}) = 1$, then if $\mathfrak{g} = \mathcal{H}_{2k+1}$ admits such metric then we have $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$. This gives the result, by virtue of Theorem 6.1. □

Remark 3. In Theorem 6.1, if $\mathfrak{g}$ is a trivial central extension of $\mathcal{H}_3$, then $\mathfrak{g} = Z_1 \oplus \mathcal{H}_3$ and the metric $\langle\ ,\ \rangle$ has one of the following form:

- The restriction of $\langle\ ,\ \rangle$ to Z_1 is Euclidean and its restriction to $\mathcal{H}_3$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & -1 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \text{ where } \alpha \in \mathbb{R}.$$

- The restriction of $\langle\ ,\ \rangle$ to Z_1 is Lorentzian and its restriction to $\mathcal{H}_3$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 1 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \text{ where } \alpha \in \mathbb{R}.$$

Theorem 6.2. *A 2-step nilpotent Lie algebra $\mathfrak{g}$ admits a flat pseudo-Euclidean metric $\langle\ ,\ \rangle$ of signature $(2, n-2)$ such that $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 2$ if and only if there exist an orthonormal vectors $\{b_1, \ldots, b_k\}$ in $\mathfrak{g}$, a linearly independent isotropic vectors $\{e_1, \bar{e}_1, e_2, \bar{e}_2\}$ in $\{b_1, \ldots, b_k\}^\perp$, where $\langle e_1, e_2\rangle = \langle e_1, \bar{e}_2\rangle = \langle \bar{e}_1, e_2\rangle = \langle \bar{e}_1, \bar{e}_2\rangle = 0$ and $\langle e_1, \bar{e}_1\rangle = \langle e_2, \bar{e}_2\rangle = 1$, such that for any $i \in \{1, \ldots, k\}$ the only non vanishing brackets are*

$$[\bar{e}_1, \bar{e}_2] = z_0,$$

$$[\bar{e}_1, b_i] = \alpha_i e_1 + \beta_i e_2, \tag{6.3}$$
$$[\bar{e}_2, b_i] = \gamma_i e_1 + \delta_i e_2,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$, *and*

$$3\langle z_0, z_0\rangle = \sum_{i=1}^{k}(\gamma_i + \beta_i)^2 - 4\alpha_i\delta_i. \tag{6.4}$$

Proof. According to Proposition 5.3, $Z(\mathfrak{g})^\perp$ is abelian, and we can split $\mathfrak{g}$ into

$$\mathfrak{g} = Z_1 \oplus \mathrm{span}\{e_1, e_2\} \oplus B_1 \oplus \mathrm{span}\{\bar{e}_1, \bar{e}_2\}, \tag{6.5}$$

where $Z(\mathfrak{g}) = Z_1 \oplus \mathrm{span}\{e_1, e_2\}$, $Z(\mathfrak{g})^\perp = \mathrm{span}\{e_1, e_2\} \oplus B_1$, $(Z_1 \oplus B_1)^\perp = \mathrm{span}\{e_1, e_2, \bar{e}_1, \bar{e}_2\}$, $\mathrm{span}\{\bar{e}_1, \bar{e}_2\}$ is totally isotropic, $\langle e_1, e_2\rangle = \langle e_1, \bar{e}_2\rangle = \langle \bar{e}_1, e_2\rangle = \langle \bar{e}_1, \bar{e}_2\rangle = 0$ and $\langle e_1, \bar{e}_1\rangle = \langle e_2, \bar{e}_2\rangle = 1$.

In the proof of Proposition 5.3, we have shown that for any $x, y \in Z(\mathfrak{g})^\perp$ and $k \in \{1, 2\}$, $[x, y]$ and $[\bar{e}_k, x]$ are in $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Let $\{b_1, \dots, b_r\}$ be an orthonormal basis of B_1. Then, the non vanishing brackets are:

$$[\bar{e}_1, \bar{e}_2] = z_0,$$
$$[\bar{e}_1, b_i] = \alpha_i e_1 + \beta_i e_2,$$
$$[\bar{e}_2, b_i] = \gamma_i e_1 + \delta_i e_2,$$

where $z_0 \in Z(\mathfrak{g}), \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ and $i = 1, \dots, r$. From (2.1) and the Lie brackets above, we have for any $u \in Z(\mathfrak{g})$ and $v \in Z(\mathfrak{g})^\perp$, $u.v = 0$. Recall that $(\mathfrak{g}, \langle\ ,\ \rangle)$ is flat if and only if for any $x, y, z \in \mathfrak{g}$

$$\mathrm{L}_{[x,y]}(z) = [\mathrm{L}_x, \mathrm{L}_y](z). \tag{6.6}$$

Let $x \in Z(\mathfrak{g}) + Z(\mathfrak{g})^\perp$, $y, z \in \mathfrak{g}$ and $i \in \{1, 2\}$. We have $\langle y.z, e_i\rangle = 0$, then $y.z \in Z(\mathfrak{g}) + Z(\mathfrak{g})^\perp$. Thus $x.(y.z) = (y.z).x = 0$. On the other hand, we have $x.y,\ y.x \in Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$. Thus $(x.y).z = (y.x).z = 0$. It follows that if one of the vectors x, y or z is in $Z(\mathfrak{g}) + Z(\mathfrak{g})^\perp$, then (6.6) is satisfied. Thus $(\mathfrak{g}, \langle\ ,\ \rangle)$ is flat if and only if

$$\mathrm{L}_{[\bar{e}_1, \bar{e}_2]}\bar{e}_1 - [\mathrm{L}_{\bar{e}_1}, \mathrm{L}_{\bar{e}_2}]\bar{e}_1 = \mathrm{L}_{[\bar{e}_1, \bar{e}_2]}\bar{e}_2 - [\mathrm{L}_{\bar{e}_1}, \mathrm{L}_{\bar{e}_2}]\bar{e}_2 = 0.$$

Straightforward calculations using (2.1) give

$$z_0.\bar{e}_1 = -\frac{1}{2}\langle z_0, z_0\rangle e_2,\ \ \bar{e}_2.\bar{e}_1 = -\frac{1}{2}z_0 - \frac{1}{2}\sum_{i=1}^{r}(\beta_i + \gamma_i)b_i,\ \ \bar{e}_1.\bar{e}_1 = -\sum_{i=1}^{r}\alpha_i b_i,$$
$$\bar{e}_1 b_i = \alpha_i e_1 + \frac{1}{2}(\beta_i + \gamma_i)e_2,\ \ \bar{e}_2 b_i = \frac{1}{2}(\beta_i + \gamma_i)e_1 + \delta_i e_2.$$

Thus the condition $\mathrm{L}_{[\bar{e}_1,\bar{e}_2]}\bar{e}_1 - [\mathrm{L}_{\bar{e}_1}, \mathrm{L}_{\bar{e}_2}]\,\bar{e}_1 = 0$ is equivalent to (6.3). Similarly, we show that the second condition $\mathrm{L}_{[\bar{e}_1,\bar{e}_2]}\bar{e}_2 - [\mathrm{L}_{\bar{e}_1}, \mathrm{L}_{\bar{e}_2}]\,\bar{e}_2 = 0$ is also equivalent to (6.3). This completes the proof. □

Corollary 6.2. *If a 2-step nilpotent Lie algebra* $\mathfrak{g}$ *admits a flat pseudo-Euclidean metric of signature* $(2, n-2)$*, then* $\dim[\mathfrak{g}, \mathfrak{g}] \leqslant 3$.

7. Examples

In this section, we show that any 6-dimensional 2-step nilpotent Lie algebra, which is not a trivial central extension of $\mathcal{H}_5$, admits a flat pseudo-Euclidean metric of signature $(2, n-2)$, where $\mathcal{H}_5$ is a 5-dimensional Heisenberg Lie algebra. For this, we use the table below which give all 6-dimensional 2-step nilpotent Lie algebras (see [1, pp. 3]). Note that $\mathcal{H}_5$ (resp. $\mathcal{H}_3$) is denoted in this table by L_5^4 (resp. L_3).

Lie algebra	Nonzero commutators
$L_3 \oplus 3L_1$	$[x_1, x_2] = x_3$
$L_5^1 \oplus L_1$	$[x_1, x_2] = x_3,\ [x_1, x_4] = x_5$
$L_5^4 \oplus L_1$	$[x_1, x_3] = x_5,\ [x_2, x_4] = x_5$
$L_3 \oplus L_3$	$[x_1, x_2] = x_3,\ [x_4, x_5] = x_6$
L_6^4	$[x_1, x_2] = x_5,\ [x_1, x_3] = x_6,\ [x_2, x_4] = x_6$
$L_6^5(-1)$	$[x_1, x_3] = x_5,\ [x_1, x_4] = x_6,\ [x_2, x_4] = x_5,\ [x_2, x_3] = -x_6$
L_6^3	$[x_1, x_3] = x_6,\ [x_1, x_2] = x_4,\ [x_2, x_3] = x_5$

The result is evident for $\mathrm{L}_3 \oplus 3\mathrm{L}_1$ and L_6^4 (Theorem 6.1). Let $\langle\ ,\ \rangle$ be a pseudo-Euclidean metric of signature $(2, n-2)$ given by the matrix

$$\langle\ ,\ \rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Theorem 6.2, let us show that, for all those Lie algebras $\mathrm{L}_5^1 \oplus \mathrm{L}_1$, $\mathrm{L}_3 \oplus \mathrm{L}_3$, $\mathrm{L}_6^5(-1)$ and L_6^3 there exists a basis $\mathbb{B}$ such that the metric given in $\mathbb{B}$ by $\langle\ ,\ \rangle$ is flat.

- For $\mathrm{L}_5^1 \oplus \mathrm{L}_1$, with our notations we put $\mathbb{B} = \{e_1, \bar{e}_1, e_2, \bar{e}_2, z_0, b_1\}$ where $e_1 = x_6$, $\bar{e}_1 = x_1$, $e_2 = x_5$, $\bar{e}_2 = x_2$, $z_0 = x_3$ and $b_1 = x_4$. One can verify easily that in this basis, the Lie brackets and the metric verify the conditions (6.3) and (6.4), thus $(\mathrm{L}_5^1 \oplus \mathrm{L}_1, \langle\ ,\ \rangle)$ is flat.
- For $\mathrm{L}_3 \oplus \mathrm{L}_3$, we put $\mathbb{B} = \{e_1, \bar{e}_1, e_2, \bar{e}_2, b_1, b_2\}$ where $e_1 = x_3$, $\bar{e}_1 = x_1$, $e_2 = x_6$, $\bar{e}_2 = x_4$, $b_1 = x_1$ and $b_2 = x_5$.

- For $L_6^5(-1)$, we put $\mathbb{B} = \{e_1, \bar{e}_1, e_2, \bar{e}_2, b_1, b_2\}$ where $e_1 = x_5 + x_6$, $\bar{e}_1 = x_1$, $e_2 = -2x_6$, $\bar{e}_2 = x_2$, $b_1 = x_4$ and $b_2 = -(3 + \sqrt{15})x_4 - x_3$.
- For L_6^3, we put $\mathbb{B} = \{e_1, \bar{e}_1, e_2, \bar{e}_2, z_0, b_1\}$ where $e_1 = x_4$, $\bar{e}_1 = x_1$, $e_2 = \frac{4}{3}x_5$, $\bar{e}_2 = x_3$, $z_0 = x_6$ and $b_1 = x_2$.

For $\mathfrak{g} = L_5^4 \oplus L_1$, it is clear that this algebra can not admit flat pseudo-Euclidean metric of signature $(2, n-2)$ such that $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 1$ (Theorem 6.1). Suppose that it admits such metric with $\dim Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = 2$ (Theorem 6.2). We have $\dim[\mathfrak{g}, \mathfrak{g}] = 1$ and $\dim Z(\mathfrak{g}) = 2$. Then $\dim Z(\mathfrak{g})^\perp = 4$ and $\dim B_1 = 2$. Put $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}e_1$, thus the Lie brackets satisfy

$$[\bar{e}_1, \bar{e}_2] = \alpha e_1, \ [\bar{e}_1, b_i] = \alpha_i e_1, \ [\bar{e}_2, b_i] = \gamma_i e_1, \ i = 1, 2.$$

The condition (6.4) implies that $\gamma_1 = \gamma_2 = 0$. Then $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}^*$. The fact that $\alpha = 0$, for example, implies that $\bar{e}_2 \in Z(\mathfrak{g})$. Put $b_2' = b_2 - \frac{\alpha_2}{\alpha_1} b_1$, then $b_2' \in Z(\mathfrak{g})$, which is a contradiction. It follows that $L_5^4 \oplus L_1$ can not admit flat pseudo-Euclidean metrics of signature $(2, n-2)$.

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