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On para-Kähler and hyper-para-Kähler Lie algebras ☆

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ABSTRACT

In this study, we consider Lie algebras that admit para-Kähler and hyper-para-Kähler structures. We provide new characterizations of these Lie algebras and develop many methods for building large classes of examples. Previously, Bai considered para-Kähler Lie algebras as left symmetric bialgebras. We reconsider this viewpoint and make improvements in order to obtain some new results. The study of para-Kähler and hyper-para-Kähler is intimately linked to the study of left symmetric algebras, particularly those that admit invariant symplectic forms. In this study, we provide many new classes of left symmetric algebras and complete descriptions of all the associative algebras that admit an invariant symplectic form. We also describe all four-dimensional hyper-para-Kähler Lie algebras.

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1. Introduction

A *para-complex* structure on a $2n$-dimensional manifold M is a field K of involutive endomorphisms ($K^2 = Id_{TM}$) such that the eigendistributions $T^{\pm}M$ with eigenvalues ± 1 have constant rank n and are integrable. In the presence of a pseudo-Riemannian metric, this notion leads to the notion of para-Kähler manifolds. A *para-Kähler* structure on a manifold M is a pair (g, K), where g is a pseudo-Riemannian metric and K is a parallel skew-symmetric para-complex structure. If (g, K) is a para-Kähler structure on M, then $\omega = g \circ K$ is a symplectic structure and the ± 1-eigendistributions $T^{\pm}M$ of K are two integrable ω-Lagrangian distributions. Thus, a para-Kähler structure can be identified with a bi-Lagrangian structure $(\omega, T^{\pm}M)$, where ω is a symplectic structure and $T^{\pm}M$ are two integrable Lagrangian distributions. If (M, g, K) is a para-Kähler manifold and J is a parallel field of skew-symmetric endomorphisms such that $J^2 = -Id_{TM}$ and $JK = -KJ$, then (M, g, K, J) is called a *hyper-para-Kähler* manifold or *hyper-symplectic* manifold. The notion of an almost para-complex structure (or almost product structure) on a manifold was introduced by Rasevskii [19] and Libermann [17]. A previous study [12] provided a survey of para-Kähler geometries. Hyper-para-Kähler structures were introduced by Hitchin [14] and they have become an important area of research recently, mainly because of their applications in theoretical physics (especially in dimension 4). For example, [9] provides a discussion of the relationship between hyper-para-Kähler metrics and the $N = 2$ superstring. When the manifold is a Lie group G, the metric and the para-complex structure are considered left-invariant, where they are both determined by their restrictions to the Lie algebra $\mathfrak{g}$ of G. In this situation, $(\mathfrak{g}, g_e, K_e)$ is called a *para-Kähler Lie algebra*. We also recover the notion of *hyper-para-Kähler Lie algebra* when we start from a left invariant hyper-para-Kähler structure on a Lie group. Para-Kähler and hyper-para-Kähler Lie algebras have been studied widely [1,3,5–7].

In the present study, we consider para-Kähler and hyper-para-Kähler Lie algebras. We obtain some known results using a new approach, which we consider simplifies both the presentation and the proofs. In the final part of this study, we provide some new results that facilitate a better understanding of these algebras and the construction of rich classes of non-trivial new examples. The basic tools used in the study of para-Kähler and hyper-para-Kähler Lie algebras are two types of algebras: left symmetric algebras, which have been studied widely, and a less well-known class of left symmetric algebras endowed with invariant symplectic forms, which are referred to as special symplectic Lie algebras in [4]. We refer to these algebras as symplectic left symmetric algebras. Our study also considers the construction of large classes of left symmetric algebras, symplectic left symmetric algebras, and symplectic Lie algebras.

The remainder of this paper is organized as follows. In Sections 2 and 3, we recall some basic definitions and we use a new approach to obtain some known characterizations of para-Kähler Lie algebras. In particular, we consider the notion of left symmetric bialgebras introduced by Bai [5]. We introduce the notion of quasi-S-matrices as a generalization of the S-matrices introduced by Bai. Proposition 3.7 describes the Lie algebra structure of the para-Kähler Lie algebra associated with a quasi-S-matrix, which plays a crucial role in Sections 6–7. We also show (see Remark 2 (b)) that a quasi-S-matrix on a left symmetric algebra U defines a Lie triple system on U^* (see [15,18,20] for the definition and properties of Lie triple systems). In Section 4, we develop some general methods for building new examples of para-Kähler Lie algebras. In Section 5, we provide a new characterization of hyper-para-Kähler Lie algebras based on a notion of the compatibility between two left symmetric products on a given vector space (see Theorem 5.1 and Definition 5.1). Sections 6–7 consider quasi-S-matrices on symplectic Lie algebras, symplectic left symmetric algebras, and pseudo-Riemannian flat Lie algebras. On a symplectic Lie algebra with its canonical left symmetric product, the set of quasi-S-matrices is in bijection with the set of solutions of an equation that generalizes the modified Yang–Baxter equation (see Proposition 6.1). As a consequence, we obtain a method for building a new class of para-Kähler Lie algebras (see Theorem 6.1) as well as a new class of Lie algebras with a Lie triple system (see Remark 4). On a symplectic left symmetric algebra or a pseudo-Riemannian flat Lie algebra, the set of S-matrices is in bijection with the set of operators that generalize $\mathcal{O}$-operators (see Proposition 7.1). Thus, we obtain a method for building a new class of para-Kähler and hyper-para-Kähler Lie algebras (see Theorems 7.1–7.2). We also determine a new class of Lie algebras with a Lie triple system (see Remark 5). In Section 8, we provide all the four-dimensional para-Kähler Lie algebras. We use a method that differs from the one described in [1], which has the advantage of simplifying the calculations greatly. In Section 9, we provide a complete description of associative symplectic left symmetric algebras (see Theorems 9.1–9.2).

Notations For a Lie algebra $\mathfrak{g}$, its bracket will be denoted by $[\ ,\ ]$ and for any $u \in \mathfrak{g}$, ad_u is the endomorphism of $\mathfrak{g}$ given by $\mathrm{ad}_u(v) = [u, v]$. If $A : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an endomorphism, the Nijenhuis torsion of A is given by

$$N_A(u, v) := [Au, Av] - A[Au, v] - A[u, Av] + A^2[u, v]. \qquad (1)$$

If $(U, .)$ is an algebra, for any $u \in U$, $\mathrm{L}_u, \mathrm{R}_u : U \longrightarrow U$ denote the left and the right multiplication by u given by $\mathrm{L}_u(v) = u.v$ and $\mathrm{R}_u(v) = v.u$, respectively. The commutator of $(U, .)$ is the bracket on U given by $[u, v] = u.v - v.u$. The curvature of $(U, .)$ is the tensor K given by

$$\mathrm{K}(u, v) := [\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]}.$$

Then, for any $u, v, w \in U$, we have the Bianchi identity

$$\oint [u, [v, w]] = \oint \mathrm{K}(u, v)w, \qquad (2)$$

where $\oint$ denotes the cyclic sum. The product on U is called Lie-admissible if its commutator is a Lie bracket, i.e., for any $u, v, w \in U$,

$$\oint [u,[v,w]] = \oint \mathrm{K}(u,v)w = 0.$$

Let V be a vector space and F is an endomorphism of V. We denote $F^t : V^* \longrightarrow V^*$ as the dual endomorphism. For any $X \in V$ and $\alpha \in V^*$, we denote $\alpha(X)$ by $\prec\alpha, X\succ$. The phase space of V is the vector space $\Phi(V) := V \oplus V^*$ endowed with the two nondegenerate bilinear forms $\langle\ ,\ \rangle_0$ and Ω_0 given by

$$\langle u+\alpha, v+\beta\rangle_0 = \prec\alpha, v\succ + \prec\beta, u\succ \quad \text{and} \quad \Omega_0(u+\alpha, v+\beta) = \prec\beta, u\succ - \prec\alpha, v\succ.$$

We denote $K_0 : \Phi(V) \longrightarrow \Phi(V)$ as the endomorphism given by $K_0(u+\alpha) = u - \alpha$.

Let $\omega \in \wedge^2 V^*$, which is nondegenerate. We denote $\flat : V \longrightarrow V^*$ as the isomorphism given by $\flat(v) = \omega(v, .)$. Put $T(V) := V \times V$ and define $\langle\ ,\ \rangle_1$, Ω_1, K_1, J_1 on $T(V)$ by

$$\Omega_1[(u,v),(w,z)] = \omega(z,u) - \omega(v,w),\ \langle (u,v),(w,z)\rangle_1 = \omega(z,u) + \omega(v,w),$$
$$K_1(u,v) = (u,-v) \quad \text{and} \quad J_1(u,v) = (-v,u).$$

Finally, if $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a representation of a Lie algebra, we denote $\rho^* : \mathfrak{g} \longrightarrow \mathrm{End}(V^*)$ as the dual representation given by $\rho^*(X)(\alpha) = -\rho(X)^t(\alpha)$.

2. Some definitions

In this section, we recall some definitions of different types of algebraic structures, which are used throughout this study.

- A complex structure on a Lie algebra $\mathfrak{g}$ is an isomorphism $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ that satisfies $J^2 = -\mathrm{Id}_\mathfrak{g}$ and $N_J = 0$. A complex structure J is called abelian if for any $u, v \in \mathfrak{g}$,

  $$[Ju, Jv] = [u,v].$$

  A para-complex structure on a Lie algebra $\mathfrak{g}$ is an isomorphism $K : \mathfrak{g} \longrightarrow \mathfrak{g}$ that satisfies $K^2 = \mathrm{Id}_G$, $N_K = 0$ and $\dim\ker(K + \mathrm{Id}_\mathfrak{g}) = \dim\ker(K - \mathrm{Id}_\mathfrak{g})$. A para-complex structure K is called abelian if for any $u, v \in \mathfrak{g}$,

  $$[Ku, Kv] = -[u,v].$$

  A complex product structure on $\mathfrak{g}$ is a couple (J, K), where J is a complex structure, K is a para-complex structure, and $KJ = -JK$.
- A *pseudo-Riemannian Lie algebra* is a finite-dimensional Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a bilinear symmetric nondegenerate form $\langle\ ,\ \rangle$. The associated *Levi-Civita product* is the product on $\mathfrak{g}$, $(u,v) \mapsto u.v$, which is given by Koszul's formula

  $$2\langle u.v, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle. \tag{3}$$

This product is determined entirely by the fact that it is Lie-admissible, i.e., $[u, v] = u.v - v.u$, and for any $u \in \mathfrak{g}$, the left multiplication by u is skew-symmetric with respect to $\langle\ ,\ \rangle$. We say that $(\mathfrak{g}, [\ ,\ ])$ is flat when its Levi-Civita product has a vanishing curvature.

- An algebra $(U, .)$ is called *left symmetric* if

$$\mathrm{ass}(u, v, w) = \mathrm{ass}(v, u, w),$$

where $\mathrm{ass}(u, v, w) = (u.v).w - u.(v.w)$. This relation is equivalent to the vanishing of the curvature of $(U, .)$. Relation (2) implies that a left symmetric product is Lie-admissible. If $(U, .)$ is a left symmetric algebra, then the Lie algebra $(U, [\ ,\ ])$ has two representations, i.e., $\mathrm{ad}_U : U \longrightarrow \mathrm{End}(U)$, $u \mapsto \mathrm{ad}_u$ and $\mathrm{L}_U : U \longrightarrow \mathrm{End}(U)$, $u \mapsto \mathrm{L}_u$.

Any associative algebra is a left symmetric algebra and if a left symmetric algebra is abelian then it is associative.

- A *symplectic Lie algebra* is a Lie algebra $(\mathfrak{g}, \omega)$ endowed with a bilinear skew-symmetric nondegenerate form ω such that for any $u, v, w \in \mathfrak{g}$,

$$\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0.$$

According to a well-known result [11], the product $\mathbf{a} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$\omega(\mathbf{a}(u, v), w) = -\omega(v, [u, w]) \tag{4}$$

induces a left symmetric algebra structure that satisfies $\mathbf{a}(u, v) - \mathbf{a}(v, u) = [u, v]$ on $\mathfrak{g}$. We denote $\mathbf{a}$ as the *left symmetric product* associated with $(\mathfrak{g}, \omega)$.

- A *symplectic left symmetric algebra* is a left symmetric algebra $(U, .)$ endowed with a bilinear skew-symmetric nondegenerate form ω, which is invariant, i.e., for any $u, v, w \in U$,

$$\omega(u.v, w) + \omega(v, u.w) = 0.$$

This implies that $(U, [\ ,\ ], \omega)$ is a symplectic Lie algebra. Symplectic left symmetric algebras, which are called special symplectic Lie algebras in [6], play a central role in the study of hyper-para-Kähler Lie algebras (see Section 5).

3. Para-Kähler Lie algebras

The notion of a para-Kähler Lie algebra is very subtle and it has many equivalent definitions, which depend on whether we emphasize its pseudo-Riemannian metric and the associated Levi-Civita product, or its symplectic form and the associated left symmetric product. There are also many characterizations, i.e., at least three characterizations in [5], where a para-Kähler Lie algebra can be characterized as the phase space of a

Lie algebra, as a matched pair of Lie algebras, or as a left symmetric bialgebra. In this section, we employ the pseudo-Riemannian viewpoint and we give a new characterization based on the Bianchi identity (2). This characterization has the advantage that it leads easily to the notions of left symmetric bialgebra and the S-matrix introduced in [5]. At the end of this section, we introduce a generalization of the notion of S-matrix and we give a precise description of the para-Kähler Lie algebras associated with these generalized S-matrices.

A *para-Kähler Lie algebra* is a pseudo-Riemannian Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ endowed with an isomorphism $K : \mathfrak{g} \longrightarrow \mathfrak{g}$ that satisfies $K^2 = \mathrm{Id}_{\mathfrak{g}}$, K is skew-symmetric with respect to $\langle\ ,\ \rangle$, and K is invariant with respect to the Levi-Civita product, i.e., $\mathrm{L}_u \circ K = K \circ \mathrm{L}_u$ for any $u \in \mathfrak{g}$. A para-Kähler Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle, K)$ has a natural bilinear skew-symmetric nondegenerate form Ω, which is defined by $\Omega(u, v) = \langle Ku, v\rangle$, and we can easily see that:

1. $(\mathfrak{g}, K)$ is a para-complex Lie algebra,
2. $(\mathfrak{g}, \Omega)$ is a symplectic Lie algebra,
3. $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$ where $\mathfrak{g}^1 = \ker(K - \mathrm{Id}_{\mathfrak{g}})$ and $\mathfrak{g}^{-1} = \ker(K + \mathrm{Id}_{\mathfrak{g}})$,
4. $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are subalgebras isotropic with respect to $\langle\ ,\ \rangle$ and Lagrangian with respect to Ω,
5. for any $u \in \mathfrak{g}$, $u.\mathfrak{g}^1 \subset \mathfrak{g}^1$ and $u.\mathfrak{g}^{-1} \subset \mathfrak{g}^{-1}$ (the dot is the Levi-Civita product).

A para-Kähler Lie algebra carries two products: the Levi-Civita product and the left symmetric product $\mathbf{a}$ associated with $(\mathfrak{g}, \Omega)$. The following proposition clarifies their relationship, where the proof is a simple computation.

Proposition 3.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle, \Omega, K)$ be a para-Kähler Lie algebra. Then, for any $u, v \in \mathfrak{g}^1$ and $\alpha, \beta \in \mathfrak{g}^{-1}$,*

$$u.v = \mathbf{a}(u, v) \quad \textit{and} \quad \alpha.\beta = \mathbf{a}(\alpha, \beta).$$

In particular, $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are left symmetric algebras.

Let $(\mathfrak{g}, \langle\ ,\ \rangle, \Omega, K)$ be a para-Kähler Lie algebra. For any $u \in \mathfrak{g}^{-1}$, let u^* denote the element of $(\mathfrak{g}^1)^*$ given by $\prec u^*, v\succ = \langle u, v\rangle$. The map $u \mapsto u^*$ realizes an isomorphism between $\mathfrak{g}^{-1}$ and $(\mathfrak{g}^1)^*$. Thus, we can identify $(\mathfrak{g}, \langle\ ,\ \rangle, \Omega, K)$ relative to the phase space $(\Phi(\mathfrak{g}^1), \langle\ ,\ \rangle_0, \Omega_0, K_0)$. Given this identification and according to Proposition 3.1, the Levi-Civita product induces a product on $\mathfrak{g}^1$ and $(\mathfrak{g}^1)^*$, which coincides with the affine product $\mathbf{a}$. Thus, both $\mathfrak{g}^1$ and $(\mathfrak{g}^1)^*$ carry a left symmetric algebra structure. For any $u \in \mathfrak{g}^1$ and for any $\alpha \in (\mathfrak{g}^1)^*$, we denote $\mathrm{L}_u : \mathfrak{g}^1 \longrightarrow \mathfrak{g}^1$ and $\mathrm{L}_\alpha : (\mathfrak{g}^1)^* \longrightarrow (\mathfrak{g}^1)^*$ as the left multiplication by u and α, respectively, i.e., for any $v \in \mathfrak{g}^1$ and any $\beta \in (\mathfrak{g}^1)^*$,

$$\mathrm{L}_u v = u.v = \mathbf{a}(u, v) \quad \text{and} \quad \mathrm{L}_\alpha \beta = \alpha.\beta = \mathbf{a}(\alpha, \beta).$$

The right multiplications R_u and R_α are defined in a similar manner. The following proposition shows that the Levi-Civita product and the affine product on $\mathfrak{g}$ identified with $\Phi(\mathfrak{g}^1)$ are determined entirely by their restrictions to $\mathfrak{g}^1$ and $(\mathfrak{g}^1)^*$. The proof of this proposition is straightforward.

Proposition 3.2. *Let $\mathfrak{g}$ be a para-Kähler Lie algebra identified with $\Phi(\mathfrak{g}^1)$, as described above. Then,*

1. *For any $u \in \mathfrak{g}^1$ and for any $\alpha \in (\mathfrak{g}^1)^*$,*
$$u.\alpha = -\mathrm{L}_u^t \alpha \quad \textit{and} \quad \alpha.u = -\mathrm{L}_\alpha^t u.$$
2. *For any $u \in \mathfrak{g}^1$ and for any $\alpha \in (\mathfrak{g}^1)^*$,*
$$\mathbf{a}(u,\alpha) = \mathrm{R}_\alpha^t u - \mathrm{ad}_u^t \alpha \quad \textit{and} \quad \mathbf{a}(\alpha,u) = -\mathrm{ad}_\alpha^t u + \mathrm{R}_u^t \alpha,$$
where $\mathrm{ad}_u : \mathfrak{g}^1 \longrightarrow \mathfrak{g}^1$ and $\mathrm{ad}_\alpha : (\mathfrak{g}^1)^* \longrightarrow (\mathfrak{g}^1)^*$ are given by $\mathrm{ad}_u v = [u,v]$ and $\mathrm{ad}_\alpha \beta = [\alpha,\beta]$.

Conversely, let U be a finite dimensional vector space and U^* is its dual space. We suppose that both U and U^* have the structure of a left symmetric algebra. We extend the products on U and U^* to $\Phi(U)$, for any $X, Y \in U$ and for any $\alpha, \beta \in U^*$, by putting

$$(X+\alpha).(Y+\beta) = X.Y - \mathrm{L}_\alpha^t Y - \mathrm{L}_X^t \beta + \alpha.\beta. \tag{5}$$

We consider the two bilinear maps $\rho : U \times U^* \longrightarrow \mathrm{End}(U)$ and $\rho^* : U^* \times U \longrightarrow \mathrm{End}(U^*)$ defined by

$$\rho(X,\alpha) = [\mathrm{L}_X, \mathrm{L}_\alpha^t] + \mathrm{L}_{\mathrm{L}_\alpha^t X} + \mathrm{L}_{\mathrm{L}_X^t \alpha}^t \quad \text{and} \quad \rho^*(\alpha,X) = [\mathrm{L}_\alpha, \mathrm{L}_X^t] + \mathrm{L}_{\mathrm{L}_X^t \alpha} + \mathrm{L}_{\mathrm{L}_\alpha^t X}^t. \tag{6}$$

Note that the endomorphism $\rho^*(\alpha,X)$ is the dual of $\rho(X,\alpha)$. Now, we give a new characterization of para-Kähler Lie algebras using the Bianchi identity.

Proposition 3.3. *According to the hypothesis above, the product on $\Phi(U)$ given by (5) is Lie-admissible if and only if*

$$\rho(X,\alpha)Y = \rho(Y,\alpha)X \quad \textit{and} \quad \rho^*(\alpha,X)\beta = \rho^*(\beta,X)\alpha \tag{7}$$

for any $X, Y \in U$ and any $\alpha, \beta \in U^*$. Moreover, this product is left symmetric if and only if $\rho(X,\alpha) = 0$, for any $X \in U$ and any $\alpha \in U^*$.

Proof. According to the Bianchi identity (2), the product given by (5) is Lie-admissible if and only if, for any $u, v, w \in \Phi(U)$,

$$\mathrm{K}(u,v)w + \mathrm{K}(v,w)u + \mathrm{K}(w,u)v = 0,$$

where K is the curvature of the product. Since the products on U and U^* are left symmetric, for any $X,Y,Z\in U$ and for any $\alpha,\beta,\gamma\in U^*$,

$$\mathrm{K}(X,Y)Z=\mathrm{K}(\alpha,\beta)\gamma=0.$$

Thus, the Bianchi identity is equivalent to

$$\mathrm{K}(X,Y)\alpha+\mathrm{K}(Y,\alpha)X+\mathrm{K}(\alpha,X)Y=0, \qquad (*)$$
$$\mathrm{K}(X,\alpha)\beta+\mathrm{K}(\alpha,\beta)X+\mathrm{K}(\beta,X)\alpha=0, \qquad (**)$$

for any $X,Y\in U$ and for any $\alpha,\beta\in U^*$. Now, we obviously have

$$\mathrm{K}(X,Y)\alpha=\left([\mathrm{L}_X,\mathrm{L}_Y]-\mathrm{L}_{[X,Y]}\right)^t\alpha=0.$$

By contrast, a direct computation yields

$$\mathrm{K}(Y,\alpha)X=\rho(Y,\alpha)X \quad \text{and} \quad \mathrm{K}(\alpha,X)Y=-\rho(X,\alpha)Y.$$

So $(*)$ is equivalent to $\rho(X,\alpha)Y=\rho(Y,\alpha)X$. A similar computation shows that $(**)$ is equivalent to $\rho^*(\alpha,X)\beta=\rho^*(\beta,X)\alpha$. The second part of the proposition follows easily from the above. □

Definition 3.1. Two left symmetric products on U and U^* that satisfy (7) are called *Lie-extendible*.

Thus, we obtain the following result.

Theorem 3.1. *Let $(U,.)$ and $(U^*,.)$ be two Lie-extendible left symmetric products. Then, $(\Phi(U),\langle\ ,\ \rangle_0,K_0)$ endowed with the Lie algebra bracket associated with the product given by (5) is a para-Kähler Lie algebra. Moreover, all para-Kähler Lie algebras are obtained in this manner.*

Example 1. Let $(U,.)$ be a left symmetric algebra. Then, the left symmetric product on U and the trivial left symmetric product on U^* are Lie-extendible such that $(\Phi(U),\langle\ ,\ \rangle_0,K_0)$ endowed with the Lie algebra bracket associated with the left symmetric product

$$(X+\alpha)\triangleright(Y+\beta)=X.Y-\mathrm{L}_X^t\beta \tag{8}$$

is a para-Kähler Lie algebra. We denote $[\ ,\ ]^\triangleright$ as the Lie bracket associated with $\triangleright$. We have

$$[X+\alpha,Y+\beta]^\triangleright=[X,Y]-\mathrm{L}_X^t\beta+\mathrm{L}_Y^t\alpha.$$

This is simply the semi-direct product of $(U, [\ ,\ ])$ with U^* endowed with the trivial bracket, and the action of U on U^* is given by L_U^*. Moreover, it is easy to check that $(\Phi(U), [\ ,\ ]^{\triangleright}, \langle\ ,\ \rangle_0)$ is a flat pseudo-Riemannian Lie algebra and that $(\Phi(U), \triangleright, \Omega_0)$ is a symplectic left symmetric algebra (also see Proposition 4.3 [6]).

In [5], Bai gave a characterization of para-Kähler Lie algebras, which is similar to that used for Lie bialgebras, where he called these structures *left symmetric bialgebras*. Thus, we present this viewpoint in a new manner using Proposition 3.3.

We consider two left symmetric algebras $(U, .)$ and $(U^*, .)$. The products on U and U^*, respectively, define by duality, two maps $\mu : U^* \longrightarrow U^* \otimes U^*$ and $\xi : U \longrightarrow U \otimes U$. Because Lie algebras U and U^* have two representations $\Psi_U : U \longrightarrow \mathrm{End}(U \otimes U)$ and $\Psi_{U^*} : U^* \longrightarrow \mathrm{End}, (U^* \otimes U^*)$ given by

$$\Psi_U = \mathrm{L}_U \otimes \mathrm{ad}_U \quad \text{and} \quad \Psi_{U^*} = \mathrm{L}_{U^*} \otimes \mathrm{ad}_{U^*}.$$

For any $X, Y \in U$ and for any $\alpha, \beta \in U^*$, a direct computation yields

$$\begin{aligned} \prec\beta, \rho(X,\alpha)Y - \rho(Y,\alpha)X\succ &= \Psi_U(X)(\xi(Y))(\alpha,\beta) - \Psi_U(Y)(\xi(X))(\alpha,\beta) \\ &\quad - \xi([X,Y])(\alpha,\beta), \\ \prec\rho^*(\alpha,X)\beta - \rho^*(\beta,X)\alpha, Y\succ &= \Psi_{U^*}(\alpha)(\mu(\beta))(X,Y) - \Psi_{U^*}(\beta)(\mu(\alpha))(X,Y) \\ &\quad - \mu([\alpha,\beta])(X,Y). \end{aligned}$$

By using Proposition 3.3, we recover a result of Bai (see [5] Theorem 4.1).

Proposition 3.4. *The product on $\Phi(U)$ given by (5) is Lie-admissible if and only if ξ is a 1-cocycle of $(U, [\ ,\ ])$ with respect to the representation Ψ_U and μ is a 1-cocycle of $(U^*, [\ ,\ ])$ with respect to the representation Ψ_{U^*}, i.e., for any $X, Y \in U$, $\alpha, \beta \in U^*$,*

$$\begin{aligned} \xi([X,Y]) &= \Psi_U(X)(\xi(Y)) - \Psi_U(Y)(\xi(X)), \\ \mu([\alpha,\beta]) &= \Psi_{U^*}(\alpha)(\mu(\beta)) - \Psi_{U^*}(\beta)(\mu(\alpha)). \end{aligned}$$

Now, we consider the case where ξ is a co-boundary. Indeed, let $(U, .)$ be a left symmetric algebra and $\xi : U \longrightarrow U \otimes U$ is a co-boundary of $(U, [\ ])$ with respect to Ψ_U, i.e., $\xi = \delta \mathrm{r}$, where $\mathrm{r} \in U \otimes U$. By duality, ξ define a product on U^* by

$$\prec\alpha.\beta, X\succ = \mathrm{r}(\mathrm{L}_X^t\alpha, \beta) + \mathrm{r}(\alpha, \mathrm{ad}_X^t\beta) = \mathrm{L}_X\mathrm{r}(\alpha,\beta) - \prec\mathrm{L}^t_{\mathrm{r}_\#(\alpha)}\beta, X\succ, \tag{9}$$

where $\mathrm{r}_\# : U^* \longrightarrow U$ is given by $\prec\beta, \mathrm{r}_\#(\alpha)\succ = \mathrm{r}(\alpha,\beta)$ and $\mathrm{L}_X\mathrm{r}(\alpha,\beta) = \mathrm{r}(\mathrm{L}_X^t\alpha,\beta) + \mathrm{r}(\alpha, \mathrm{L}_X^t\beta)$.

According to Proposition 3.3, to obtain a para-Kähler Lie algebra structure on $\Phi(U)$, $(U^*, .)$ must be a left symmetric algebra and the couple $(U, .)$, $(U^*, .)$ must be Lie-extendible. Note that because $\xi = \delta\mathrm{r}$, then the first equation in (7) holds. Next, we determine the conditions under which the second equation in (7) holds and $(U^*,.)$ is a left symmetric algebra. Put $\mathrm{r}=\mathfrak{a}+\mathfrak{s}$, where $\mathfrak{a}$ is skew-symmetric and $\mathfrak{s}$ is symmetric, and define $\mathrm{L}(\mathfrak{a})\in U^*\otimes U\otimes U$ as

$$\mathrm{L}(\mathfrak{a})(X,\alpha,\beta)=\mathrm{L}_X\mathfrak{a}(\alpha,\beta).$$

It follows immediately from (9) that for any $\alpha,\beta\in U^*$ and $X\in U$,

$$\begin{aligned}\mathrm{L}^t_\alpha X&=\mathrm{r}_\#\circ\mathrm{L}^t_X\alpha+[X,\mathrm{r}_\#(\alpha)]\quad\text{and}\\ \prec[\alpha,\beta],X\succ&=\prec\mathrm{L}^t_{\mathrm{r}_\#(\beta)}\alpha-\mathrm{L}^t_{\mathrm{r}_\#(\alpha)}\beta,X\succ+2\mathrm{L}_X\mathfrak{a}(\alpha,\beta).\end{aligned}\tag{10}$$

We consider the two representations $\mathrm{Q}:U\longrightarrow\mathrm{End}(U\otimes U\otimes U)$ and $\mathrm{P}:U\longrightarrow\mathrm{End}(U^*\otimes U\otimes U)$ given by

$$\mathrm{Q}=\mathrm{L}_U\otimes\mathrm{L}_U\otimes\mathrm{ad}_U\quad\text{and}\quad\mathrm{P}=\mathrm{L}^*_U\otimes\mathrm{L}_U\otimes\mathrm{L}_U.$$

We also define $\Delta(\mathrm{r})\in U\otimes U\otimes U\simeq\mathrm{End}(U^*\otimes U^*,U)$ by

$$\Delta(\mathrm{r})(\alpha,\beta)=\mathrm{r}_\#([\alpha,\beta])-[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)].\tag{11}$$

Proposition 3.5. *For any $X,Y\in U$ and $\alpha,\beta,\gamma\in U^*$, we have*

$$\begin{aligned}\prec\rho^*(\alpha,X)\beta-\rho^*(\beta,X)\alpha,Y\succ&=2P(X)(\mathrm{L}(\mathfrak{a}))(Y,\alpha,\beta),\\ \prec\mathrm{ass}(\alpha,\beta,\gamma)-\mathrm{ass}(\beta,\alpha,\gamma),X\succ&=\prec\gamma,-\mathrm{Q}(X)(\Delta(\mathrm{r}))(\alpha,\beta)\\ &\qquad+\mathrm{r}_\#\left(\rho^*(\alpha,X)\beta-\rho^*(\beta,X)\alpha\right)\succ.\end{aligned}$$

Proof. First, let us compute the associator of $\alpha,\beta,\gamma\in U^*$ with respect to the product given by (9). For any $X\in U$, we have

$$\begin{aligned}\prec\mathrm{ass}(\alpha,\beta,\gamma),X\succ&=\prec\alpha.(\beta.\gamma),X\succ-\prec(\alpha.\beta).\gamma,X\succ\\ &=\mathrm{r}(\mathrm{L}^t_X\alpha,\beta.\gamma)+\mathrm{r}(\alpha,\mathrm{ad}^t_X(\beta.\gamma))-\mathrm{r}(\mathrm{L}^t_X(\alpha.\beta),\gamma)-\mathrm{r}(\alpha.\beta,\mathrm{ad}^t_X\gamma)\\ &=\prec\beta.\gamma,\mathrm{r}_\#(\mathrm{L}^t_X\alpha)\succ+\prec\mathrm{ad}^t_X(\beta.\gamma),\mathrm{r}_\#(\alpha)\succ-\prec\gamma,\mathrm{r}_\#(\mathrm{L}^t_X(\alpha.\beta))\succ\\ &\quad-\prec\mathrm{ad}^t_X\gamma,\mathrm{r}_\#(\alpha.\beta)\succ\\ &=\mathrm{r}(\mathrm{L}^t_{\mathrm{r}_\#(\mathrm{L}^t_X\alpha)}\beta,\gamma)+\mathrm{r}(\beta,\mathrm{ad}^t_{\mathrm{r}_\#(\mathrm{L}^t_X\alpha)}\gamma)+\mathrm{r}(\mathrm{L}^t_{[X,\mathrm{r}_\#(\alpha)]}\beta,\gamma)\\ &\quad+\mathrm{r}(\beta,\mathrm{ad}^t_{[X,\mathrm{r}_\#(\alpha)]}\gamma)-\prec\gamma,\mathrm{r}_\#(\mathrm{L}^t_X(\alpha.\beta))\succ-\prec\gamma,[X,\mathrm{r}_\#(\alpha.\beta)]\succ\\ &=\prec\gamma,\mathrm{r}_\#\left(\mathrm{L}^t_{\mathrm{r}_\#(\mathrm{L}^t_X\alpha)}\beta\right)\succ+\prec\gamma,[\mathrm{r}_\#(\mathrm{L}^t_X\alpha),\mathrm{r}_\#(\beta)]\succ\\ &\quad+\prec\gamma,\mathrm{r}_\#\left(\mathrm{L}^t_{[X,\mathrm{r}_\#(\alpha)]}\beta\right)\succ+\prec\gamma,[[X,\mathrm{r}_\#(\alpha)],\mathrm{r}_\#(\beta)]\succ\\ &\quad-\prec\gamma,\mathrm{r}_\#(\mathrm{L}^t_X(\alpha.\beta))\succ-\prec\gamma,[X,\mathrm{r}_\#(\alpha.\beta)]\succ.\end{aligned}$$

In addition,

$$\begin{aligned}Q(X)(\Delta(\mathrm{r}))(\alpha,\beta) &= [X,\Delta(\mathrm{r})(\alpha,\beta)] - \Delta(\mathrm{r})(\mathrm{L}_X^*\alpha,\beta) - \Delta(\mathrm{r})(\alpha,\mathrm{L}_X^*\beta)\\ &= [X,\mathrm{r}_\#([\alpha,\beta])] - [X,[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)]] + \mathrm{r}_\#([\mathrm{L}_X^t\alpha,\beta])\\ &\quad - [\mathrm{r}_\#\left(\mathrm{L}_X^t\alpha\right),\mathrm{r}_\#(\beta)] + \mathrm{r}_\#([\alpha,\mathrm{L}_X^t\beta]) - [\mathrm{r}_\#(\alpha),\mathrm{r}_\#\left(\mathrm{L}_X^t\beta\right)].\end{aligned}$$

Thus,

$$\prec \mathrm{ass}(\alpha,\beta,\gamma) - \mathrm{ass}(\beta,\alpha,\gamma), X\succ + \prec\gamma, Q(X)(\Delta(\mathrm{r}))(\alpha,\beta)\succ = \prec\gamma,\mathrm{r}_\#(A)\succ,$$

where

$$\begin{aligned}A = {}&\mathrm{L}^t_{\mathrm{r}_\#(\mathrm{L}_X^t\alpha)}\beta - \mathrm{L}^t_{\mathrm{r}_\#(\mathrm{L}_X^t\beta)}\alpha + \mathrm{L}^t_{[X,\mathrm{r}_\#(\alpha)]}\beta - \mathrm{L}^t_{[X,\mathrm{r}_\#(\beta)]}\alpha - \mathrm{L}_X^t([\alpha,\beta]) + [\mathrm{L}_X^t\alpha,\beta]\\ &+ [\alpha,\mathrm{L}_X^t\beta].\end{aligned}$$

By using the first relation in (10), we obtain

$$A = \mathrm{L}^t_{\mathrm{L}^t_\alpha X}\beta - \mathrm{L}^t_{\mathrm{L}^t_\beta X}\alpha - \mathrm{L}^t_X([\alpha,\beta]) + [\mathrm{L}^t_X\alpha,\beta] + [\alpha,\mathrm{L}^t_X\beta].$$

Now, according to (6), we have

$$\begin{aligned}\rho^*(\alpha,X)\beta &= [\mathrm{L}_\alpha,\mathrm{L}^t_X]\beta + \mathrm{L}_{\mathrm{L}^t_X\alpha}\beta + \mathrm{L}^t_{\mathrm{L}^t_\alpha X}\beta\\ &= \alpha.(\mathrm{L}^t_X\beta) - \mathrm{L}^t_X(\alpha.\beta) + (\mathrm{L}^t_X\alpha).\beta + \mathrm{L}^t_{\mathrm{L}^t_\alpha X}\beta,\end{aligned}$$

so $A = \rho^*(\alpha,X)\beta - \rho^*(\beta,X)\alpha$ and the second assertion follows. Furthermore, by using the second relation in (10), for any $Y \in U$, we obtain

$$\prec A, Y\succ = -2\mathrm{L}_{X.Y}\mathfrak{a}(\alpha,\beta) + 2\mathrm{L}_Y\mathfrak{a}(\mathrm{L}^t_X\alpha,\beta) + 2\mathrm{L}_Y\mathfrak{a}(\alpha,\mathrm{L}^t_X\beta),$$

and the first assertion follows. □

Therefore, we obtain the following result.

Theorem 3.2. *Let $(U,.)$ be a left symmetric algebra and $\mathrm{r} = \mathfrak{a} + \mathfrak{s} \in U \otimes U$. Then, the product given by (9) is left symmetric and the left symmetric products on (U,U^*) are Lie-extendible if and only if for any $X \in U$*

$$\mathrm{Q}(X)(\Delta(\mathrm{r})) = 0 \quad \textit{and} \quad \mathrm{P}(X)(\mathrm{L}(\mathfrak{a})) = 0.$$

In this case, $(\Phi(U), \langle\ ,\ \rangle_0, \Omega_0, K_0)$ endowed with the Lie bracket associated with the product given by (5) is a para-Kähler Lie algebra.

We obtain the following corollary immediately.

Corollary 3.1. *If $\mathfrak{a}$ is L_U-invariant, i.e.* $\mathrm{L}(\mathfrak{a}) = 0$, *then the product given by* (9) *is left symmetric and the left symmetric products on* (U, U^*) *are Lie-extendible if and only if* $\Delta(\mathrm{r})$ *is Q-invariant.*

In fact, the statement in Theorem 3.2 is the same as that in Theorem 5.4 in [5]. To demonstrate this, let us investigate the relationship between $\Delta(\mathrm{r})$ and $[[\mathrm{r}, \mathrm{r}]]$, which appear in Bai's Theorem. Let $(U, .)$ be a left symmetric algebra and $\mathrm{r} = \sum_i a_i \otimes b_i$. In [5], Bai defines $[[\mathrm{r}, \mathrm{r}]]$ by

$$[[\mathrm{r}, \mathrm{r}]] = \mathrm{r}_{13}.\mathrm{r}_{12} - \mathrm{r}_{23}.\mathrm{r}_{21} + [\mathrm{r}_{23}, \mathrm{r}_{12}] - [\mathrm{r}_{13}, \mathrm{r}_{21}] - [\mathrm{r}_{13}, \mathrm{r}_{23}],$$

where

$$\mathrm{r}_{13}.\mathrm{r}_{12} = \sum_{i,j} a_i.a_j \otimes b_j \otimes b_i, \quad \mathrm{r}_{23}.\mathrm{r}_{21} = \sum_{i,j} b_j \otimes a_i.a_j \otimes b_i,$$
$$[\mathrm{r}_{23}, \mathrm{r}_{12}] = \sum_{i,j} a_j \otimes [a_i, b_j] \otimes b_i, \quad [\mathrm{r}_{13}, \mathrm{r}_{21}] = \sum_{i,j} [a_i, b_j] \otimes a_j \otimes b_i,$$
$$[\mathrm{r}_{13}, \mathrm{r}_{23}] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j].$$

Proposition 3.6. *For any* $\alpha, \beta, \gamma \in U^*$, *we have*

$$[[\mathrm{r}, \mathrm{r}]](\alpha, \beta, \gamma) = \prec \gamma, \Delta(\mathrm{r})(\alpha, \beta) \succ.$$

Proof. Recall that according to (10), for any $X \in U$,

$$\prec [\alpha, \beta], X \succ = \prec \mathrm{L}^t_{\mathrm{r}_\#(\beta)} \alpha - \mathrm{L}^t_{\mathrm{r}_\#(\alpha)} \beta, X \succ + 2\mathrm{L}_X \mathfrak{a}(\alpha, \beta),$$

$\mathfrak{a} = \frac{1}{2} \sum_i (a_i \otimes b_i - b_i \otimes a_i)$ is the skew-symmetric part of r. We have

$$\mathrm{r}_\#(\alpha) = \sum_i \prec \alpha, a_i \succ b_i \quad \text{and} \quad \mathrm{r}_\#(\beta) = \sum_i \prec \beta, a_i \succ b_i.$$

Thus,

$$-\prec \gamma, [\mathrm{r}_\#(\alpha), \mathrm{r}_\#(\beta)] \succ = -\sum_{i,j} \prec \alpha, a_i \succ \prec \beta, a_j \succ \prec \gamma, [b_i, b_j] \succ = -[\mathrm{r}_{13}, \mathrm{r}_{23}](\alpha, \beta, \gamma).$$

Now,

$$\mathrm{r}(\mathrm{L}^t_{\mathrm{r}_\#(\beta)} \alpha, \gamma) = \sum_j \prec \beta, a_j \succ \mathrm{r}(\mathrm{L}^t_{b_j} \alpha, \gamma) = \sum_{i,j} \prec \beta, a_j \succ \prec \mathrm{L}^t_{b_j} \alpha, a_i \succ \prec \gamma, b_i \succ$$
$$= \sum_{i,j} (b_j.a_i) \otimes a_j \otimes b_i(\alpha, \beta, \gamma).$$

In the same manner, we obtain

$$-\mathrm{r}(\mathrm{L}^t_{\mathrm{r}_\#(\alpha)}\beta,\gamma) = -\sum_{i,j} a_j \otimes (b_j.a_i) \otimes b_i(\alpha,\beta,\gamma).$$

Furthermore,

$$\begin{aligned}
2\mathrm{r}(\mathrm{L}\mathfrak{a}(\alpha,\beta),\gamma) &= 2\sum_i \prec \mathrm{L}\mathfrak{a}(\alpha,\beta), a_i\succ\prec\gamma, b_i\succ \\
&= 2\sum_i \mathfrak{a}(\mathrm{L}^t_{a_i}\alpha,\beta)\prec\gamma,b_i\succ + 2\sum_i \mathfrak{a}(\alpha,\mathrm{L}^t_{a_i}\beta)\prec\gamma,b_i\succ, \\
2\sum_i \mathfrak{a}(\mathrm{L}^t_{a_i}\alpha,\beta)\prec\gamma,b_i\succ &= \sum_{i,j}\big(\prec\mathrm{L}^t_{a_i}\alpha,a_j\succ\prec\beta,b_j\succ - \prec\mathrm{L}^t_{a_i}\alpha,b_j\succ\prec\beta,a_j\succ\big)\prec\gamma,b_i\succ \\
&= \sum_{i,j}\left((a_i.a_j)\otimes b_j\otimes b_i - (a_i.b_j)\otimes a_j\otimes b_i\right)(\alpha,\beta,\gamma), \\
&= \mathrm{r}_{13}.\mathrm{r}_{12}(\alpha,\beta,\gamma) - \sum_{i,j}(a_i.b_j)\otimes a_j\otimes b_i(\alpha,\beta,\gamma), \\
2\sum_i \mathfrak{a}(\alpha,\mathrm{L}^t_{a_i}\beta)\prec\gamma,b_i\succ &= \sum_{i,j}\left(a_j\otimes(a_i.b_j)\otimes b_i - b_j\otimes(a_i.a_j)\otimes b_i\right)(\alpha,\beta,\gamma) \\
&= \sum_{i,j} a_j\otimes(a_i.b_j)\otimes b_i(\alpha,\beta,\gamma) - \mathrm{r}_{23}.\mathrm{r}_{21}(\alpha,\beta,\gamma).
\end{aligned}$$

By combining all of the above, we obtain the desired formula. □

Remark 1.

1. This proposition shows that the statement in Theorem 3.2 is the same as that in Theorem 5.4 in [5]. However, our proof is easier to obtain because the expression of $\Delta(\mathrm{r})$ is simpler to handle than that of $[[\mathrm{r},\mathrm{r}]]$. The practical nature of $\Delta(\mathrm{r})$ is crucial later in the present study, particularly in Sections 6–7.
2. Let $(U,.)$ be a left symmetric algebra and $\mathrm{r}\in U\otimes U$. According to Proposition 3.6, $[[r,r]]=0$ iff $\mathrm{r}_\#$ is a Lie algebra endomorphism. This generalizes Theorem 6.6 in [5], as stated in the case when r is symmetric.
   Now, suppose that r is symmetric and $\mathrm{r}_\#$ is an isomorphism. By using (10), we can easily see that for any $X,Y,Z\in U$,

   $$\prec\mathrm{r}_\#^{-1}(Z),\Delta(\mathrm{r})(\mathrm{r}_\#^{-1}(X),\mathrm{r}_\#^{-1}(Y)) = B(X,Y.Z) - B(Y,X.Z) - B(Z,[X,Y]),$$

   where $B\in U^*\otimes U^*$ is given by $B(X,Y) = \prec\mathrm{r}_\#^{-1}(X),Y\succ$. So $[[r,r]]=0$ iff B is a 2-cocycle of $(U,.)$. This was proved in a different way in Theorem 6.3 in [5].

Now, let us introduce a key notion of our study, i.e., the notion of a quasi-S-matrix as a generalization of the S-matrix that first appeared in [5].

Let U be a left symmetric algebra. A *quasi-S-matrix* of U is an $\mathrm{r} \in U \otimes U$ such that its skew-symmetric part is L_U-invariant and $[[\mathrm{r}, \mathrm{r}]]$ is Q-invariant. Recall that an *S-matrix* of U is an $\mathrm{r} \in U \otimes U$, which is symmetric and it satisfies

$$[[r, r]] = 0. \tag{12}$$

In the following, we focus on the Lie algebra structure on $\Phi(U)$ associated with a quasi-S-matrix. We show that Lie algebra can be described in a precise and simple manner. Indeed, let r be a quasi-S-matrix. Then, according to Theorem 3.2, the product on U^* given by (9) is left symmetric and $(\Phi(U), [\ ,\ ]^r, \langle\ ,\ \rangle_0, K_0)$ is a para-Kähler Lie algebra, where

$$[X + \alpha, Y + \beta]^r = [X, Y] - \mathrm{L}^t_X\beta - \mathrm{L}^t_\alpha Y + \mathrm{L}^t_Y\alpha + \mathrm{L}^t_\beta X + [\alpha, \beta].$$

In Example 1, we showed that $\Phi(U)$ carries a left symmetric product $\triangleright$ and its associated Lie bracket $[\ ,\ ]^\triangleright$ induces a para-Kähler Lie algebra structure on $\Phi(U)$. We define a new bracket on $\Phi(U)$ by putting

$$[X + \alpha, Y + \beta]^{\triangleright,r} = [X + \alpha, Y + \beta]^\triangleright + \Delta(\mathrm{r})(\alpha, \beta). \tag{13}$$

The following proposition was inspired by a result that appeared in [13] in the context of Lie bialgebras and R-matrices (see Proposition 4.2.1.1 in [13]).

Proposition 3.7. *$(\Phi(U), [\ ,\ ]^{\triangleright,r})$ is a Lie algebra and the linear map $\xi : (\Phi(U), [\ ,\ ]^{\triangleright,r}) \longrightarrow (\Phi(U), [\ ,\ ]^r)$, $X + \alpha \mapsto X - \mathrm{r}_\#(\alpha) + \alpha$ is an isomorphism of Lie algebras.*

Proof. Clearly, ξ is bijective. Let us show that ξ preserves the Lie brackets. It is clear that for any $X, Y \in U$, $\xi\left([X, Y]^{\triangleright,r}\right) = [\xi(X), \xi(Y)]^r$. Now, for any $X \in U$, $\alpha \in U^*$,

$$\begin{aligned}
\xi\left([X, \alpha]^{\triangleright,r}\right) &= \xi(-\mathrm{L}^t_X\alpha) \\
&= \mathrm{r}_\#(\mathrm{L}^t_X\alpha) - \mathrm{L}^t_X\alpha \\
&\overset{(10)}{=} \mathrm{L}^t_\alpha X - [X, \mathrm{r}_\#(\alpha)] - \mathrm{L}^t_X\alpha \\
&= [X, -\mathrm{r}_\#(\alpha) + \alpha]^r \\
&= [\xi(X), \xi(\alpha)]^r.
\end{aligned}$$

In addition, for any $\alpha, \beta \in U^*$,

$$\begin{aligned}
\xi\left([\alpha, \beta]^{\triangleright,r}\right) &= \xi(\Delta(\mathrm{r})(\alpha, \beta)) \\
&= \Delta(\mathrm{r})(\alpha, \beta),
\end{aligned}$$

$$
\begin{aligned}
[\xi(\alpha),\xi(\beta)]^r &= [-\mathrm{r}_\#(\alpha)+\alpha,-\mathrm{r}_\#(\beta)+\beta]^r\\
&= [\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)]+[\alpha,\beta]+\mathrm{L}^t_{\mathrm{r}_\#(\alpha)}\beta-\mathrm{L}^t_{\mathrm{r}_\#(\beta)}\alpha+\mathrm{L}^t_\alpha\mathrm{r}_\#(\beta)-\mathrm{L}^t_\beta\mathrm{r}_\#(\alpha)\\
&\overset{(10)}{=} [\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)]+\mathrm{r}_\#(\mathrm{L}^t_{\mathrm{r}_\#(\beta)}\alpha)-\mathrm{r}_\#(\mathrm{L}^t_{\mathrm{r}_\#(\alpha)}\beta)+[\mathrm{r}_\#(\beta),\mathrm{r}_\#(\alpha)]\\
&\quad -[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)]\\
&= \mathrm{r}_\#([\alpha,\beta])-[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)]\\
&= \Delta(\mathrm{r})(\alpha,\beta). \qquad \square
\end{aligned}
$$

We can now transform the para-Kähler structure associated with r from $(\Phi(U),[\ ,\ ]^r,\langle\ ,\ \rangle_0,K_0)$ to $\Phi(U)$ via ξ and we obtain the following proposition.

Proposition 3.8. *Let $(U,.)$ be a left symmetric algebra and $\mathrm{r}=\mathfrak{a}+\mathfrak{s}\in U\otimes U$ is a quasi-S-matrix. Then, $(\Phi(U),[\ ,\ ]^{\triangleright,r},\langle\ ,\ \rangle_r,K_r)$ is a para-Kähler Lie algebra, where*

$$\langle X+\alpha,Y+\beta\rangle_r=\langle X+\alpha,Y+\beta\rangle_0-2\mathfrak{s}(\alpha,\beta)\quad\textit{and}$$
$$K_r(X+\alpha)=K_0(X+\alpha)-2\mathrm{r}_\#(\alpha).$$

Remark 2.

(*a*) In fact, using a similar method, we can generalize the result of Diatta [13]. Let $(\mathfrak{g},[\ ,\ ])$ be a Lie algebra and $\mathrm{r}\in\mathfrak{g}\wedge\mathfrak{g}$. On $\mathfrak{g}^*$ and $\Phi(\mathfrak{g})$, we define two brackets $[\ ,\ ]^*$ and $[\ ,\ ]^{\mathrm{r}}$, respectively, by

$$[\alpha,\beta]^*=\mathrm{ad}^*_{\mathrm{r}_\#(\alpha)}\beta-\mathrm{ad}^*_{\mathrm{r}_\#(\beta)}\alpha\quad\text{and}$$
$$[X+\alpha,Y+\beta]^{\mathrm{r}}=[X,Y]+[\alpha,\beta]^*-\mathrm{ad}^t_X\beta-\mathrm{ad}^t_\alpha Y+\mathrm{ad}^t_Y\alpha+\mathrm{ad}^t_\beta X,$$

and $[\mathrm{r},\mathrm{r}]\in\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\simeq\mathrm{End}(\mathfrak{g}^*\otimes\mathfrak{g}^*,\mathfrak{g})$ by

$$[\mathrm{r},\mathrm{r}](\alpha,\beta)=\mathrm{r}_\#([\alpha,\beta]^*)-[\mathrm{r}_\#(\alpha),\mathrm{r}_\#(\beta)].$$

It is well known that $[\ ,\ ]^*$ is a Lie bracket iff $[\mathrm{r},\mathrm{r}]$ is ad-invariant. In this case, $[\ ,\ ]^{\mathrm{r}}$ is a Lie bracket. Define a new bracket on $\Phi(\mathfrak{g})$ by putting

$$[X+\alpha,Y+\beta]^{\diamond,r}=[X,Y]+\mathrm{ad}^*_X\beta-\mathrm{ad}^*_Y\alpha+[\mathrm{r},\mathrm{r}](\alpha,\beta).$$

By using the same argument employed in the proof of Proposition 3.7, we can see that $(\Phi(\mathfrak{g}),[\ ,\ ]^{\diamond,r})$ is a Lie algebra and the linear map $\xi:(\Phi(\mathfrak{g}),[\ ,\ ]^{\diamond,r})\longrightarrow(\Phi(\mathfrak{g}),[\ ,\ ]^r)$, $X+\alpha\mapsto X-\mathrm{r}_\#(\alpha)+\alpha$ is an isomorphism of Lie algebras. When $[\mathrm{r},\mathrm{r}]=0$, we recover the result of Diatta.

(*b*) Let $(U,.)$ be a left symmetric algebra and $\mathrm{r}=\mathfrak{a}+\mathfrak{s}\in U\otimes U$ is a quasi-S-matrix. The Lie algebra $(\Phi(U),[\ ,\ ]^{\triangleright,r})$ is a $\mathbb{Z}_2$-graded Lie algebra and hence $L:U^*\times U^*\times U^*\longrightarrow U^*$ given by

$$L(\alpha,\beta,\gamma)=\mathrm{L}^*_{\Delta(\mathrm{r})(\alpha,\beta)}\gamma$$

is a Lie triple system (e.g., see [15,18,20] for a definition and the properties of Lie triple systems).

4. Some classes of para-Kähler Lie algebras

In this section, we develop some methods for building para-Kähler Lie algebras based on the following proposition, where we employ the notations given in the previous section, particularly Proposition 3.8.

Proposition 4.1. *Let $(U,.)$ be a left symmetric algebra and $\mathrm{r}=\mathfrak{a}+\mathfrak{s}\in U\otimes U$, which is L_U-invariant. Then, $\mathrm{L}(\mathfrak{a})=0$, $[[\mathrm{r},\mathrm{r}]]=\Delta(\mathrm{r})=0$ and $(\Phi(U),[\ ,\ ]^\triangleright,\langle\ ,\ \rangle_r,K_r)$ a para-Kähler Lie algebra. Moreover, the Levi-Civita product of $(\Phi(U),[\ ,\ ]^\triangleright,\langle\ ,\ \rangle_r)$ is $\triangleright$ given by (8).*

Proof. Since r is L_U-invariant, then $\mathfrak{a}$ and thus $\mathrm{L}(\mathfrak{a})=0$. The vanishing of $\Delta(\mathrm{r})$ is immediate. Thus, we can apply Proposition 3.8. To conclude, we can easily check that $\triangleright$ is actually the Levi-Civita product of $(\Phi(U),[\ ,\ ]^\triangleright,\langle\ ,\ \rangle_r)$. □

As a consequence of this proposition, we obtain the following large class of para-Kähler Lie algebras.

Proposition 4.2. *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be a pseudo-Riemannian flat Lie algebra, i.e., the Levi-Civita product "." is left symmetric. Denote $\flat:\mathfrak{g}\longrightarrow\mathfrak{g}^*$ as the isomorphism associated with $\langle\ ,\ \rangle$. Then, $(\Phi(\mathfrak{g}),[\ ,\ ]^\triangleright,\langle\ ,\ \rangle_\flat,K_\flat)$ is a para-Kähler Lie algebra, where*

$$\langle X+\alpha,Y+\beta\rangle_\flat=\langle X+\alpha,Y+\beta\rangle_0-2\langle\flat^{-1}(\alpha),\flat^{-1}(\beta)\rangle\quad\textit{and}$$
$$K_\flat(X+\alpha)=K_0(X+\alpha)-2\flat^{-1}(\alpha).$$

Moreover, the Levi-Civita product of $(\Phi(\mathfrak{g}),[\ ,\ ]^\triangleright,\langle\ ,\ \rangle_\flat)$ is $\triangleright$ given by (8).

Proof. The Levi-Civita product defines a left symmetric algebra structure on U, and $\mathrm{r}\in U\otimes U$ defined by $\mathrm{r}(\alpha,\beta)=\langle\flat^{-1}(\alpha),\flat^{-1}(\beta)$ is L_U-invariant, which we can conclude by using Proposition 4.1. □

Next, we provide some methods for building pseudo-Riemannian flat Lie algebras.

Let $(U,[.,.],\omega)$ be a symplectic Lie algebra and B is a nondegenerate bi-invariant bilinear symmetric form on U. The isomorphism D defined by

$$\omega(X,Y)=\mathrm{B}(D(X),Y)$$

is an invertible derivation and hence U is nilpotent (see [16]). The nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ given by

$$\langle X,Y\rangle=\mathrm{B}(D(X),D(Y))$$

satisfies

$$\langle X.Y, Z\rangle + \langle Y, X.Z\rangle = 0,$$

where the dot designates the left symmetric product associated with ω given by (4). Thus, $(U, \langle\ ,\ \rangle)$ is a flat pseudo-Riemannian Lie algebra (see [8]). Therefore, any symplectic quadratic Lie algebra (U, B, ω) leads to a flat pseudo-Riemannian Lie algebra $(U, \langle\ ,\ \rangle)$.

More generally, let $(\mathfrak{g}, [\ ,\ ], B)$ be a quadratic Lie algebra and $\mathfrak{r} \in \mathfrak{g} \wedge \mathfrak{g}$ is a solution of the classical Yang–Baxter equation. The product on $\mathfrak{g}^*$ given by $\alpha.\beta = \mathrm{ad}^*_{\mathfrak{r}_\#(\alpha)}\beta$ is left symmetric and thus it induces a Lie bracket $[\ ,\ ]_\mathfrak{r}$ on $\mathfrak{g}^*$. In fact, this product is the Levi-Civita product of B^* (the induced bilinear nondegenerate symmetric form on $\mathfrak{g}^*$). Thus, $(\mathfrak{g}^*, [\ ,\ ]_\mathfrak{r}, B^*)$ is a flat pseudo-Riemannian Lie algebra (see [10]).

Now, let us give a method for building symplectic quadratic Lie algebras.

Let $n \in \mathbb{N}^*$ and $\mathcal{A}$ be a vector space with a basis $\{e_1, \ldots, e_n\}$. On $\mathcal{A}$, we consider the product defined by

$$e_ie_j = e_je_i = e_{i+j} \text{ if } i + j \leqq n, \quad e_ie_j = e_je_i = 0 \text{ if } i + j > n.$$

The vector space $\mathcal{A}$ endowed with this product is a commutative and associative algebra.

Let $(\mathcal{L}, [.,])$ be an arbitrary Lie algebra. Then, the following product

$$[X \otimes a, Y \otimes b]_\mathcal{T} := [X, Y] \otimes ab,$$

defines the structure of a Lie algebra on the vector space $\mathcal{T} := \mathcal{L} \otimes \mathcal{A}$. Moreover, the endomorphism δ of $\mathcal{T}$ defined by

$$\delta(X \otimes e_i) := iX \otimes e_i$$

for any $X \in \mathcal{L}$ and any $i \in \{1, \ldots, n\}$, is an invertible derivation of $\mathcal{T}$.

Now, on the vector space $U := \mathcal{T} \oplus \mathcal{T}^*$, we define the structure of a symplectic quadratic algebra in the following manner. For any $s, t \in \mathcal{T}$ and any $f, h \in \mathcal{T}^*$, put

$$\begin{aligned} [t + f, s + h]_U &= [t, s]_\mathcal{T} - h \circ \mathrm{ad}_\mathcal{T}(t) + f \circ \mathrm{ad}_\mathcal{T}(s), \\ \mathrm{B}(t + f, s + h) &= f(s) + h(t), \\ D(t + f) &= \delta(t) - f \circ \delta, \\ \omega(t + f, s + h) &= \mathrm{B}(D(t + f), s + h). \end{aligned}$$

We can easily check that (U, B, ω) is a symplectic quadratic Lie algebra.

Proposition 4.3. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra, $\mathrm{b} \in \wedge^2\mathfrak{g}$ is a solution of the classical Yang–Baxter equation on $(\mathfrak{g}, [\ ,\ ])$, i.e., $[\mathrm{b}, \mathrm{b}] = 0$, and $\mathrm{r} = \mathfrak{s} + \mathfrak{a} \in \mathfrak{g}^* \otimes \mathfrak{g}^*$, such that*

$\mathrm{ad}^*_{\mathrm{b}_\#(\alpha)}\mathrm{r} = 0$*, for any* $\alpha \in \mathfrak{g}^*$*. Then,* $(\Phi(\mathfrak{g}), [\ ,\ ]^{\mathrm{b}}, \langle\ ,\ \rangle^r, K^r)$ *is a para-Kähler Lie algebra, where*

$$[X+\alpha, Y+\beta]^{\mathrm{b}} = \mathrm{ad}^*_{\mathrm{b}_\#(\alpha)}\beta - \mathrm{ad}^*_{\mathrm{b}_\#(\beta)}\alpha + [\mathrm{b}_\#(\alpha), Y] + [X, \mathrm{b}_\#(\beta)],$$
$$\langle X+\alpha, Y+\beta\rangle^r = \langle X+\alpha, Y+\beta\rangle_0 - 2\mathfrak{s}(X,Y) \quad \text{and}$$
$$K^r(X+\alpha) = -K_0(X+\alpha) - 2\mathrm{r}_\#(X).$$

Moreover, the Levi-Civita product associated with $(\Phi(\mathfrak{g}), [\ ,\ ]^{\mathrm{b}}, \langle\ ,\ \rangle^r)$ *is left symmetric and it is given by*

$$(X+\alpha) \triangleright_{\mathrm{b}} (Y+\beta) = \mathrm{ad}^*_{\mathrm{b}_\#(\alpha)}\beta + [\mathrm{b}_\#(\alpha), Y].$$

Proof. It is well known that the product given by $\alpha.\beta = \mathrm{ad}^*_{\mathrm{b}_\#(\alpha)}\beta$ on $\mathfrak{g}^*$ is left symmetric and that the condition $\mathrm{ad}^*_{\mathrm{b}_\#(\alpha)}\mathrm{r} = 0$ is equivalent to r is invariant with respect to this product on $\mathfrak{g}^*$. Thus, $(\mathfrak{g}^*, ., \mathrm{r})$ satisfies the hypothesis of Proposition 4.1 and the proposition follows. □

There is an interesting case in this situation, as follows.

Corollary 4.1. *Let* $(\mathfrak{g}, [\ ,\ ])$ *be a Lie algebra,* $\mathrm{b} \in \wedge^2\mathfrak{g}$ *is a solution of the classical Yang–Baxter equation on* $(\mathfrak{g}, [\ ,\ ])$*, and* $\mathrm{k} \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ *is the Killing form. Then,* $(\Phi(\mathfrak{g}), [\ ,\ ]^{\mathrm{b}}, \langle\ ,\ \rangle^{\mathrm{k}}, K^{\mathrm{k}})$ *is a para-Kähler Lie algebra.*

5. Hyper-para-Kähler Lie algebras

Hyper-para-Kähler Lie algebras, which are also known as hyper-symplectic Lie algebras, comprise a subclass of the class of para-Kähler Lie algebras. Based on the previous sections, we give a new characterization of these Lie algebras. This characterization leads to a notion of compatibility between two left symmetric algebra structures on a given vector space. Since a hyper-para-Kähler Lie algebra has a complex product structure, we also obtain a characterization of these structures.

A *hyper-para-Kähler Lie algebra* is a para-Kähler Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle, K)$ endowed with an endomorphism J such that $J^2 = -\mathrm{Id}_{\mathfrak{g}}$, $JK = -KJ$, J is skew-symmetric with respect to $\langle\ ,\ \rangle$ and J is invariant with respect to the Levi-Civita product. According to Theorem 3.1, a para-Kähler Lie algebra can be identified in the phase space of Lie-extendible left symmetric algebras, and thus it is natural to understand how hyper-para-Kähler Lie algebras can be described in this setting.

Proposition 5.1. *Let* $(U, .)$ *and* $(U^*, .)$ *be a couple of Lie-extendible left symmetric algebras,* $(\Phi(U), \langle\ ,\ \rangle_0, K_0)$*, the associated para-Kähler Lie algebra, and* $J : \Phi(U) \longrightarrow \Phi(U)$*, an endomorphism. Then,* $(\Phi(U), \langle\ ,\ \rangle_0, K_0, J)$ *is a hyper-para-Kähler Lie algebra if and only if a bilinear nondegenerate* $\omega \in \wedge^2 U^*$ *exists such that:*

(*i*) *for any* $X \in U$, $\alpha \in U^*$, $JX = \flat(X)$ *and* $J\alpha = -\flat^{-1}(\alpha)$, *where* $\flat : U \longrightarrow U^*$ *is the isomorphism given by* $\flat(X) = \omega(X, .)$,

(*ii*) $(U, ., \omega)$ *and* $(U, \circ, \omega)$ *are symplectic left symmetric algebras, where* $\circ$ *is given by*

$$X \circ Y = \flat^{-1}(\flat(X).\flat(Y)).$$

Proof. From the relation $JK_0 = -K_0J$, we deduce that for any $X \in U$, $JX \in U^*$, and thus J defines an isomorphism $\flat : U \longrightarrow U^*$. Moreover, from $J^2 = -\mathrm{Id}_{\mathfrak{g}}$, we deduce that $J\alpha = -\flat^{-1}\alpha$ for any $\alpha \in U^*$. The skew-symmetry of J implies that $\omega \in \wedge^2 U^*$ given by

$$\omega(X, Y) = \prec\flat(X), Y\succ$$

is skew-symmetric and it is actually nondegenerate. Now, J is invariant if and only if for any $X, Y, \in U$ and any $\alpha, \beta \in U^*$,

$$-\mathrm{L}^t_X(JY) = J(X.Y),\ X.J(\alpha) = -J(\mathrm{L}^t_X\alpha),\ \alpha.J(X) = -J(\mathrm{L}^t_\alpha X) \quad \text{and} \quad -\mathrm{L}^t_\alpha(J\beta) = J(\alpha.\beta).$$

This is equivalent to

$$\mathrm{L}^t_X \circ \flat + \flat \circ \mathrm{L}_X = \flat^{-1} \circ \mathrm{L}^t_X + \mathrm{L}_X \circ \flat^{-1} = 0 \quad \text{and}$$
$$\mathrm{L}^t_\alpha \circ \flat^{-1} + \flat^{-1} \circ \mathrm{L}_\alpha = \flat \circ \mathrm{L}^t_\alpha + \mathrm{L}_\alpha \circ \flat = 0,$$

for any $X \in U$, $\alpha \in U^*$. Now, it is obvious that these relations are equivalent to

$$\mathrm{L}^t_X \circ \flat + \flat \circ \mathrm{L}_X = 0 \quad \text{and} \quad \flat \circ \mathrm{L}^t_\alpha + \mathrm{L}_\alpha \circ \flat = 0, \tag{14}$$

for any $X \in U$, $\alpha \in U^*$. We can see easily that this is equivalent to

$$\omega(X.Y, Z) + \omega(Y, X.Z) = 0 \quad \text{and} \quad \omega(X \circ Y, Z) + \omega(Y, X \circ Z) = 0,$$

for any $X, Y, Z \in U$. Thus, $(U, ., \omega)$ and $(U, \circ, \omega)$ are symplectic left symmetric algebras. Obviously, the converse is true. □

Now, let U be a vector space, and $\omega \in \wedge^2 U^*$ nondegenerate and $\bullet$, $\circ$ are two products on U such that $(U, \bullet, \omega)$ and $(U, \circ, \omega)$ are symplectic left symmetric algebras. Define $J_0 : \Phi(U) \longrightarrow \Phi(U)$ by $J_0X = \flat(X)$ and $J_0\alpha = -\flat^{-1}(\alpha)$, and denote $\flat(\circ)$ as the product on the U^* image by $\flat$ of $\circ$. Let the dot denote the product on $\Phi(U)$, which extends $(U, \bullet)$ and $(U^*, \flat(\circ))$ by (5). By using (14), it is easy to see that for any $X, Y \in U$, $\alpha, \beta \in U^*$,

$$(X + \alpha).(Y + \beta) = X \bullet Y + \flat^{-1}(\alpha) \circ Y - \left(\mathrm{L}^\circ_{\flat^{-1}(\alpha)}\right)^t \beta - \left(\mathrm{L}^\bullet_X\right)^t \beta. \tag{15}$$

In Proposition 3.3, we showed that this product is Lie-admissible iff (7) holds. Given ω, we can identify $\Phi(U)$ as $T(U)$. Indeed, we define $\xi : T(U) \longrightarrow \Phi(U)$ by

$$\xi(X,0) = X \quad \text{and} \quad \xi(0,X) = \flat(X).$$

We have $\Omega_1 = \xi^*\Omega_0$, $\langle\ ,\ \rangle_1 = \xi^*\langle\ ,\ \rangle_0$, $K_1 = \xi^{-1} \circ K_0 \circ \xi$ and $J_1 = \xi^{-1} \circ J_0 \circ \xi$. It is easy to check that

$$(X,Y).(Z,T) := \xi^{-1}(\xi(X,Y).\xi(Z,T)) = (X \bullet Z, X \bullet T) + (Y \circ Z, Y \circ T). \tag{16}$$

Now, by using (14), we can see easily that for any $X \in U$ and any $\alpha \in U^*$,

$$\rho(X,\alpha) = -\mathrm{K}^{\bullet,\circ}(X,\flat^{-1}(\alpha)) \quad \text{and} \quad \rho^*(\alpha,X) = \flat \circ \mathrm{K}^{\bullet,\circ}(X,\flat^{-1}(\alpha)) \circ \flat^{-1},$$

where

$$\mathrm{K}^{\bullet,\circ}(X,Y) = [\mathrm{L}^{\bullet}_X, \mathrm{L}^{\circ}_Y] - (\mathrm{L}^{\circ}_{X\bullet Y} - \mathrm{L}^{\bullet}_{Y\circ X}).$$

(To distinguish between $\bullet$ and $\circ$, we denote $\mathrm{L}^{\bullet}_X$ as the left multiplication by X associated with $\bullet$, and so on). Thus, by using Proposition 3.3, we obtain the following proposition, which actually does not involves ω.

Proposition 5.2. *Let U be a vector space and $\bullet$, $\circ$ are two left symmetric products on U. The following assertions are equivalent.*

1. *The product given by (16) is Lie-admissible.*
2. *For any $X,Y,Z \in U$, $\mathrm{K}^{\bullet,\circ}(X,Y)Z = \mathrm{K}^{\bullet,\circ}(Z,Y)X$ and $\mathrm{K}^{\bullet,\circ}(X,Y)Z = \mathrm{K}^{\bullet,\circ}(X,Z)Y$.*

Moreover, the product given by (16) is left symmetric if and only if $\mathrm{K}^{\bullet,\circ}$ vanishes identically.

We can see easily that the second assertion in this proposition is equivalent to

$$Y \circ [X,Z]^{\bullet} - [Y \circ X, Z]^{\bullet} - [X, Y \circ Z]^{\bullet} = (Z \bullet Y) \circ X - (X \bullet Y) \circ Z, \tag{17}$$

$$Y \bullet [X,Z]^{\circ} - [Y \bullet X, Z]^{\circ} - [X, Y \bullet Z]^{\circ} = (Z \circ Y) \bullet X - (X \circ Y) \bullet Z, \tag{18}$$

for any $X,Y,Z \in U$.

Now, let us state an important formula. Let $\bullet$ and $\circ$ be two algebra structures on a vector space U. A straightforward computation gives the following formula:

$$\mathrm{K}^{\bullet+\circ}(X,Y) = \mathrm{K}^{\bullet}(X,Y) + \mathrm{K}^{\circ}(X,Y) + \mathrm{K}^{\bullet,\circ}(X,Y) - \mathrm{K}^{\bullet,\circ}(Y,X), \tag{19}$$

where K^x is the curvature of x.

Definition 5.1. Two left symmetric algebras structures $\bullet$ and $\circ$ on U are called compatible if they satisfy (17)–(18), or equivalently $\mathrm{K}^{\bullet,\circ}$ satisfies the second assertion in Proposition 5.2.

The following proposition is an immediate consequence of (19).

Proposition 5.3. *Let $\bullet$, $\circ$ be two compatible left symmetric algebra structures on U. Then, for any $a, b \in \mathbb{R}$, $(U, a\ \bullet + b\ \circ)$ is a left symmetric algebra.*

Remark 3. Let $\bullet$, $\circ$ be two compatible left symmetric algebra structures on U. As a consequence of Proposition 5.3, the bracket $a[\ ,\ ]^\bullet + b[\ ,\ ]^\circ$ is a Lie bracket and hence the two dual Poisson structures on U^* associated with $[\ ,\ ]^\bullet$ and $[\ ,\ ]^\circ$ are compatible (e.g., see [21] for a definition of compatible Poisson structures).

Finally, we obtain a characterization of hyper-para-Kähler Lie algebras. In fact, our method can be easily generalized to obtain a characterization of complex product structures. The characterization given in the following theorem completes the study of complex product structures provided in [3].

Theorem 5.1.

1. *Let $\bullet$, $\circ$ be two compatible left symmetric algebra structures on U. Then, $(T(U), K_1, J_1)$ endowed with the Lie algebra structure associated with the product given by (16) is a complex product Lie algebra. Moreover, all complex product Lie algebras are obtained in this manner.*
2. *Let $\bullet$, $\circ$ be two compatible left symmetric algebra structures on U and $\omega \in \wedge^2 U^*$ such that $(U, \omega, \bullet)$ and $(U, \omega, \circ)$ are symplectic left symmetric algebras. Then, $(T(U), \langle\ ,\ \rangle_1, K_1, J_1)$ endowed with the Lie algebra structure associated with the product given by (16) is a hyper-para-Kähler Lie algebra. Moreover, all hyper-para-Kähler Lie algebras are obtained in this manner.*

Proof.

1. To show that $(T(U), K_1, J_1)$ is a complex product Lie algebra, it suffices to show that the Nijenhuis torsion of K_1 and J_1 vanishes, which is easy to check. Conversely, let $(\mathfrak{g}, K, J)$ be a complex product Lie algebra. We have $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, where $\mathfrak{g}^i = \ker(K - i\mathrm{Id}_\mathfrak{g})$ and J defines an isomorphism $\phi : \mathfrak{g}^1 \longrightarrow \mathfrak{g}^{-1}$. We consider the product "." on $\mathfrak{g}$ given by

$$(u^1 + u^{-1}).(v^1 + v^{-1}) = u^1 \circ v^1 + \phi(u^1 \circ \phi^{-1}(v^{-1})) + \phi^{-1}(u^{-1} \star \phi(v^1)) + u^{-1} \star v^{-1},$$

where $\circ$ and $\star$ are the products on $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$, respectively, which are given by

$$u^1 \circ v^1 = -\pi_1 J[u^1, Jv^1] \quad \text{and} \quad u^{-1} \star v^{-1} = -\pi_{-1} J[u^{-1}, Jv^{-1}],$$

where π_i is the projection on $\mathfrak{g}^i$. In [3], it was shown that $\circ, \star$ are left symmetric and "." is Lie-admissible. Put $U = \mathfrak{g}^1$, $\bullet = \phi^{-1}(\star)$ and define $\xi : T(U) \longrightarrow \mathfrak{g}$ by $\xi(X, 0) = X$ and $\xi(0, X) = \phi(X)$. We then obtain the desired isomorphism.

2. This is a consequence of the method given above. □

Example 2. Let $(U, .)$ be a left symmetric algebra. Then, "." is compatible with itself so $(T(U), K_1, J_1)$ endowed with the Lie algebra bracket associated with the left symmetric product

$$(X, Y).(Z, T) = (X.Z + Y.Z, Y.T + X.T)$$

is a complex product Lie algebra. Moreover, if U carries ω such that $(U, ., \omega)$ is a symplectic left symmetric algebra, then $(T(U), \langle\ ,\ \rangle_1, K_1, J_1)$ is a hyper-para-Kähler Lie algebra.

The following proposition is immediate.

Proposition 5.4. *Let $\bullet$, $\circ$ be two compatible left symmetric algebra structures on U and $(T(U), K_1, J_1)$ is the associated complex product structure. Then, the following are equivalent.*

(*i*) *K_1 is abelian.*
(*ii*) *J_1 is abelian.*
(*iii*) *Both $\bullet$ and $\circ$ are commutative, and hence associative.*

According to (17)–(18), two associative and commutative algebra structures $\bullet$ and $\circ$ on U are compatible if for any $X, Y, Z \in U$,

$$(Z \bullet Y) \circ X - (X \bullet Y) \circ Z = (Z \circ Y) \bullet X - (X \circ Y) \bullet Z = 0. \tag{20}$$

In this case, $(T(U), K_1, J_1)$ endowed with the bracket associated with the product given by (16) is an abelian complex product structure. A similar result is given in [3] with a different product.

6. Quasi-S-matrices on symplectic Lie algebras

In Section 2, we showed that finding the set of quasi-S-matrices on a given left symmetric algebra yields a large class of para-Kähler Lie algebras. In this section, we investigate the set of quasi-S-matrices with respect to the left symmetric product associated with a symplectic Lie algebra.

Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and $\flat : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is the isomorphism given by $\flat(X) = \omega(X, .)$. The product $\mathbf{a}$ given by (4) is left symmetric. We associate any endomorphism $A : \mathfrak{g} \longrightarrow \mathfrak{g}$ with the tensor $\mathrm{YB}(A) \in \mathrm{End}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, which is given by

$$\mathrm{YB}(A)(X, Y) = A[AX, Y] + A[X, AY] - [AX, AY]. \tag{21}$$

The following proposition provides a useful characterization of quasi-S-matrices on $(\mathfrak{g}, \mathbf{a})$.

Proposition 6.1. *Let* $\mathrm{r} \in \mathfrak{g} \otimes \mathfrak{g}$ *and* $\mathfrak{a}$ *is its skew-symmetric part. Put* $A = \mathrm{r}_\# \circ \flat$ *and* $T = \mathfrak{a}_\# \circ \flat$. *Then, the following assertions hold.*

- (i) *The tensor* r *is a quasi-S-matrix of* $(\mathfrak{g}, \mathbf{a})$ *if and only if* T *and* $\mathrm{YB}(A)$ *are* ad-*invariant.*
- (ii) *If* r *is symmetric, then it is an S-matrix of* $(\mathfrak{g}, \mathbf{a})$ *if and only if* $\mathrm{YB}(A) = 0$.
- (iii) *If* r *is symmetric and invertible, then it is an S-matrix of* $(\mathfrak{g}, \mathbf{a})$ *if and only if* A^{-1} *is a derivation of the Lie algebra* $\mathfrak{g}$.

Proof. From (4), for any $X \in \mathfrak{g}$, we find that

$$\flat \circ \mathrm{ad}_X = \mathrm{L}^*_X \circ \flat. \tag{22}$$

Moreover, for any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned} \Delta(\mathrm{r})(\flat X, \flat Y) &= \mathrm{r}_\#([\flat X, \flat Y]) - [\mathrm{r}_\#(\flat X), \mathrm{r}_\#(\flat Y)] \\ &\overset{(10)}{=} A \circ \flat^{-1}\left(\mathrm{L}^t_{AY} \flat X\right) - A \circ \flat^{-1}\left(\mathrm{L}^t_{AX} \flat Y\right) \\ &\quad + 2A \circ \flat^{-1}\left(\mathrm{L}(\mathfrak{a})(\flat X, \flat Y)\right) - [AX, AY] \\ &\overset{(22)}{=} \mathrm{YB}(A)(X, Y) + 2A \circ \flat^{-1}\left(\mathrm{L}(\mathfrak{a})(\flat X, \flat Y)\right). \end{aligned}$$

From this relation and (22), we can see easily that (i) and (ii) hold. Now, it is easy to show that if A is invertible, then $\mathrm{YB}(A) = 0$ if and only if A^{-1} is a derivation of the Lie algebra $\mathfrak{g}$ and (iii) holds. □

Let $\mathfrak{g}$ be a Lie algebra. The modified Yang–Baxter equation is the equation

$$\mathrm{YB}(A)(X, Y) = t[X, Y], \quad \text{for all } X, Y \in \mathfrak{g}, \tag{23}$$

where $t \in \mathbb{R}$ is a fixed parameter and the unknown A is an endomorphism of $\mathfrak{g}$. When $t = 0$, we obtain the operator form of the classical Yang–Baxter equation. The following proposition is an immediate consequence of Proposition 6.1.

Proposition 6.2. *Let* $(\mathfrak{g}, \omega)$ *be a symplectic Lie algebra and* A *is a solution of the modified Yang–Baxter equation, which is skew-symmetric with respect to* ω. *Then,* $\mathrm{r} = A \circ \flat^{-1}$ *is a quasi-S-matrix of* $(\mathfrak{g}, \mathbf{a})$.

Theorem 6.1. *Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and $A : \mathfrak{g} \longrightarrow \mathfrak{g}$. Put $A = A^s + A^a$, where A^s and A^a are the symmetric and the skew-symmetric parts of A (with respect to ω), respectively. If both* $\mathrm{YB}(A)$ *and A^s are* ad*-invariant, then the product $\circ$ on $\mathfrak{g}$ given by $X \circ Y = X.[(A^s - A^a)Y] - (AX).Y$ is left symmetric and $(T(\mathfrak{g}), \langle\ ,\ \rangle_A, K_A)$ endowed with the Lie bracket given by*

$$[(X,Y),(Z,T)]^A = ([X,Z] + \mathrm{YB}(A)(Y,T), [X,T] + [Z,Y])$$

is a para-Kähler Lie algebra, where

$$\langle (X,Y),(Z,T)\rangle_A = \omega(T,X) + \omega(Y,Z) + 2\omega(A^aY,T) \quad \textit{and}$$
$$K_A(X,Y) = (X - 2AY, -Y),$$

and the dot is the left symmetric product associated with $(\mathfrak{g}, \omega)$.

Proof. According to Proposition 6.1, r given by $\mathrm{r}(\alpha,\beta) = -\omega(A\flat^{-1}(\alpha), \flat^{-1}(\beta))$ is a quasi-S-matrix with respect to the left symmetric product associated with ω. By virtue of Corollary 3.1 and Proposition 3.8, on $\mathfrak{g}^*$, r defines a left symmetric Lie algebra structure by (9) and $(\Phi(\mathfrak{g}), [\ ,\ ]^{\rhd,r}\langle\ ,\ \rangle_r, K_r)$ is a para-Kähler Lie algebra. Now, we consider the linear isomorphism $\mu : T(\mathfrak{g}) \longrightarrow \Phi(\mathfrak{g})$, $(X,Y) \mapsto (X, \flat(Y))$. Thus, $(T(\mathfrak{g}), [\ ,\ ]^\mu, \mu^*\langle\ ,\ \rangle_r, \mu^{-1} \circ K_r \circ \mu)$ is a para-Kähler Lie algebra and $[\ ,\ ]^\mu$ is a pull-back by μ of $[\ ,\ ]^{\rhd,r}$. We can check easily that this bracket is the Lie bracket given in the statement of the theorem, $\langle\ ,\ \rangle_A = \mu^*\langle\ ,\ \rangle_r$ and $K_A = \mu^{-1} \circ K_r \circ \mu$.

Let us compute the pull-back by $\flat$ of the left symmetric product on $\mathfrak{g}^*$ given by (9). We have

$$\begin{aligned}
\prec\alpha, X \circ Y\succ &= -\prec\flat(X).\flat(Y), \flat^{-1}(\alpha)\succ \\
&\overset{(9)}{=} -\mathrm{r}(\mathrm{L}^t_{\flat^{-1}(\alpha)}\flat(X), \flat(Y)) - \mathrm{r}(\flat(X), \mathrm{ad}^t_{\flat^{-1}(\alpha)}\flat(Y)) \\
&\overset{(22)}{=} \omega(A[\flat^{-1}(\alpha), X], Y) - \omega(AX, \flat^{-1}(\mathrm{ad}^t_{\flat^{-1}(\alpha)}\flat(Y))) \\
&= \omega([\flat^{-1}(\alpha), X], A^sY) + \omega([X, \flat^{-1}(\alpha)], A^aY) + \omega(Y, [\flat^{-1}(\alpha), AX]) \\
&= \prec\alpha, X.[(A^s - A^a)Y] - (AX).Y\succ. \qquad \square
\end{aligned}$$

Remark 4. Actually, this theorem and Remark 2 (b) suggest the following more general result. Let $\mathfrak{g}$ be a Lie algebra and A is an endomorphism of $\mathfrak{g}$ such that $\mathrm{YB}(A)$ is ad-invariant. Then, we can check that the bracket $[\ ,\ ]^A$ on $T(\mathfrak{g})$ is a Lie bracket and hence $L^A : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$L^A(X,Y,Z) = [\mathrm{YB}(A)(X,Y), Z]$$

is a Lie triple system.

Example 3. Let $(U,.)$ be a left symmetric algebra with an invertible derivation D. On the vector space $\Phi(U) := U \oplus U^*$, we know that we have a left symmetric structure defined by:

$$(X+\alpha) \rhd (Y+\beta) := X.Y - L_X^t\beta, \quad \forall X, Y \in U, \alpha, \beta \in U^*.$$

Moreover, $(\Phi(U), \rhd, \Gamma_0)$ is a symplectic left symmetric algebra. Now, it is easy to verify that the endomorphism Δ of $\Phi(U)$ defined by:

$$\Delta(X+\alpha) := D(X) - \alpha \circ D, \forall X \in U, \alpha \in U^*,$$

is an invertible derivation of $(\Phi(U), \rhd)$, which is skew-symmetric with respect to Γ_0. In the following, we construct left symmetric algebras with an invertible derivation. Let $n \in \mathbb{N}^*$ and A is a vector space with a basis $\{e_1, \ldots, e_n\}$. On A, we consider the product defined by:

$$e_ie_j = e_je_i := e_{i+j} \text{ if } i+j \leq n, \;\; e_ie_j = e_je_i := 0 \text{ if } i+j > n.$$

The vector space A endowed with this product is a commutative associative algebra. Let $(V, \star)$ be a symmetric algebra, then $(U := V \otimes A, .)$ is a left symmetric algebra, where the product "." is defined by:

$$v \otimes e_i.w \otimes e_j := v \star w \otimes e_ie_j, \;\; \forall v, w \in V, i, j \in \{1, \ldots, n\}.$$

Moreover, the endomorphism D of U defined by:

$$D(v \otimes e_i) := iv \otimes e_i, \;\; \forall v \in V, i \in \{1, \ldots, n\},$$

is an invertible derivation of $(U,.)$. Finally, by using the first construction, we obtain a symplectic left symmetric algebra $(\Phi(U), \rhd, \Gamma_0)$ with an invertible derivation Δ, which is skew-symmetric with respect to Γ_0.

7. Quasi-S-matrices on a left symmetric algebra U with an invariant isomorphism $\Theta : U \longrightarrow U^*$

In this section, we investigate the set of quasi-S-matrices on a left symmetric algebra U with an invariant isomorphism $\Theta : U \longrightarrow U^*$. The most important classes are symplectic left symmetric algebras and flat pseudo-Riemannian Lie algebras.

Let $(U,.)$ be a left symmetric algebra and $\Theta : U \longrightarrow U^*$ is an isomorphism that is invariant, i.e., for any $X \in U$,

$$\Theta \circ \mathrm{L}_X = \mathrm{L}_X^* \circ \Theta. \tag{24}$$

With any endomorphism $A : U \longrightarrow U$, we associate the tensors $\delta(A), \mathcal{O}(A) \in \mathrm{End}(U \otimes U, U)$ given by

$$\delta(A)(X,Y) = X.A(Y) - Y.A(X) - A([X,Y]) \quad \text{and}$$
$$\mathcal{O}(A)(X,Y) = [AX, AY] - (A(AX.Y) - A(AY.X)). \tag{25}$$

We can see easily that

$$\mathcal{O}(A) = N_A + A \circ \delta(A), \tag{26}$$

where N_A is the Nijenhuis torsion of A. The following proposition gives a useful characterization of quasi-S-matrices and S-matrices on $(U, .)$. The second assertion of this proposition was obtained by Bai (see Corollary 6.8 [5]).

Proposition 7.1. *Let* $\mathrm{r} \in U \otimes U$ *and* $\mathfrak{a}$ *is its skew-symmetric part. Put* $A = \mathrm{r}_\# \circ \Theta$ *and* $T = \mathfrak{a}_\# \circ \Theta$. *Then, the following assertions hold.*

(*i*) *The tensor* r *is a quasi-S-matrix of* $(U, .)$ *if and only if* T *is* L_U*-invariant and* $\mathcal{O}(A)$ *is* $\mathrm{L}_U^* \otimes \mathrm{L}_U^* \otimes \mathrm{ad}$*-invariant.*
(*ii*) *If* r *is symmetric, then it is an S-matrix of* $(U, .)$ *if and only if* $\mathcal{O}(A) = 0$.
(*iii*) *If* r *is symmetric and invertible, then it is an S-matrix of* $(U, .)$ *if and only if* $\delta(A^{-1}) = 0$.

Proof. For any $X, Y \in U$, we have

$$\begin{aligned}
\Delta(\mathrm{r})(\Theta X, \Theta Y) &= \mathrm{r}_\#([\Theta X, \Theta Y]) - [\mathrm{r}_\#(\Theta X), \mathrm{r}_\#(\Theta Y)] \\
&\overset{(10)}{=} A \circ \Theta^{-1}\left(\mathrm{L}^t_{AY}\Theta X\right) - A \circ \Theta^{-1}\left(\mathrm{L}^t_{AX}\Theta Y\right) \\
&\quad + 2A \circ \Theta^{-1}\left(\mathrm{L}(\mathfrak{a})(\Theta X, \Theta Y)\right) - [AX, AY] \\
&\overset{(24)}{=} -[AX, AY] - A(AY.X) + A(AX.Y) + 2A \circ \Theta^{-1}\left(\mathrm{L}(\mathfrak{a})(\Theta X, \Theta Y)\right) \\
&= -\mathcal{O}(A)(X,Y) + 2A \circ \Theta^{-1}\left(\mathrm{L}(\mathfrak{a})(\Theta X, \Theta Y)\right).
\end{aligned}$$

From this relation and (24), we can see easily that (*i*) and (*ii*) hold. Now, it is easy to show that if A is invertible, then $\mathcal{O}(A) = 0$ if and only if $\delta(A^{-1}) = 0$ and (*iii*) holds. □

According to the terminology used by Bai [5], if $\mathcal{O}(A) = 0$, then A is called an $\mathcal{O}$-operator for the Lie algebra $(U, [\ ,\ ])$ with respect to the representation L_U.

There are two interesting cases, as follows.

(*i*) The isomorphism Θ is skew-symmetric. In this case, $(U, ., \omega)$ is a symplectic left symmetric algebra where $\omega(X,Y) = \prec\Theta(X), Y\succ$.

(*ii*) The isomorphism Θ is symmetric. In this case, $(U, ., \langle\ ,\ \rangle)$ is a flat pseudo-Riemannian Lie algebra where $\langle X, Y\rangle = \prec\Theta(X), Y\succ$.

Proposition 7.2. *Let $(U, .)$ be a left symmetric algebra and $\Theta : U \longrightarrow U^*$ is an invariant isomorphism, and $\mathrm{r} \in U \otimes U$. Put $A = \mathrm{r}_\# \circ \Theta$ and denote by $\circ$ the product on U pull-back by Θ of the product on U^* given by (9). Then, the following assertions hold.*

1. *If Θ is skew-symmetric, then for any $X, Y \in U$,*

$$X \circ Y = [AX, Y] + A(Y.X) + Q(X, Y), \tag{27}$$

   where $Q : U \times U \longrightarrow U$ is defined by

$$\prec\alpha, Q(X, Y)\succ = -\omega(\delta(A^s - A^a)(\Theta^{-1}(\alpha), Y), X), \quad \forall \alpha \in U^*,$$

   and A^s and A^a are the symmetric and skew-symmetric parts of A with respect to the 2-form ω associated with Θ, respectively.
2. *If Θ is symmetric, then for any $X, Y \in U$,*

$$X \circ Y = Y.AX + AX.Y - A(Y.X) + P(X, Y), \tag{28}$$

   where $P : U \times U \longrightarrow U$ is defined by

$$\prec\alpha, P(X, Y)\succ = \langle\delta(A^s - A^a)(\Theta^{-1}(\alpha), Y), X\rangle, \quad \forall \alpha \in U^*,$$

   and A^s and A^a are the symmetric and the skew-symmetric parts of A with respect to the 2-form $\langle\ ,\ \rangle$ associated with Θ, respectively.
3. *If Θ is skew-symmetric and r is a quasi-S-matrix, then $(U, \circ, \omega)$ is a symplectic left symmetric algebra if and only if $\delta(A^a) = 0$.*

Proof.

1. Suppose that Θ is skew-symmetric and define ω by $\omega(X, Y) = \prec\Theta(X), Y\succ$. Thus, for any $\alpha, \beta \in U^*$,

$$\mathrm{r}(\alpha, \beta) = -\omega(A \circ \Theta(\alpha), \beta).$$

   We have

$$\begin{aligned} \prec\alpha, X \circ Y\succ &= \prec\alpha, \Theta^{-1}(\Theta(X).\Theta(Y))\succ \\ &= -\prec\Theta(X).\Theta(Y), \Theta^{-1}(\alpha)\succ \\ &\overset{(9)}{=} -\mathrm{r}(\mathrm{L}^t_{\Theta^{-1}(\alpha)}\Theta(X), \Theta(Y)) - \mathrm{r}(\Theta(X), \mathrm{ad}^t_{\Theta^{-1}(\alpha)}\Theta(Y)) \end{aligned}$$

$$
\begin{aligned}
&\overset{(24)}{=} \mathrm{r}(\Theta(\Theta^{-1}(\alpha).X), \Theta(Y)) + \omega(AX, \Theta^{-1}\left(\mathrm{ad}^t_{\Theta^{-1}(\alpha)}\Theta(Y)\right))\\
&= \omega(Y, A(\Theta^{-1}(\alpha).X)) - \omega(Y, [\Theta^{-1}(\alpha), AX])\\
&= \omega((A^s - A^a)Y, \Theta^{-1}(\alpha).X) - \omega(Y, [\Theta^{-1}(\alpha), AX])\\
&= -\omega(\Theta^{-1}(\alpha).(A^s - A^a)Y, X) - \omega(Y, [\Theta^{-1}(\alpha), AX])\\
&= -\omega(\delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X) - \omega(Y.(A^s - A^a)(\Theta^{-1}(\alpha)), X)\\
&\quad - \omega((A^s - A^a)([\Theta^{-1}(\alpha), Y], X) - \omega(Y, [\Theta^{-1}(\alpha), AX])\\
&= -\omega(\delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X) + \omega(\Theta^{-1}(\alpha), A(Y.X))\\
&\quad - \omega([\Theta^{-1}(\alpha), Y], AX) - \omega(Y, [\Theta^{-1}(\alpha), AX])\\
&\overset{(a)}{=} -\omega(\delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X) + \prec\alpha, A(Y.X)) + [AX, Y]\succ.
\end{aligned}
$$

In (a), we employ the fact that ω is a 2-cocycle with respect to the Lie bracket.

2. Suppose that Θ is symmetric and define $\langle\ ,\ \rangle$ as $\langle X, Y\rangle = \prec\Theta(X), Y\succ$. Therefore, for any $\alpha, \beta \in U^*$,

$$\mathrm{r}(\alpha, \beta) = \langle A \circ \Theta(\alpha), \beta\rangle.$$

For any $\alpha \in U^*$ and $X, Y \in U$, we have

$$
\begin{aligned}
\prec\alpha, X \circ Y\succ &= \prec\alpha, \Theta^{-1}(\Theta(X).\Theta(Y))\succ\\
&= \prec\Theta(X).\Theta(Y), \Theta^{-1}(\alpha)\succ\\
&\overset{(9)}{=} \mathrm{r}(\mathrm{L}^t_{\Theta^{-1}(\alpha)}\Theta(X), \Theta(Y)) + \mathrm{r}(\Theta(X), \mathrm{ad}^t_{\Theta^{-1}(\alpha)}\Theta(Y))\\
&\overset{(24)}{=} -\mathrm{r}(\Theta(\Theta^{-1}(\alpha).X), \Theta(Y)) + \langle AX, \Theta^{-1}\left(\mathrm{ad}^t_{\Theta^{-1}(\alpha)}\Theta(Y)\right)\rangle\\
&= -\langle Y, A(\Theta^{-1}(\alpha).X)\rangle + \langle Y, [\Theta^{-1}(\alpha), AX]\rangle\\
&= -\langle (A^s - A^a)Y, \Theta^{-1}(\alpha).X\rangle + \langle Y, [\Theta^{-1}(\alpha), AX]\rangle\\
&= \langle \Theta^{-1}(\alpha).(A^s - A^a)Y, X\rangle + \langle Y, [\Theta^{-1}(\alpha), AX]\rangle\\
&= \langle \delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X\rangle + \langle Y.(A^s - A^a)(\Theta^{-1}(\alpha)), X\rangle\\
&\quad + \langle (A^s - A^a)([\Theta^{-1}(\alpha), Y], X\rangle + \langle Y, [\Theta^{-1}(\alpha), AX]\rangle\\
&= \langle \delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X\rangle - \langle \Theta^{-1}(\alpha), A(Y.X)\rangle\\
&\quad + \langle [\Theta^{-1}(\alpha), Y], AX\rangle + \langle Y, [\Theta^{-1}(\alpha), AX]\rangle\\
&= \langle \delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X\rangle - \alpha(A(Y.X))\\
&\quad - \langle Y.\Theta^{-1}(\alpha), AX\rangle - \langle Y, AX.\Theta^{-1}(\alpha)\rangle\\
&= \langle \delta((A^s - A^a)(\Theta^{-1}(\alpha), Y), X\rangle - \prec\alpha, A(Y.X)\succ\\
&\quad + \prec\alpha, Y.AX + AX.Y\succ.
\end{aligned}
$$

3. Suppose that Θ is skew-symmetric and r is a quasi-S-matrix. For any $X, Y, Z \in U$, we have

$$\begin{aligned}\omega(X \circ Y, Z) + \omega(Y, X \circ Z) &= \omega([AX, Y], Z) + \omega(A(Y.X), Z) + \omega(Q(X, Y), Z)\\ &\quad + \omega(Y, [AX, Z]) + \omega(Y, A(Z.X)) + \omega(Y, Q(X, Z))\\ &= -\omega(X, (A^s - A^a)[Y, Z] - Y.(A^s - A^a)Z\\ &\quad + Z.(A^s - A^a)Y) - \Theta(Z)(Q(X, Y)) + \Theta(Y)(Q(X, Z))\\ &= -\omega(X, \delta((A^s - A^a))(Y, Z)) + \omega(\delta((A^s - A^a))(Y, Z), X)\\ &\quad - \omega(\delta((A^s - A^a))(Z, Y), X)\\ &= -3\omega(X, \delta(A^s - A^a))(Y, Z)).\end{aligned}$$

To conclude, we remark that since r is a quasi-S-matrix, then its skew-symmetric part is L_U-invariant and hence $\delta(A^s) = 0$. □

The proof of the following two theorems is similar to that of Theorem 6.1. The second part of Theorem 7.1 is based on the third part of Proposition 7.2 and (26).

Theorem 7.1. *Let $(U, ., \omega)$ be a symplectic left symmetric algebra and A is an endomorphism of U. We denote A^s and A^a as the symmetric and the skew-symmetric parts of A with respect to ω, respectively. The following assertions hold.*

1. *If $\mathcal{O}(A)$ is $\mathrm{L}_U^* \otimes \mathrm{L}_U^* \otimes \mathrm{ad}$-invariant and A^s is L_U-invariant, then:*
   - *(i) the product on U given by (27) is left symmetric,*
   - *(ii) $(T(U), [\ ,\ ]^A, K_A, J_A)$ is a complex product structure and $(T(U), [\ ,\ ]^A, \langle\ ,\ \rangle_A, K_A)$ is a para-Kähler Lie algebra.*
2. *If A^s is L_U-invariant, $\delta(A^a) = 0$ and N_A is $\mathrm{L}_U^* \otimes \mathrm{L}_U^* \otimes \mathrm{ad}$-invariant, then $(T(U), [\ ,\ ]^A, \langle\ ,\ \rangle_A, K_A, J_A)$ is a hyper-para-Kähler Lie algebra.*

In the above, $[\ ,\ ]^A, \langle\ ,\ \rangle_A, K_A, J_A$ are given by

$$\begin{aligned}[(X, Y), (Z, T)]^A &= ([X, Z] + \mathcal{O}(A)(T, Y), X.T - Z.Y),\\ J_A(X, Y) &= (-Y + AX - A^2Y, X - AY),\\ \langle (X, Y), (Z, T)\rangle_A &= \omega(T, X) + \omega(Y, Z) + 2\omega(A^aY, T), \quad K_A(X, Y) = (X - 2AY, -Y).\end{aligned}$$

Theorem 7.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Riemannian Lie algebra and A is an endomorphism of $\mathfrak{g}$. We denote A^s and A^a as the symmetric and the skew-symmetric parts of A with respect to $\langle\ ,\ \rangle$, respectively. If $\mathcal{O}(A)$ is $\mathrm{L}_{\mathfrak{g}}^* \otimes \mathrm{L}_{\mathfrak{g}}^* \otimes \mathrm{ad}$-invariant and A^a is $\mathrm{L}_{\mathfrak{g}}$-invariant, then the product given by (28) on $\mathfrak{g}$ is left symmetric. Moreover, $(T(\mathfrak{g}), [\ ,\ ]^A, K_A, J_A)$ is a complex product structure and $(T(\mathfrak{g}), [\ ,\ ]^A, \langle\ ,\ \rangle_A, K_A)$ is a para-Kähler Lie algebra, where*

$$[(X,Y),(Z,T)]^A = ([X,Z] + \mathcal{O}(A)(T,Y), X.T - Z.Y),$$
$$J_A(X,Y) = (-Y + AX - A^2Y, X - AY),$$
$$\langle (X,Y),(Z,T)\rangle_A = \langle T,X\rangle + \langle Y,Z\rangle + 2\langle A^sY,T\rangle, \quad K_A(X,Y) = (X - 2AY, -Y).$$

In this case, the dot is the Levi-Civita product and $\mathrm{L}_\mathfrak{g}$ *is its associated representation.*

Remark 5. As in Remark 4, we obtain the following more general result. Let $(U,.)$ be a left symmetric algebra and A is an endomorphism of U such that $\mathcal{O}(A)$ is $\mathrm{L}_U^* \otimes \mathrm{L}_U^* \otimes$ ad-invariant. Then, we can check that the bracket $[\ ,\ ]^A$ on $T(U)$ is a Lie bracket and hence $L^A : \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ given by

$$L^A(X,Y,Z) = \mathcal{O}(A)(X,Y).Z$$

is a Lie triple system.

8. Four-dimensional hyper-para-Kähler Lie algebras

In this section, we determine all of the four-dimensional hyper-para-Kähler Lie algebras up to an isomorphism. First, we need to determine the two-dimensional symplectic left symmetric algebras and the compatible couples of these algebras. Four-dimensional hyper-para-Kähler Lie algebras were classified in [1] using a vast computation (see also [2]), but we employ a new method that reduces the calculations significantly.

Let $(U,.,\omega)$ be a symplectic left symmetric algebra. We have

$$\mathrm{R}_{u.v} = \mathrm{R}_v \circ \mathrm{R}_u + [\mathrm{L}_u, \mathrm{R}_v], \tag{29}$$
$$\mathrm{L}_{[u,v]} = [\mathrm{L}_u, \mathrm{L}_v], \tag{30}$$
$$\mathrm{L}_u + \mathrm{L}_u^{\mathrm{a}} = 0, \tag{31}$$

where $\mathrm{L}_u^{\mathrm{a}}$ is the adjoint of L_u with respect to ω. Put

$$U.U = \mathrm{span}\{u.v, u,v \in U\},$$
$$D(U.U) = \mathrm{span}\{u.v - v.u, u,v \in U\},$$
$$S(U.U) = \mathrm{span}\{u.v + v.u, u,v \in U\}.$$

Clearly, we have

$$U.U = D(U.U) + S(U.U) \quad \text{and} \quad (U.U)^\perp = \{u \in U, \mathrm{R}_u = 0\}. \tag{32}$$

The sign $\perp$ designates the orthogonal with respect to ω.

Proposition 8.1. *Let* $(U,.,\omega)$ *be an abelian symplectic left symmetric algebra. Then,* $U.U \subset (U.U)^\perp$.

Proof. For any $u \in U$, we have $\mathrm{R}_u = \mathrm{L}_u$ and thus from (29)–(30) for any $u, v \in U$, we obtain

$$\mathrm{R}_{u.v} = \mathrm{R}_u \circ \mathrm{R}_v = \mathrm{R}_v \circ \mathrm{R}_u.$$

Moreover, $\mathrm{R}_u^{\mathrm{a}} = -\mathrm{R}_u$ and thus $\mathrm{R}_{u.v} = 0$, and the proposition follows from (32). □

Proposition 8.2. *Let $(U, ., \omega)$ be a two-dimensional non-trivial abelian symplectic left symmetric algebra. Then, a basis $\{e_1, e_2\}$ of U exists such that*

$$\omega = e_1^* \wedge e_2^*, \, \mathrm{R}_{e_1} = \mathrm{L}_{e_1} = 0 \quad \textit{and} \quad e_2.e_2 = ae_1, \quad a \neq 0.$$

Proof. From Proposition 8.1, we find that $U.U = (U.U)^\perp = \mathrm{span}\{e_1\}$. Select e_2 such that $\omega(e_1, e_2) = 1$ and the proposition follows. □

Proposition 8.3. *Let $(U, ., \omega)$ be a two-dimensional non-abelian symplectic left symmetric algebra. Then, a basis $\{e_1, e_2\}$ of U exists such that*

$$\omega = e_1^* \wedge e_2^*, \, e_1.e_1 = 0, e_2.e_2 = ae_2 \quad \textit{and} \quad e_1.e_2 = -e_2.e_1 = ae_1, \quad a \neq 0.$$

Proof. By necessity, we have $\dim D(U.U) = 1$. We distinguish two cases, as follows.

1. **First case**: $\dim U.U = 1$. In this case, $U.U = D(U.U) = \mathrm{span}\{e_1\}$. If $S(U.U) = \{0\}$, then we can select e_2 such that $\omega(e_1, e_2) = 1$. Since $e_1.e_1, e_2.e_2 \in S(U.U)$, then $e_1.e_1 = e_2.e_2 = 0$. Moreover, since $U.U = (U.U)^\perp$, then $\mathrm{R}_{e_1} = 0$. Now, $e_1.e_2 \in S(U.U)$ and then $e_1.e_2 = 0$. Thus, the product is trivial.
   If $S(U.U) \neq \{0\}$, then $U.U = D(U.U) = S(U.U) = \mathrm{span}\{e_1\}$. Select e_2 such that $\omega(e_1, e_2) = 1$. We have $\mathrm{R}_{e_1} = 0$, $e_2.e_2 = ae_1$ and $e_1.e_2 = be_1$. The relation $\mathrm{L}_{[e_1,e_2]}e_2 = [\mathrm{L}_{e_1}, \mathrm{L}_{e_2}](e_2)$ implies that $b = 0$ and hence $[e_1, e_2] = 0$, which is impossible. In conclusion, this case is impossible.
2. **Second case**: $\dim U.U = 2$. In this case, $U.U = D(U.U) \oplus S(U.U)$. Select a basis $\{e_1, e_2\}$ of U such that $e_1 \in D(U.U)$, $e_2 \in S(U.U)$, and $\omega(e_1, e_2) = 1$. Since $e_1 \in D(U.U)^\perp$ and $e_2 \in S(U.U)^\perp$, then we obtain

$$\begin{aligned} \omega(e_1.e_2 + e_2.e_1, e_2) &= 0, \\ \omega(e_1.e_2 + e_2.e_1, e_1) &= 2\omega(e_1.e_2, e_1), \\ &= -2\omega(e_2, e_1.e_1) = 0. \end{aligned}$$

   Thus, $e_1.e_2 = -e_2.e_1$ and hence $[e_1, e_2] = 2e_1.e_2$. So $e_1.e_2 = -e_2.e_1 = ae_1$. In addition, $e_1.e_1, e_2.e_2 \in S(U.U)$ so $e_1.e_1 = be_2$ and $e_2.e_2 = ce_2$. Now, the relation $\mathrm{L}_{e_2}^{\mathrm{a}} = -\mathrm{L}_{e_2}$ implies that $c = a$ and the relation $\mathrm{L}_{[e_1,e_2]} = [\mathrm{L}_{e_1}, \mathrm{L}_{e_2}]$ implies that $b = 0$, and thus the proposition follows. □

Remark 6. From Propositions 8.2 and 8.3, we can deduce that if $(U, ., \omega)$ is an abelian symplectic left symmetric algebra, so $D(U.U) = 0$ and thus $U.U = S(U.U)$ is an ω-isotropic one dimensional vector space. However, if $(U, ., \omega)$ is a non-abelian symplectic left symmetric algebra, then $U = U.U = D(U.U) \oplus S(U.U)$, where $D(U.U)$ and $S(U.U)$ are one-dimensional ω-isotropic vector spaces. This remark plays a crucial role in the proof of Theorem 8.1.

Recall that two symplectic left symmetric structures $(U, \star, \omega)$ and $(U, \circ, \omega)$ are called compatible if $K^{\star,\circ}$ satisfies the second assertion of Proposition 5.2. It is obvious that if $(U, \star, \omega)$ is a symplectic left symmetric algebra, then it is compatible with $(U, \alpha\star, \omega)$ for any $\alpha \in \mathbb{K}$. We refer to this case as trivially compatible.

Theorem 8.1. *Let $(U, \star, \omega)$ and $(U, \circ, \omega)$ be two symplectic left symmetric structures over a two-dimensional vector space U. Then, $(U, \star, \omega)$ and $(U, \circ, \omega)$ are non-trivially compatible if and only if one of the following holds.*

1. *A basis $\{e_1, e_2\}$ exists such that*

$$\mathrm{L}^{\star}_{e_1} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \mathrm{L}^{\star}_{e_2} = \begin{pmatrix} -a & -b \\ 0 & a \end{pmatrix}, \; \mathrm{L}^{\circ}_{e_1} = 0 \quad \text{and} \quad \mathrm{L}^{\circ}_{e_2} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

   $\omega = e_1^* \wedge e_2^*$ with $a \neq 0$ and $b \neq 0$.
2. *A basis $\{e_1, e_2\}$ exists such that*

$$\mathrm{L}^{\star}_{e_1} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \; \mathrm{L}^{\star}_{e_2} = \begin{pmatrix} -a & b \\ 0 & a \end{pmatrix}, \; \mathrm{L}^{\circ}_{e_1} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \; \mathrm{L}^{\circ}_{e_2} = \begin{pmatrix} -c & -b \\ 0 & c \end{pmatrix}.$$

   $\omega = e_1^* \wedge e_2^*$, with $a \neq 0$, $b \neq 0$ and $c \neq 0$.

Proof. The proof is based on an adequate use of the fact that the sum of two compatible symplectic left symmetric structures is left symmetric (see Proposition 5.3) and the use of Propositions 8.2–8.3 and Remark 6.

First, we can check that if $\star$ and $\circ$ have one of the forms above, then they are symplectic and compatible. Suppose that $(U, \star, \omega)$ and $(U, \circ, \omega)$ are non-trivially compatible. We distinguish three cases, as follows.

1. Both $\star$ and $\circ$ are abelian. Then, $\star + \circ$ defines an abelian symplectic left symmetric algebra structure on U, and thus by virtue of Proposition 8.2, $a \neq 0$ and a basis $\{e_1, e_2\}$ of U exists such that $\omega = e_1^* \wedge e_2^*$

$$e_1 \star e_1 + e_1 \circ e_1 = e_1 \star e_2 + e_1 \circ e_2 = 0, \; e_2 \star e_2 + e_2 \circ e_2 = ae_1. \qquad (*)$$

   Moreover, $(U \star U)^{\perp} = U \star U$ and $(U \circ U)^{\perp} = U \circ U$.

Suppose that $e_1 \circ e_1 \neq 0$. Then, from (∗), as given above, we obtain $U \star U = U \circ U = \mathrm{span}\{e_1\}$, and by (32) $\mathrm{L}^\star_{e_1} = \mathrm{L}^\circ_{e_1} = 0$, which contradicts $e_1 \circ e_1 \neq 0$. Thus, $e_1 \circ e_1 = e_1 \star e_1 = 0$. The same argument shows that $e_1 \circ e_2 = e_1 \star e_2 = 0$ and hence $\mathrm{L}^\star_{e_1} = \mathrm{L}^\circ_{e_1} = 0$, which by virtue of (32) implies that $U \star U = U \circ U = \mathrm{span}\{e_1\}$. Thus, $b \neq 0$ and $c \neq 0$ exists such that $e_2 \star e_2 = be_1$ and $e_2 \circ e_2 = ce_1$. Finally, $\circ = \frac{b}{c}\star$ and this case is not possible.

2. The product $\star$ is not abelian and $\circ$ is abelian. Then, $\star + \circ$ defines a non-abelian symplectic left symmetric algebra structure on U, and hence by virtue of Proposition 8.3, $a \neq 0$ and a basis $\{e_1, e_2\}$ of U exists such that $\omega = e_1^* \wedge e_2^*$ and

$$e_1 \star e_1 + e_1 \circ e_1 = 0, \; e_2 \star e_2 + e_2 \circ e_2 = ae_2,$$
$$e_1 \star e_2 + e_1 \circ e_2 = -e_2 \star e_1 - e_2 \circ e_1 = ae_1. \qquad (**)$$

Moreover, $U = D(U \star U) \oplus S(U \star U)$ and $U \circ U = S(U \circ U) = (U \circ U)^\perp$. By adding the two last relations in (∗∗), we obtain

$$e_1 \star e_2 + e_2 \star e_1 = 2e_1 \circ e_2.$$

So $e_1 \circ e_2 \in S(U \star U)$. If $e_1 \circ e_2 \neq 0$, it spans $U \circ U$ and hence from the second relation in (∗∗), we deduce that $e_2 \in S(U \star U)$ and thus $U \circ U = \mathrm{span}\{e_2\}$. Therefore, by (32), $\mathrm{L}^\circ_{e_2} = 0$, which contradicts $e_1 \circ e_2 \neq 0$. Thus, $e_1 \circ e_2 = 0$.
Now, suppose that $e_1 \circ e_1 \neq 0$. From the second relation in (∗∗), we deduce that $U \circ U = \mathrm{span}\{e_2\}$ and thus $\mathrm{L}^\circ_{e_2} = 0$. We deduce that

$$e_1 \star e_1 = -e_1 \circ e_1 = be_2, \; e_1 \star e_2 = -e_2 \star e_1 = ae_1.$$

From the relation $\mathrm{L}^\star_{[e_1,e_2]} e_1 = [\mathrm{L}^\star_{e_1}, \mathrm{L}^\star_{e_2}] e_1$, we deduce that $b = 0$ and $\circ = 0$, and thus we must have $e_1 \circ e_1 = 0$.
Therefore, we have shown that $\mathrm{L}^\circ_{e_1} = 0$ and thus $U \circ U = \mathrm{span}\{e_1\}$, $e_2 \circ e_2 = be_1$. We deduce that

$$e_1 \star e_1 = 0, \; e_2 \star e_2 = ae_2 - be_1, \; e_1 \star e_2 = -e_2 \star e_1 = ae_1.$$

Therefore, we find that $\star$ and $\circ$ satisfy the first form in the theorem.

3. Both $\star$ and $\circ$ are non-abelian. According to Proposition 5.3, $\star + \circ$ and $\star - \circ$ define two symplectic left symmetric algebra structures on U, one of which must be non-abelian. Thus, we can suppose that $\star + \circ$ is non-abelian by replacing $\circ$ by $-\circ$ if this is necessary. By virtue of Proposition 8.3, $a \neq 0$ and a basis $\{e_1, e_2\}$ of U exists such that $\omega = e_1^* \wedge e_2^*$, and

$$e_1 \star e_1 + e_1 \circ e_1 = 0, \; e_2 \star e_2 + e_2 \circ e_2 = ae_2,$$
$$e_1 \star e_2 + e_1 \circ e_2 = -e_2 \star e_1 - e_2 \circ e_1 = ae_1. \qquad (***)$$

Moreover, $U = D(U \star U) \oplus S(U \star U) = D(U \circ U) \oplus S(U \circ U)$. Put

$$v = e_1 \star e_2 + e_2 \star e_1 = -e_1 \circ e_2 - e_2 \circ e_1.$$

If $v \neq 0$, then it spans $S(U \star U)$ and $S(U \circ U)$; thus, from (∗∗∗) above, we obtain $S(U \star U) = S(U \circ U) = \text{span}\{e_2\}$. Therefore,

$$\mathrm{L}^{\star}_{e_1} = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix}, \mathrm{L}^{\star}_{e_2} = \begin{pmatrix} -c & 0 \\ d & c \end{pmatrix}, \mathrm{L}^{\circ}_{e_1} = \begin{pmatrix} 0 & a-c \\ -b & 0 \end{pmatrix}, \mathrm{L}^{\circ}_{e_2} = \begin{pmatrix} c-a & 0 \\ -d & a-c \end{pmatrix}.$$

The relations

$$\mathrm{L}^{\star}_{[e_1,e_2]} = [\mathrm{L}^{\star}_{e_1}, \mathrm{L}^{\star}_{e_2}] \quad \text{and} \quad \mathrm{L}^{\circ}_{[e_1,e_2]} = [\mathrm{L}^{\circ}_{e_1}, \mathrm{L}^{\circ}_{e_2}] \tag{$****$}$$

are equivalent to $d^2 = 4bc = 4b(c-a)$, which is equivalent to $d = b = 0$. This implies that $\circ = \frac{a-c}{c}\star$. This case is impossible and hence $v = 0$.
If $w = e_1 \star e_1 = -e_1 \circ e_1 \neq 0$, then it spans $S(U \star U)$ and $S(U \circ U)$; thus, from (∗∗∗), we obtain $S(U \star U) = S(U \circ U) = \text{span}\{e_2\}$. Therefore,

$$\mathrm{L}^{\star}_{e_1} = \begin{pmatrix} 0 & c \\ b & 0 \end{pmatrix}, \mathrm{L}^{\star}_{e_2} = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}, \mathrm{L}^{\circ}_{e_1} = \begin{pmatrix} 0 & a-c \\ -b & 0 \end{pmatrix}, \mathrm{L}^{\circ}_{e_2} = \begin{pmatrix} c-a & 0 \\ 0 & a-c \end{pmatrix}.$$

In this case, (∗∗∗∗) implies that $b = 0$ and hence $\circ = \frac{a-c}{c}\star$. This case is impossible and hence $w = 0$. In summary, we have shown that

$$e_1 \star e_2 + e_2 \star e_1 = -e_1 \circ e_2 - e_2 \circ e_1 = e_1 \star e_1 = -e_1 \circ e_1 = 0.$$

So

$$\mathrm{L}^{\star}_{e_1} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \mathrm{L}^{\star}_{e_2} = \begin{pmatrix} -c & d \\ 0 & c \end{pmatrix}, \mathrm{L}^{\circ}_{e_1} = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}, \mathrm{L}^{\circ}_{e_2} = \begin{pmatrix} c-a & -d \\ 0 & a-c \end{pmatrix}.$$

Thus, $\star$ and $\circ$ satisfy the second form in the theorem.
Finally, a direct computation shows that for $\star$ and $\circ$, as in the first form in the theorem, we have

$$K^{\star,\circ}(e_1, e_1) = K^{\star,\circ}(e_1, e_2) = K^{\star,\circ}(e_2, e_1) = 0 \quad \text{and} \quad K^{\star,\circ}(e_2, e_2) = -2\begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix},$$

and for the second form,

$$K^{\star,\circ}(e_1, e_1) = K^{\star,\circ}(e_1, e_2) = K^{\star,\circ}(e_2, e_1) = 0 \quad \text{and} \quad K^{\star,\circ}(e_2, e_2) = 2\begin{pmatrix} 0 & ab+bc \\ 0 & 0 \end{pmatrix}.$$

In both cases, $K^{\star,\circ}$ satisfies the second assertion of Proposition 5.2 and the theorem is proved. □

Based on the above, we now give all the four-dimensional hyper-para-Kähler Lie algebras.

Theorem 8.2. *Any* $(\mathfrak{g}, \Omega, K, J)$ *four-dimensional hyper-para-Kähler Lie algebra is isomorphic to* $(\mathbb{R}^4, [\ ,\ ], \Omega_0, K_0, J_0)$*, where in the canonical basis* (e_1, e_2, f_1, f_2)*, we have*

$$\Omega_0 = f_2^* \wedge e_1^* + e_2^* \wedge f_1^*,\ K_0 e_i = e_i,\ K_0 f_i = -f_i,\ J_0 e_i = f_i,\ J_0 f_i = -e_i,$$

and the non-vanishing Lie brackets have one of the following expressions:

(*i*) $[e_2, f_2] = a(f_1 - e_1)$, $a \neq 0$.
(*ii*) $[e_1, e_2] = 2ae_1$, $[e_1, f_2] = a(e_1 + f_1)$, $[e_2, f_1] = -a(e_1 + f_1)$, $[e_2, f_2] = a(f_2 - e_2)$ *and* $[f_1, f_2] = 2af_1$, $a \neq 0$.
(*iii*) $[e_1, e_2] = 2ae_1$, $[e_1, f_2] = af_1$ *and* $[e_2, f_2] = -b(e_1 + f_1)$, $a \neq 0$, $b \neq 0$.
(*iv*) $[e_1, e_2] = 2ae_1$, $[e_1, f_2] = af_1 + ce_1$, $[e_2, f_1] = -c(f_1 + e_1)$, $[e_2, f_2] = b(e_1 + f_1) - ce_2 + af_2$ *and* $[f_1, f_2] = 2cf_1$, $a \neq 0$, $b \neq 0$, $c \neq 0$.

9. Symplectic associative algebras

In this section, we consider an important subclass of the class of symplectic left symmetric algebras. In order to introduce this subclass, we begin by providing a geometric interpretation of symplectic left symmetric algebras.

Let $(U, ., \omega)$ be a symplectic left symmetric algebra. The product is Lie-admissible and the bracket $[u, v] = u.v - v.u$ is a Lie bracket on U. Moreover, since

$$\omega(u.v, w) + \omega(v, u.w) = 0$$

for any $u, v, w \in U$, a direct computation yields

$$\omega([u, v], w) + \omega([v, w], u) + \omega([w, u], v) = 0,$$

and thus $(U, [\ ,\ ], \omega)$ is a symplectic Lie algebra. Let G be the simply connected Lie group associated with $(U, [\ ,\ ])$. For any $u \in U$, we denote u^ℓ as the left invariant vector field on G associated with u. The formulae

$$\omega^\ell(u^\ell, v^\ell) = \omega(u, v) \quad \text{and} \quad \nabla_{u^\ell} v^\ell = (u.v)^\ell$$

define a left invariant symplectic form on G, and a flat and torsion-free left invariant connection. Moreover, ∇ is symplectic, i.e., $\nabla \omega^\ell = 0$. The connection ∇ is right invariant iff for any $u, v, w \in U$,

$$[u^\ell, \nabla_{v^\ell} w^\ell] = \nabla_{[u^\ell, v^\ell]} w^\ell + \nabla_{v^\ell} [u^\ell, w^\ell].$$

A straightforward computation shows that this relation is equivalent to the associativity of the left symmetric product on U.

A *symplectic associative algebra* is a symplectic left symmetric algebra that is associative. We have seen that there is a correspondence between the set of symplectic associative algebras and symplectic Lie groups endowed with a bi-invariant affine structure, for which the symplectic form is parallel.

In the following, we give accurate descriptions of symplectic associative algebras (see Theorems 9.1–9.2).

Let $(U, ., \omega)$ be an associative symplectic algebra. Then, for any $u, v \in U$,

$$\mathrm{L}_{uv} = \mathrm{L}_u \circ \mathrm{L}_v.$$

Since $\mathrm{L}_u^{\mathrm{a}} = -\mathrm{L}_u$, then for any $u \in U$, we obtain

$$\mathrm{L}_{uv} = \mathrm{L}_u \circ \mathrm{L}_v = -\mathrm{L}_v \circ \mathrm{L}_u = -\mathrm{L}_{vu}. \tag{33}$$

Proposition 9.1. *Let $(U, ., \omega)$ be an associative symplectic algebra. Then, $U^4 = 0$ and $\mathcal{J} = U^2 + (U^2)^\perp$ is a co-isotropic two-sided ideal of U that satisfies $\mathcal{J}^2 = 0$.*

Proof. It is obvious that $\mathcal{J}$ is a co-isotropic two-sided ideal. For any $u, v, w \in U$,

$$\begin{aligned} \mathrm{L}_{uvw} &= \mathrm{L}_u \circ \mathrm{L}_v \circ \mathrm{L}_w \\ &\overset{(33)}{=} \mathrm{L}_w \circ \mathrm{L}_u \circ \mathrm{L}_v \\ &= \mathrm{L}_w \circ \mathrm{L}_{uv} \\ &\overset{(33)}{=} -\mathrm{L}_{uv} \circ \mathrm{L}_w \\ &= -\mathrm{L}_{uvw}, \end{aligned}$$

and thus $\mathrm{L}_{uvw} = 0$ and $U^4 = 0$. Recall that

$$(U^2)^\perp = \{u \in U, \mathrm{R}_u = 0\}.$$

So $U^2.(U^2)^\perp = (U^2)^\perp.(U^2)^\perp = 0$. In addition, for any $u \in (U^2)^\perp$ and any $v, w \in U$,

$$u.v.w \overset{(33)}{=} -v.u.w = 0,$$

so $(U^2)^\perp.U^2 = 0$. In conclusion, $\mathcal{J}^2 = 0$. □

According to this proposition, to study associative symplectic algebras, we need to distinguish two cases that depend on the triviality of U^3.

Model of associative symplectic algebras with $U^3 = 0$ Let V be a vector space and $(I, \mathfrak{s})$ is a symplectic vector space. Let $\mathfrak{m} : V^* \longrightarrow V \odot V$ and $\mathfrak{n} : I \longrightarrow V \odot V$ be two linear maps ($V \odot V$ is the space of bilinear symmetric forms on V^*).

The space $U_1 = V \oplus I \oplus V^*$ carries a symplectic form ω, for which I and $V \oplus V^*$ are orthogonal, $\omega_{|I\times I} = \mathfrak{s}$, and for any $u \in V$, $\alpha \in V^*$, $\omega(\alpha, u) = -\omega(u, \alpha) = \alpha(u)$. Define a product on U such that $V^*.V^*, I.V^* \subset V$ by

$$\prec\gamma, \alpha.\beta\succ = \mathfrak{m}(\alpha)(\beta, \gamma) \quad \text{and} \quad \prec\beta, i.\alpha\succ = \mathfrak{n}(i)(\alpha, \beta), \tag{34}$$

for any $\alpha, \beta, \gamma \in V^*$, and $i \in I$ (all the others products vanish).

It is easy to see that $(U_1, ., \omega)$ is an associative symplectic algebra and $U_1^3 = 0$. We refer to this algebra as an *associative symplectic algebra of type one.* In fact, all associative symplectic algebras with $U^3 = 0$ are of this form.

Theorem 9.1. *Any associative symplectic algebra with* $U^3 = 0$ *is isomorphic to an associative symplectic algebra of type one.*

Proof. The condition that $U^3 = 0$ is equivalent to $U^2 \subset (U^2)^\perp$. Put $V = U^2$ and select I a complement of V in $(U^2)^\perp$. The restriction on ω to I defines a symplectic form, such as $\mathfrak{s}$. The orthogonal $I^\perp$ of I is a symplectic space that contains V as a Lagrangian subspace, so we can choose a Lagrangian complement W of V in $I^\perp$. The linear map $W \longrightarrow V^*$, $u \mapsto \omega(u, .)$ is an isomorphism, so we can identify (U, ω) as $V \oplus I \oplus V^*$ endowed with the symplectic form described above. Since $V.U = U.V = 0$ and for any $u \in I$, $\mathrm{R}_u = 0$ and thus we find that the product on U is given by (34), which completes the proof. □

Model of associative symplectic algebras with $U^3 \neq 0$ Let $V = V_0 \oplus V_1$ and $I = I_0 \oplus I_1$ be two vector spaces such that $(I_0, \mathfrak{s}_0)$, $(I_1, \mathfrak{s}_1)$ are symplectic vector spaces. Put $V^* = V_0^0 \oplus V_1^0$ where V_0^0 and V_1^0 are the annihilators of V_0 and V_1, respectively.

Let $\mathfrak{a} : V_1 \longrightarrow V_0 \odot V_0$, $\mathfrak{b} : I_0 \longrightarrow V_0 \odot V_0$, $\mathfrak{c} : I_1 \longrightarrow V \odot V$ and $\mathfrak{d} : V^* \longrightarrow V \odot V$ be linear maps ($V_0 \odot V_0$ (resp. $V_1 \odot V_1$) in the space of bilinear symmetric forms on V_1^0 (resp. V_0^0)). Finally, let $F : V^* \times V_1^0 \longrightarrow I_0$ be a bilinear map.

The space $U_2 = V \oplus I \oplus V^*$ carries a symplectic form ω, for which I and $V \oplus V^*$ are orthogonal, $\omega_{|I\times I} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, and for any $u \in V$, $\alpha \in V^*$, $\omega(\alpha, u) = -\omega(u, \alpha) = \alpha(u)$.

Now, we define a product on U_2 that satisfies

$$V_1.V_1^0, I_0.V_1^0, V^*.I_0 \subset V_0,\ I_1.V^* \subset V, V^*.V_0^0 \subset V,\ V^*.V_1^0 \subset V \oplus I_0,$$

and it is given by

$$\prec\beta_1, v_1.\alpha_1\succ = \mathfrak{a}(v_1)(\alpha_1, \beta_1),\ v_1 \in V_1, \alpha_1, \beta_1 \in V_1^0,$$
$$\prec\beta_1, i_0.\alpha_1\succ = \mathfrak{b}(i_0)(\alpha_1, \beta_1),\ i_0 \in I_0, \alpha_1, \beta_1 \in V_1^0,$$

$$\begin{aligned}
\prec\beta_1, \alpha.i_0\succ &= \mathfrak{s}_0(i_0, F(\alpha, \beta_1)),\ i_0 \in I_0, \alpha \in V^*, \beta_1 \in V_1^0,\\
\prec\beta, i_1.\alpha\succ &= \mathfrak{c}(i_1)(\alpha, \beta),\ i_1 \in I_1, \alpha, \beta \in V^*,\\
\alpha.\beta_0 &= E(\alpha, \beta_0),\ \alpha \in V^*, \beta_0 \in V_0^0,\\
\alpha.\beta_1 &= E(\alpha, \beta_1) + F(\alpha, \beta_1),\ \alpha \in V^*, \beta_1 \in V_1^0.\\
\prec\gamma, E(\alpha, \beta)\succ &= \mathfrak{d}(\alpha)(\beta, \gamma).
\end{aligned}$$

With this product U_2, becomes an algebra for which the left multiplication is symplectic. Now, this product is associative iff for any $\alpha \in V^*$, $\beta_1, \gamma_1, \mu_1 \in V_1^0$, $\beta_0 \in V_0^0$,

$$\mathfrak{a}(E_1(\alpha, \beta_1))(\gamma_1, \mu_1) + \mathfrak{b}(F(\alpha, \beta_1))(\gamma_1, \mu_1) = \mathfrak{s}_0(F(\beta_1, \gamma_1), F(\alpha, \mu_1)), \tag{35}$$

$$\mathfrak{a}(E_1(\alpha, \beta_0))(\gamma_1, \mu_1) = \mathfrak{s}_0(F(\beta_0, \gamma_1), F(\alpha, \mu_1)). \tag{36}$$

In this case, $U_2^3 = 0$ iff

$$\mathfrak{s}_0(F(\beta_1, \gamma_1), F(\alpha, \mu_1)) = \mathfrak{s}_0(F(\beta_0, \gamma_1), F(\alpha, \mu_1)) = 0.$$

When (35) and (36) hold, and $U_2^3 \neq 0$, we refer to $(U_2, ., \omega)$ as an *associative symplectic algebra of type two.*

Theorem 9.2. *Any associative symplectic algebra with $U^3 \neq 0$ is isomorphic to an associative algebra of type two.*

Proof. Since $U^4 = 0$ and $U^3 \subset U^2$, then

$$U^3 \subset U^2 \subset (U^3)^\perp \quad \text{and} \quad U^3 \subset (U^2)^\perp \subset (U^3)^\perp.$$

Put $V_0 = U^3$ and select a complement V_1 of V_0 in $V = U^2 \cap (U^2)^\perp$. Select I_0 and I_1 as two subspaces of U such that

$$U^2 = V \oplus I_0 \quad \text{and} \quad (U^2)^\perp = V \oplus I_1.$$

We find that $I_0 \cap I_1 = \{0\}$, $\omega(I_0, I_1) = 0$, I_0, I_1 are symplectic and $I = I_0 \oplus I_1$ is also symplectic. Denote $\mathfrak{s}_0$ and $\mathfrak{s}_1$ as the restrictions on ω to I_0 and I_1, respectively. Now, $I^\perp$ is symplectic and it contains V as a Lagrangian subspace, so we can select a Lagrangian subspace W of the $I^\perp$ complement of V. The linear map $W \longrightarrow V^*$, $u \mapsto \omega(u, .)$ realizes an isomorphism. Finally, we identify

$$U = V \oplus (I_1 \oplus I_2) \oplus V^*$$

with the symplectic form given by

$$\omega(V, V) = \omega(V, I) = \omega(V^*, V^*) = \omega(V^*, I) = 0 \quad \text{and} \quad \omega_{|I\times I} = \omega_1 \oplus \omega_2,$$

and for any $u \in V$, $\alpha \in V^*$, $\omega(\alpha, u) = -\omega(u, \alpha) = \alpha(u)$. Denote V_0^0 and V_1^0 as the annihilators of V_0 and V_1, respectively.

Next, we consider the product's properties. In Proposition 9.1, we showed that $\mathcal{J} = V \oplus (I_0 \oplus I_1)$ satisfies $\mathcal{J}^2 = 0$. Obviously, for any $u \in V_0$, $\mathrm{L}_u = \mathrm{R}_u = 0$, and from (32), for any $u \in V \oplus I_1$, we have $\mathrm{R}_u = 0$. Since $U^2.V^* \subset V_0$ and because the symplectic form is invariant, we obtain $U^2.V_0^0 = 0$. Since $V^*.U^2 \subset V_0$, then from the invariance of the symplectic form, we find that $V^*.V_0^0 \subset V$. Thus, we can put

$$\begin{aligned}
\prec\beta_1, v_1.\alpha_1\succ &= \mathfrak{a}(v_1)(\alpha_1, \beta_1),\ v_1 \in V_1, \alpha_1, \beta_1 \in V_1^0,\\
\prec\beta_1, i_0.\alpha_1\succ &= \mathfrak{b}(i_0)(\alpha_1, \beta_1),\ i_0 \in I_0, \alpha_1, \beta_1 \in V_1^0,\\
\prec\beta_1, \alpha.i_0\succ &= \mathfrak{s}_0(i_0, F(\alpha, \beta_1)),\ i_0 \in I_0, \alpha \in V^*, \beta_1 \in V_1^0,\\
\prec\beta, i_1.\alpha\succ &= \mathfrak{c}(i_1)(\alpha, \beta),\ i_1 \in I_1, \alpha, \beta \in V^*,\\
\alpha.\beta_0 &= E(\alpha, \beta_0),\ \alpha \in V^*, \beta_0 \in V_0^0,\\
\alpha.\beta_1 &= E(\alpha, \beta_1) + F(\alpha, \beta_1),\ \alpha \in V^*, \beta_1 \in V_1^0.
\end{aligned}$$

The invariance of the symplectic form implies that $\mathfrak{a}(v_1)$, $\mathfrak{b}$, $\mathfrak{c}(i_1)$, and $\mathfrak{d}(\alpha)$ are symmetric, where $\mathfrak{d}(\alpha)(\beta, \gamma) = \gamma(E(\alpha, \beta))$. The associativity of this product is equivalent to $\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma$ for any $\alpha, \beta, \gamma \in V^*$. Obviously, this is true when $\gamma \in V_0^0$, so the associativity is equivalent to

$$\mu_1(\alpha.(\beta_0.\gamma_1)) = \mu_1((\alpha.\beta_0).\gamma_1) \quad \text{and} \quad \prec\mu_1, \alpha.(\beta_1.\gamma_1)\succ = \prec\mu_1, (\alpha.\beta_1).\gamma_1\succ$$

for any $\alpha \in V^*$, $\alpha_1, \beta_1, \mu_1 \in V_1^0$ and $\beta_0 \in V_0^0$, which is equivalent to

$$\begin{aligned}
&\mathfrak{s}_0(F(\beta_0, \gamma_1), F(\alpha, \mu_1)) = \mathfrak{a}(E_1(\alpha, \beta_0))(\gamma_1, \mu_1),\\
&\mathfrak{s}_0(F(\beta_1, \gamma_1), F(\alpha, \mu_1)) = \mathfrak{a}(E_1(\alpha, \beta_1))(\gamma_1, \mu_1) + \mathfrak{b}(F(\alpha, \beta_1))(\gamma_1, \mu_1).
\end{aligned}$$

This completes the proof. □

Corollary 9.1. *Let $(U, ., \omega)$ be an associative symplectic algebra. Then,*

- (*i*) *If $\dim U = 2$, then $(U, ., \omega)$ is isomorphic to an associative symplectic algebra of type one of the form $V \oplus V^*$ with $\dim V = 1$.*
- (*ii*) *If $\dim U = 4$, then $(U, ., \omega)$ is isomorphic to an associative symplectic algebra of type one, either of the form $V \oplus V^*$ with $\dim V = 2$ or $V \oplus I \oplus V^*$ with $\dim V = 1$.*

A six-dimensional associative algebra $U^3 \neq 0$ is isomorphic to $V \oplus I_0 \oplus V^*$ with $\dim V = 2$ and $V = V_0 \oplus V_1$. Select a basis (e_0, e_1) of V such that $e_i \in V_i$ and a basis (f_1, f_2) of I_0 such that $\mathfrak{s}_1(f_1, f_2) = 1$. Eqs. (35) and (36) are equivalent to

$$\begin{aligned}\mathfrak{a}(E_1(e_0^*,e_1^*))(e_1^*,e_1^*)+\mathfrak{b}(F(e_0^*,e_1^*))(e_1^*,e_1^*)&=\mathfrak{s}(F(e_1^*,e_1^*),F(e_0^*,e_1^*)),\\ \mathfrak{a}(E_1(e_0^*,e_0^*))(e_1^*,e_1^*)&=\mathfrak{s}(F(e_0^*,e_1^*),F(e_0^*,e_1^*)),\\ \mathfrak{a}(E_1(e_1^*,e_1^*))(e_1^*,e_1^*)+\mathfrak{b}(F(e_1^*,e_1^*))(e_1^*,e_1^*)&=\mathfrak{s}(F(e_1^*,e_1^*),F(e_1^*,e_1^*)),\\ \mathfrak{a}(E_1(e_1^*,e_0^*))(e_1^*,e_1^*)&=\mathfrak{s}(F(e_0^*,e_1^*),F(e_1^*,e_1^*)).\end{aligned}$$

Put

$$F(e_0^*,e_1^*)=af_1+bf_2 \quad\text{and}\quad F(e_1^*,e_1^*)=cf_1+df_2.$$

Thus,

$$\begin{aligned}\mathfrak{a}(E_1(e_0^*,e_1^*))(e_1^*,e_1^*)+a\mathfrak{b}(f_1)(e_1^*,e_1^*)+b\mathfrak{b}(f_2)(e_1^*,e_1^*)&=cb-ad,\\ \mathfrak{a}(E_1(e_0^*,e_0^*))(e_1^*,e_1^*)&=0,\\ \mathfrak{a}(E_1(e_1^*,e_1^*))(e_1^*,e_1^*)+c\mathfrak{b}(f_1)(e_1^*,e_1^*)+d\mathfrak{b}(f_2)(e_1^*,e_1^*)&=0,\\ \mathfrak{a}(E_1(e_1^*,e_0^*))(e_1^*,e_1^*)&=ad-cb,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&a\mathfrak{b}(f_1)(e_1^*,e_1^*)+b\mathfrak{b}(f_2)(e_1^*,e_1^*)=-\mathfrak{a}(E_1(e_1^*,e_0^*))(e_1^*,e_1^*)-\mathfrak{a}(E_1(e_0^*,e_1^*))(e_1^*,e_1^*),\\ &c\mathfrak{b}(f_1)(e_1^*,e_1^*)+d\mathfrak{b}(f_2)(e_1^*,e_1^*)=-\mathfrak{a}(E_1(e_1^*,e_1^*))(e_1^*,e_1^*),\\ &\qquad\mathfrak{a}(E_1(e_0^*,e_0^*))(e_1^*,e_1^*)=0,\\ &\qquad\mathfrak{a}(E_1(e_1^*,e_0^*))(e_1^*,e_1^*)=ad-cb.\end{aligned}$$

Put $\mathfrak{a}(e_1)(e_1^*,e_1^*)=\alpha\neq 0$, $E_1(e_0^*,e_0^*)=0$ and $\delta=ad-cb\neq 0$, $E_1(e_i^*,e_j^*)=a_{ij}e_1$. Thus,

$$\begin{aligned}\mathfrak{b}(f_1)(e_1^*,e_1^*)&=-\alpha\delta^{-1}(aa_{11}-c(a_{10}+a_{01}))\quad\text{and}\\ \mathfrak{b}(f_2)(e_1^*,e_1^*)&=\alpha\delta^{-1}(ba_{11}-d(a_{10}+a_{01})).\end{aligned}$$

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