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On the local structure of noncommutative deformations

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ABSTRACT

Let $(M, \pi, \mathcal{D})$ be a Poisson manifold endowed with a flat, torsion-free contravariant connection. We show that if $\mathcal{D}$ is an $\mathcal{F}$-connection then there exists a tensor $\mathbf{T}$ such that $\mathcal{D}\mathbf{T}$ is the metacurvature tensor introduced by E. Hawkins in his work on noncommutative deformations. We compute $\mathbf{T}$ and the metacurvature tensor in this case and show that if $\mathbf{T} = 0$ then near any regular point π and $\mathcal{D}$ are defined in a natural way by a Lie algebra action and a solution of the classical Yang–Baxter equation. Moreover, when $\mathcal{D}$ is the contravariant Levi-Civita connection associated to π and a Riemannian metric, the Lie algebra action can be chosen in such a way that it preserves the metric. This solves the inverse problem of a result of the second author.

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1. Introduction and main result

In [1,2] Hawkins showed that if a deformation of the graded algebra $\Omega^*(M)$ of differential forms on a Riemannian manifold M comes from a spectral triple describing M, then the Poisson tensor π (which characterizes the deformation) and the Riemannian metric satisfy the following conditions:

(H_1) the associated metric contravariant connection $\mathcal{D}$ is flat;
(H_2) the metacurvature of $\mathcal{D}$ vanishes;
(H_3) π is compatible with the Riemannian volume μ, i.e., $d(i_\pi \mu) = 0$.

The metric contravariant connection associated naturally to any pair of a (pseudo-)Riemannian metric and a Poisson tensor is the contravariant analogue of the classical Levi-Civita connection; it has appeared first in [3]. The metacurvature, introduced in [2], is a (2, 3)-type tensor field (symmetric in the contravariant indices and antisymmetric in the covariant indices) associated naturally to any flat, torsion-free contravariant connection.

The main result of Hawkins [2, Theorem 6.6 and also Lemma 6.5] states that if (M, π, g) is a triple satisfying (H_1)–(H_3) with M compact, then around any regular point $x_0 \in M$ the Poisson tensor can be written as

$$\pi = \sum_{i,j} a^{ij} X_i \wedge X_j \tag{1}$$

where the matrix (a^{ij}) is constant and invertible and $\{X_1, \dots, X_s\}$ is a family of linearly independent commuting Killing vector fields.

On the other hand, the second author showed in [4] that if $\zeta : \mathfrak{g} \to \mathfrak{X}^1(M)$ is an action of a finite-dimensional real Lie algebra $\mathfrak{g}$ on a smooth manifold M and $r \in \wedge^2 \mathfrak{g}$ is a solution of the classical Yang–Baxter equation, then:

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(a) The map $\mathcal{D}^r : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ given by

$$\mathcal{D}^r_\alpha \beta := \sum_{i,j=1}^n a^{ij} \alpha(\zeta(u_i)) \mathcal{L}_{\zeta(u_j)} \beta, \tag{2}$$

where $\{u_1, \dots, u_n\}$ is any basis of $\mathfrak{g}$ and a^{ij} are the components of r in this basis, depends only on r and ζ and defines a flat, torsion-free contravariant connection with respect to the Poisson tensor $\pi^r := \zeta(r)$.

(b) If M is Riemannian and ζ preserves the metric, $\mathcal{D}^r$ is nothing else but the metric contravariant connection associated to the metric and π^r.

(c) If $\mathfrak{g}$ acts freely on M, the metacurvature of $\mathcal{D}^r$ vanishes.

In this setting, (1) can be re-expressed by saying that there exists a free action $\zeta : \mathfrak{g} \to \mathfrak{X}^1(U)$ of a finite-dimensional abelian Lie algebra $\mathfrak{g}$ on an open neighborhood $U \subseteq M$ of x_0 which preserves g, and a solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation such that $\pi = \pi^r$. Moreover, since ζ preserves g, then $\mathcal{D} = \mathcal{D}^r$ by (b). It follows that $\mathcal{D}$ is a Poisson connection, i.e., $\mathcal{D}\pi = 0$ and hence an $\mathcal{F}^{\text{reg}}$-connection (see [5]).

Given a flat, torsion-free $\mathcal{F}^{\text{reg}}$-connection $\mathcal{D}$ on a Poisson manifold (M, π), we shall see that there exists a $(2, 2)$-type tensor field $\mathbf{T}$ on the dense open set of regular points such that

(i) $\mathcal{D}\mathbf{T} = \mathcal{M}$ where $\mathcal{M}$ is the metacurvature of $\mathcal{D}$;

(ii) $\mathbf{T}$ vanishes if and only if the exterior differential of any parallel 1-form is also parallel.

By looking at the proof of the second author's result closely, one observes that in proving (c) the second author showed that $\mathcal{D}^r$ is an $\mathcal{F}^{\text{reg}}$-connection and that whenever a 1-form is $\mathcal{D}^r$-parallel then so is its exterior differential, meaning that $\mathbf{T}$ vanishes. Accordingly, (c) can be rephrased as follows:

(c′) If $\mathfrak{g}$ acts freely on M, $\mathcal{D}^r$ is an $\mathcal{F}^{\text{reg}}$-connection and $\mathbf{T}$ vanishes (and hence so does $\mathcal{M}$).

Note that in the case studied by Hawkins $\mathbf{T}$ vanishes since as we saw above the action ζ is free. So it is natural to consider the following problem, inverse of the second author's result: *Given a smooth manifold M endowed with a Poisson tensor π and a Riemannian metric g such that the associated metric contravariant connection is a flat $\mathcal{F}^{\text{reg}}$-connection and such that $\mathbf{T} = 0$, is there a free action of a finite-dimensional Lie algebra $\mathfrak{g}$ preserving g and a solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation such that $\pi = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$?*

The main result of this paper answers in the affirmative to that question in a more general setting. More precisely,

Theorem 1.1. *Let $(M, \pi, \mathcal{D})$ be a Poisson manifold endowed with a flat, torsion-free contravariant connection.*

(1) *If $\mathcal{D}$ is an $\mathcal{F}^{\text{reg}}$-connection and $\mathbf{T} = 0$, then for any regular point x_0 with rank $2r$, there exists a free action $\zeta : \mathfrak{g} \to \mathfrak{X}(U)$ of a $2r$-dimensional real Lie algebra $\mathfrak{g}$ on a neighborhood U of x_0, and an invertible solution $r \in \wedge^2 \mathfrak{g}$ of the classical Yang–Baxter equation, such that $\pi = \pi^r$ and $\mathcal{D} = \mathcal{D}^r$.*

(2) *Moreover, if $\mathcal{D}$ is the metric contravariant connection associated to π and a Riemannian metric g, then the action can be chosen in such a way that its fundamental vector fields are Killing.*

The paper is organized as follows. In Section 2, we recall some standard facts about contravariant connections and the metacurvature tensor; we also define the tensor $\mathbf{T}$. Section 3 is devoted to the computation of the metacurvature tensor (and the tensor $\mathbf{T}$ as well) in the case of an $\mathcal{F}^{\text{reg}}$-connection. In Section 4, we give a proof of Theorem 1.1.

Notation 1.2. For a smooth manifold M, $C^\infty(M)$ will denote the space of smooth functions on M, $\Gamma(V)$ will denote the space of smooth sections of a vector bundle V over M, $\Omega^p(M) := \Gamma(\wedge^p T^*M)$ will denote the space of differential p-forms, and $\mathfrak{X}^p(M) := \Gamma(\wedge^p TM)$ will denote the space of p-vector fields.

For a Poisson tensor π on M, we will denote by $\pi_\sharp : T^*M \to TM$ the anchor map defined by $\beta(\pi_\sharp(\alpha)) = \pi(\alpha, \beta)$, and by H_f the Hamiltonian vector field of a function f, that is, $H_f := \pi_\sharp(df)$. We will also denote by $[\,,\,]_\pi$ the Koszul–Schouten bracket on differential forms (see, e.g., [6]); this is given on 1-forms by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi_\sharp(\alpha)}\beta - \mathcal{L}_{\pi_\sharp(\beta)}\alpha - d\big(\pi(\alpha, \beta)\big).$$

The symplectic foliation of (M, π) will be denoted by $\mathcal{S}$, and $T\mathcal{S} = \operatorname{Im} \pi_\sharp$ will be its associated tangent distribution. Finally, we will denote by M^{reg} the dense open set where the rank of π is locally constant.

2. Preliminaries

2.1. Contravariant connections

Contravariant connections on Poisson manifolds were defined by Vaismann [7] and studied in detail by Fernandes [8]. These connections play an important role in Poisson geometry (see for instance [8,9]) and have recently turned out to be useful in other branches of mathematics (e.g., [1,2]).

The definition of a contravariant connection mimics the usual definition of a covariant connection, except that cotangent vectors have taken the place of tangent vectors. More precisely, a *contravariant connection* on a Poisson manifold (M, π) is an $\mathbb{R}$-bilinear map

$$\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M), \quad (\alpha, \beta) \mapsto \mathcal{D}_\alpha \beta$$

such that for any $f \in \mathcal{C}^\infty(M)$,

$$\mathcal{D}_{f\alpha}\beta = f\, \mathcal{D}_\alpha \beta \quad \text{and} \quad \mathcal{D}_\alpha(f\beta) = f\, \mathcal{D}_\alpha\beta + \pi_\sharp(\alpha)(f)\beta.$$

A contravariant connection $\mathcal{D}$ is called an $\mathcal{F}$*-connection* [8] if it satisfies

$$(\forall\, a \in T^*M,\ \pi_\sharp(a) = 0) \Longrightarrow \mathcal{D}_a = 0.$$

We call $\mathcal{D}$ an $\mathcal{F}^{\mathrm{reg}}$*-connection* if the restriction of $\mathcal{D}$ to M^{reg} is an $\mathcal{F}$-connection.

The *torsion* and the *curvature* of a contravariant connection $\mathcal{D}$ are formally identical to the usual ones:

$$T(\alpha, \beta) = \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha - [\alpha, \beta]_\pi,$$
$$R(\alpha, \beta)\gamma = \mathcal{D}_\alpha\mathcal{D}_\beta\gamma - \mathcal{D}_\beta\mathcal{D}_\alpha\gamma - \mathcal{D}_{[\alpha,\beta]_\pi}\gamma.$$

These are (2, 1) and (3, 1)-type tensor fields, respectively. When $T \equiv 0$ (resp. $R \equiv 0$), $\mathcal{D}$ is called *torsion-free* (resp. *flat*).

In local coordinates $(x^1, \dots, x^d)$, the local components of the torsion and curvature tensor fields are given by

$$T_k^{ij} = \Gamma_k^{ij} - \Gamma_k^{ji} - \frac{\partial \pi^{ij}}{\partial x^k}, \tag{3}$$

$$R_l^{ijk} = \sum_{m=1}^{d} \Gamma_l^{im}\Gamma_m^{jk} - \Gamma_l^{jm}\Gamma_m^{ik} + \pi^{im}\frac{\partial \Gamma_l^{jk}}{\partial x^m} - \pi^{jm}\frac{\partial \Gamma_l^{ik}}{\partial x^m} - \frac{\partial \pi^{ij}}{\partial x^m}\Gamma_l^{mk}, \tag{4}$$

where Γ_k^{ij} are the *Christoffel symbols* defined by $\mathcal{D}_{dx^i}dx^j = \sum_{k=1}^d \Gamma_k^{ij}\, dx^k$ and π^{ij} are the components of π.

Given a (pseudo-)Riemannian metric g on a Poisson manifold (M, π), one has a contravariant version of the Levi-Civita connection: there exists a unique torsion-free contravariant connection $\mathcal{D}$ on M which is metric-compatible, i.e.,

$$\pi_\#(\alpha) \cdot \langle \beta, \gamma \rangle = \langle \mathcal{D}_\alpha\beta, \gamma \rangle + \langle \beta, \mathcal{D}_\alpha\gamma \rangle \quad \forall\, \alpha, \beta, \gamma \in \Omega^1(M),$$

where $\langle , \rangle$ denotes the metric pairing induced by g. This connection is determined by the formula

$$\langle \mathcal{D}_\alpha\beta, \gamma \rangle = \frac{1}{2}\Big\{\pi_\sharp(\alpha) \cdot \langle \beta, \gamma \rangle + \pi_\sharp(\beta) \cdot \langle \alpha, \gamma \rangle - \pi_\sharp(\gamma) \cdot \langle \alpha, \beta \rangle + \langle [\alpha, \beta]_\pi, \gamma \rangle - \langle [\beta, \gamma]_\pi, \alpha \rangle + \langle [\gamma, \alpha]_\pi, \beta \rangle\Big\}, \tag{5}$$

and is called the *metric contravariant connection* (or *contravariant Levi-Civita connection*) associated to (π, g).

2.2. The metacurvature

In this subsection we recall briefly from [2] the definition of the metacurvature tensor and give some related formulas.

Let (M, π) be a Poisson manifold. Given a torsion-free contravariant connection $\mathcal{D}$ on M, there exists a unique bracket $\{,\}$ on the space $\Omega^*(M)$ of differential forms, with the following properties:

1. $\{,\}$ is bilinear, degree 0 and antisymmetric

$$\{\sigma, \tau\} = -(-1)^{\deg(\sigma)\deg(\tau)}\{\tau, \sigma\}. \tag{6}$$

2. $\{,\}$ satisfies the product rule

$$\{\sigma, \tau \wedge \rho\} = \{\sigma, \tau\} \wedge \rho + (-1)^{\deg(\sigma)\deg(\tau)}\tau \wedge \{\sigma, \rho\}. \tag{7}$$

3. The exterior differential d is a derivation with respect to $\{,\}$, i.e.,

$$d\{\sigma, \tau\} = \{d\sigma, \tau\} + (-1)^{\deg(\sigma)}\{\sigma, d\tau\}. \tag{8}$$

4. For any $f, g \in \mathcal{C}^\infty(M)$ and any $\sigma \in \mathcal{C}^\infty(M)$,

$$\{f, g\} = \pi(df, dg) \quad \text{and} \quad \{f, \sigma\} = \mathcal{D}_{df}\sigma. \tag{9}$$

This bracket is given (on decomposable forms) by

$$\{\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_l\}$$
$$= (-1)^{k+1}\sum_{i,j}(-1)^{i+j}\{\alpha_i, \beta_j\} \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \widehat{\beta_j} \wedge \cdots \wedge \beta_l, \tag{10}$$

where the hat $\widehat{\ }$ denotes the absence of the corresponding factor, and the brackets $\{\alpha_i, \beta_j\}$ are given by the formula[1]

$$\{\alpha, \beta\} = -\mathcal{D}_\alpha d\beta - \mathcal{D}_\beta d\alpha + d\mathcal{D}_\beta \alpha + [\alpha, d\beta]_\pi. \tag{11}$$

We call the bracket $\{,\}$ *Hawkins bracket*.

Hawkins showed that $\{,\}$ satisfies the graded Jacobi identity,

$$\{\sigma, \{\tau, \rho\}\} - \{\{\sigma, \tau\}, \rho\} - (-1)^{\deg(\sigma)\deg(\tau)}\{\tau, \{\sigma, \rho\}\} = 0, \tag{12}$$

if and only if $\mathcal{D}$ is flat and a certain 5-index tensor, called the metacurvature of $\mathcal{D}$, vanishes identically. In fact, Hawkins showed that if $\mathcal{D}$ is flat, then it determines a $(2, 3)$-type tensor field $\mathcal{M}$ symmetric in the contravariant indices and antisymmetric in the covariant indices, given by

$$\mathcal{M}(df, \alpha, \beta) = \{f, \{\alpha, \beta\}\} - \{\{f, \alpha\}, \beta\} - \{\alpha, \{f, \beta\}\}. \tag{13}$$

The tensor $\mathcal{M}$ is the *metacurvature* of $\mathcal{D}$.

The following formulas, due to Hawkins, will be useful later. Let α be a parallel 1-form; since $\mathcal{D}$ is torsion-free, $[\alpha, \eta]_\pi = \mathcal{D}_\alpha \eta$ for any $\eta \in \Omega^*(M)$, and so, by (11), the Hawkins bracket of α and any 1-form β is given by

$$\{\alpha, \beta\} = -\mathcal{D}_\beta d\alpha. \tag{14}$$

Using this, one can deduce easily from (13) that for any parallel 1-forms α, β and any 1-form γ,

$$\mathcal{M}(\gamma, \beta, \alpha) = -\mathcal{D}_\gamma \mathcal{D}_\beta d\alpha. \tag{15}$$

2.3. The tensor **T**

We now define the tensor **T**, an essential ingredient in our main result.

Let (M, π) be a Poisson manifold endowed with a flat, torsion-free, contravariant $\mathcal{F}^{\text{reg}}$-connection $\mathcal{D}$. For each $x \in M^{\text{reg}}$ and any $a, b \in T_x^*M$, define

$$\mathbf{T}_x(a, b) := \{\alpha, \beta\}(x) \quad \left(\in \bigwedge^2 T_x^*M\right), \tag{16}$$

where $\{,\}$ denotes the Hawkins bracket associated to $\mathcal{D}$, and α and β are parallel 1-forms defined in a neighborhood of x such that $\alpha(x) = a$ and $\beta(x) = b$. (Such 1-forms exist, see Proposition 3.4.) This is independent of the choice of α and β since by (14) and (6) we have

$$\mathbf{T}_x(a, b) = -(\mathcal{D}_\alpha d\beta)(x) = -(\mathcal{D}_\beta d\alpha)(x). \tag{17}$$

The assignment $x \mapsto \mathbf{T}_x$ is then a smooth $(2, 2)$-type tensor field on M^{reg}, symmetric in the contravariant indices and antisymmetric in the covariant indices, which by (15) verifies $\mathcal{D}\mathbf{T} = \mathcal{M}$, and which clearly vanishes if and only if the exterior differential of any parallel 1-form is also parallel.

3. Computation of the tensors $\mathcal{M}$ and T

The metacurvature tensor is rather difficult to compute in general. In the symplectic case, Hawkins has established a simple formula for the metacurvature [2, Theorem 2.4]. Bahayou and the second author have also established in [10] a formula for the metacurvature in the case of a Lie–Poisson group endowed with a left-invariant Riemannian metric. In this section we explain how to compute the metacurvature (and the tensor **T** as well) in the case of an $\mathcal{F}^{\text{reg}}$-connection, generalizing thus Hawkins's formula.

Throughout this section, $\mathcal{D}$ will be a torsion-free contravariant connection on a d-dimensional Poisson manifold (M, π).

We begin with the following simple lemma.

Lemma 3.1. *Let $U \subseteq M$ be an open set on which the rank of π is constant. Assume that $\mathcal{D}$ is an $\mathcal{F}$-connection on U. Then, for any $\alpha, \beta \in \Omega^1(U)$, $\pi_\sharp(\beta) = 0$ implies $\pi_\sharp(\mathcal{D}_\alpha\beta) = 0$, and in this case, $\mathcal{D}_\alpha\beta = \mathcal{L}_{\pi_\sharp(\alpha)}\beta$.*

In other words, the kernel of the anchor map restricted to U is stable under $\mathcal{D}$. The next lemma shows that, around any regular point, there exists a complementary subbundle of $\operatorname{Ker}\pi_\sharp$ which is also stable under $\mathcal{D}$, provided that $\mathcal{D}$ is flat.

[1] This formula appeared first in [10].

Lemma 3.2. *If $\mathcal{D}$ is flat and is an $\mathcal{F}^{\mathrm{reg}}$-connection, then for any $x \in M^{\mathrm{reg}}$ and any $\mathcal{H}_0 \subseteq T_x^*M$ such that $T_x^*M = (\operatorname{Ker}\pi_\sharp)_x \oplus \mathcal{H}_0$, the cotangent bundle splits smoothly around x into:*

$$T^*M = (\operatorname{Ker}\pi_\sharp) \oplus \mathcal{H}$$

with $\mathcal{H}$ stable under $\mathcal{D}$, i.e. $\mathcal{D}\mathcal{H} \subseteq \mathcal{H}$, and $\mathcal{H}_x = \mathcal{H}_0$.

Proof. Let $(U; x^i, y^u)$ $(i = 1, \dots, 2r;\ u = 1, \dots, d - 2r)$ be a local chart around x such that

$$\pi = \frac{1}{2}\sum_{i,j=1}^{2r} \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

and the matrix $(\pi^{ij})_{1\le i,j\le 2r}$ is constant and invertible; let $(\bar{\pi}_{ij})_{1\le i,j\le 2r}$ denote the inverse matrix. The restriction of $\operatorname{Ker}\pi_\sharp$ to U is a (rank $d - 2r$) subbundle of $T^*_{|U}M$, so we can choose a (arbitrary) smooth decomposition

$$T^*_{|U}M = (\operatorname{Ker}\pi_\sharp) \oplus \mathcal{H}.$$

Then clearly $\operatorname{Ker}\pi_\sharp = \operatorname{span}\{dy^u\}$, and

$$\mathcal{H} = \operatorname{span}\left\{\theta^i = dx^i + \sum_{u=1}^{d-2r} B_u^i\, dy^u\right\}$$

for some functions $B_u^i \in \mathcal{C}^\infty(U)$. Since $\mathcal{D}$ is a torsion-free $\mathcal{F}$-connection on U, one has $\mathcal{D}_{dy^u} = \mathcal{D}dy^u = 0$ for all u. Thus, for any i, j,

$$\begin{aligned}\mathcal{D}_{\theta^i}\theta^j &= \mathcal{D}_{dx^i}dx^j + \sum_{u=1}^{d-2r} \pi_\sharp(dx^i)(B_u^j)\, dy^u \\ &= \left(\sum_{k=1}^{2r} \Gamma_k^{ij}\, dx^k + \sum_{u=1}^{d-2r} \Gamma_u^{ij}\, dy^u\right) + \sum_{u=1}^{d-2r}\sum_{k=1}^{2r} \pi^{ik}\frac{\partial B_u^j}{\partial x^k}\, dy^u \\ &= \sum_{k=1}^{2r} \Gamma_k^{ij}\,\theta^k + \sum_{u=1}^{d-2r}\left(\Gamma_u^{ij} + \sum_{k=1}^{2r}\left(\pi^{ik}\frac{\partial B_u^j}{\partial x^k} - \Gamma_k^{ij}B_u^k\right)\right) dy^u,\end{aligned}$$

where $\Gamma_k^{ij}, \Gamma_u^{ij}$ are the Christoffel symbols of $\mathcal{D}$. Therefore, the desired decomposition exists if and only if we may find a family of local functions $\{B_u^i\}_{i,u}$ satisfying the following system of PDEs

$$\Gamma_u^{ij} + \sum_{k=1}^{2r}\left(\pi^{ik}\frac{\partial B_u^j}{\partial x^k} - \Gamma_k^{ij}B_u^k\right) = 0 \quad \forall\, i, j,\ \forall\, u,$$

or equivalently

$$\frac{\partial B_u^j}{\partial x^i} = \sum_{k=1}^{2r}\left(\sum_{l=1}^{2r}\bar{\pi}_{il}\Gamma_k^{lj}\right)B_u^k - \sum_{l=1}^{2r}\bar{\pi}_{il}\Gamma_u^{lj} \quad \forall\, i, j,\ \forall\, u. \tag{$*$}$$

In matrix notation, this is

$$\frac{\partial}{\partial x^i}B_u = \Gamma_i B_u + Y_i^u,$$

where

$$B_u = \begin{pmatrix} B_u^1 \\ \vdots \\ \vdots \\ B_u^{2r}\end{pmatrix}; \qquad \Gamma_i = \left(\sum_{m=1}^{2r}\bar{\pi}_{im}\Gamma_l^{mk}\right)_{1\le k,\, l\le 2r}; \qquad Y_i^u = -\sum_{j=1}^{2r}\bar{\pi}_{ij}\begin{pmatrix}\Gamma_u^{j1} \\ \vdots \\ \vdots \\ \Gamma_u^{j\,2r}\end{pmatrix}.$$

Considering the B_u^i's as functions with variables x^i and parameters y^u, the system above can be solved, according to Frobenius's Theorem (see, e.g., [11, Theorem 1.1]), if and only if the integrability conditions

$$\Gamma_i\Gamma_j + \frac{\partial}{\partial x^j}\Gamma_i = \Gamma_j\Gamma_i + \frac{\partial}{\partial x^i}\Gamma_j, \qquad \Gamma_i Y_j^u + \frac{\partial}{\partial x^j}Y_i^u = \Gamma_j Y_i^u + \frac{\partial}{\partial x^i}Y_j^u,$$

hold for all i, j and all u. With indices, these are respectively

$$\sum_{m=1}^{2r} \Gamma_l^{im}\Gamma_m^{jk} - \Gamma_l^{jm}\Gamma_m^{ik} + \pi^{im}\frac{\partial \Gamma_l^{jk}}{\partial x^m} - \pi^{jm}\frac{\partial \Gamma_l^{ik}}{\partial x^m} = 0,$$
$$\sum_{m=1}^{2r} \Gamma_u^{im}\Gamma_m^{jk} - \Gamma_u^{jm}\Gamma_m^{ik} + \pi^{im}\frac{\partial \Gamma_u^{jk}}{\partial x^m} - \pi^{jm}\frac{\partial \Gamma_u^{ik}}{\partial x^m} = 0,$$

which by (4) mean that the curvature vanishes. Thus $(*)$ has solutions (which depend smoothly on the parameters and the initial values). □

Notation 3.3. Given $\mathcal{H}$ as above, the restriction of $\pi_\sharp$ to $\mathcal{H}$ defines an isomorphism from $\mathcal{H}$ onto $T\mathcal{S}$; we will denote by $\varpi^{\mathcal{H}} : T\mathcal{S} \to \mathcal{H}$ its inverse.

Proposition 3.4. *The following are equivalent:*

(a) *$\mathcal{D}$ is flat and is an $\mathcal{F}^{\mathrm{reg}}$-connection.*
(b) *For any $x \in M^{\mathrm{reg}}$ and any $a \in T_x^*M$, there exists a 1-form α defined in a neighborhood of x such that $\alpha(x) = a$ and $\mathcal{D}\alpha = 0$.*
(c) *Around any $x \in M^{\mathrm{reg}}$, there exists a smooth coframe $(\alpha^1, \ldots, \alpha^d)$ of M such that $\mathcal{D}\alpha^i = 0$ for all i. Such a coframe will be called flat.*

Proof. The equivalence (b) $\Longleftrightarrow$ (c) is obvious.

(a) $\Longrightarrow$ (b): Let $U \subseteq M$ be an open neighborhood of x on which the rank of π is constant. Over U, $T\mathcal{S}$ is a (involutive) regular distribution and $\mathcal{D}$ is a torsion-free $\mathcal{F}$-connection. So we can define a partial connection ∇ on $T_{|U}\mathcal{S}$ by setting for any $\alpha, \beta \in \Omega^1(U)$,

$$\nabla_{\pi_\sharp(\alpha)}\pi_\sharp(\beta) = \pi_\sharp(\mathcal{D}_\alpha\beta). \tag{18}$$

One verifies immediately that the curvature tensor fields R^∇ and $R^{\mathcal{D}}$ respectively of ∇ and $\mathcal{D}$ are related by:

$$R^\nabla\big(\pi_\sharp(a), \pi_\sharp(b)\big)\pi_\sharp(c) = \pi_\sharp\left(R^{\mathcal{D}}(a, b)c\right) \quad \forall\, a, b, c \in T_{|U}^*M,$$

and hence R^∇ vanishes since by hypothesis $R^{\mathcal{D}}$ does. Using Frobenius's Theorem, we can then show in a way similar to the classical case that, for any $v \in T_x\mathcal{S}$, there exists a vector field X defined on some neighborhood of x such that $X(x) = v$, X is tangent to $T\mathcal{S}$, that is, $X(y) \in T_y\mathcal{S}$ for any y near x, and $\nabla X = 0$.

Now let $a \in T_x^*M$. According to Lemma 3.2, the cotangent bundle splits smoothly around x into: $T^*M = (\operatorname{Ker}\pi_\sharp) \oplus \mathcal{H}$ with $\mathcal{H}$ stable under $\mathcal{D}$. Write $a = b + c$ with $b \in \operatorname{Ker}\pi_\sharp(x)$ and $c \in \mathcal{H}_x$. By the argument above, there exists a ∇-parallel vector field X defined in a neighborhood of x which is tangent to $T\mathcal{S}$ and such that $X(x) = \pi_\sharp(c)$. Put $\gamma = \varpi^{\mathcal{H}}(X) \in \Gamma(\mathcal{H})$; then $\gamma(x) = c$, and for any 1-form ϕ, $\pi_\sharp(\mathcal{D}_\phi\gamma) = \nabla_{\pi_\sharp(\phi)}X = 0$ implying that $\mathcal{D}\gamma = 0$. Taking $\alpha = \sum_{u=1}^s b_u\, dy^u + \gamma$, where (y^u) is a family of local functions on M such that $\operatorname{Ker}\pi_\sharp = \operatorname{span}\{dy^1, \ldots, dy^s\}$ near x, and b_u are the coordinates of b in $\{d_xy^1, \ldots, d_xy^s\}$, we obtain finally the desired 1-form.

(c) $\Longrightarrow$ (a): It is clear that if (c) holds, then $\mathcal{D}$ is flat. So we need only to show that $\mathcal{D}$ is an $\mathcal{F}^{\mathrm{reg}}$-connection. Let $x \in M^{\mathrm{reg}}$ be arbitrary, and let $(\alpha^1, \ldots, \alpha^d)$ be a flat coframe around x. For any $a \in \operatorname{Ker}\pi_\sharp(x)$ and any 1-form $\beta = \sum_i f_i\,\alpha^i$, we have $\mathcal{D}_a\beta = \sum_i \pi_\sharp(a)(f_i)\,\alpha^i + f_i\,\mathcal{D}_a\alpha^i = 0$. □

The following corollary is a refinement of the preceding proposition.

Corollary 3.5. *If $\mathcal{D}$ is flat and is an $\mathcal{F}^{\mathrm{reg}}$-connection, then around any $x \in M^{\mathrm{reg}}$ there exists an $\mathcal{S}$-foliated coordinate system with leafwise coordinates $\{x^i\}_{i=1}^{2r}$ and transverse coordinates $\{y^u\}_{u=1}^{d-2r}$ such that for any $\mathcal{H}$ as in Lemma 3.2,*

$$\mathbf{F}^* = \left(\phi_i := \varpi^{\mathcal{H}}(\partial/\partial x^i);\ dy^u\right)$$

is a flat coframe of M near x. Such a coordinate system will be called flat.

Remark 3.6. Another equivalent way of expressing that the $\mathcal{S}$-foliated coordinate system (x^i, y^u) is flat is the following: $\nabla\partial/\partial x^i = 0$ for all i, where ∇ is the (local) partial connection defined by (18).

We assume for the remainder of this section that $\mathcal{D}$ is flat and is an $\mathcal{F}^{\mathrm{reg}}$-connection.

We shall compute the tensors $\mathcal{M}$ and $\mathbf{T}$ in the coframe $\mathbf{F}^*$. To do so, we need first to determine its dual frame. With the notations of Corollary 3.5, for each i, there exist unique functions, $A_1^i, \ldots, A_{d-2r}^i$, defined in a neighborhood of x such that

$$dx^i + \sum_{u=1}^{d-2r} A_u^i\, dy^u \in \mathcal{H}. \tag{19}$$

For any i and any u we put

$$X_i := -H_{x^i} = -\pi_\sharp(dx^i), \qquad Y_u := \frac{\partial}{\partial y^u} - \sum_{i=1}^{2r} A_u^i \frac{\partial}{\partial x^i}. \tag{20}$$

Lemma 3.7. *With the above notations, (X_i, Y_u) is the dual frame to $\mathbf{F}^*$. Moreover, the vector fields X_i and Y_u are, respectively, Hamiltonian and Poisson, and verify*

$$\begin{aligned}
&[X_i, X_j] = -\sum_{k=1}^{2r} \frac{\partial \pi^{ij}}{\partial x^k} X_k; \qquad [X_i, Y_u] = \sum_{j=1}^{2r} \frac{\partial A_u^i}{\partial x^j} X_j;\\
&[Y_u, Y_v] = \sum_{i,j=1}^{2r} \bar{\pi}_{ij}\left(\frac{\partial A_u^j}{\partial y^v} - \frac{\partial A_v^j}{\partial y^u} + \sum_{k=1}^{2r} A_u^k \frac{\partial A_v^j}{\partial x^k} - A_v^k \frac{\partial A_u^j}{\partial x^k}\right) X_i.
\end{aligned} \tag{21}$$

Here, $\pi^{ij} := \pi(dx^i, dx^j)$ and $(\bar{\pi}_{ij})$ is the inverse matrix of (π^{ij}).

Proof. The fact that (X_i, Y_u) is the dual frame to $\mathbf{F}^*$ follows immediately once we note that

$$\phi_i := \varpi^{\mathcal{H}}(\partial/\partial x^i) = \sum_{j=1}^{2r} \bar{\pi}_{ij}\Big(dx^j + \sum_{u=1}^{d-2r} A_u^j \, dy^u\Big). \tag{22}$$

By definition, each of the vector fields X_i is Hamiltonian. To see that each Y_u is Poisson, observe that $[\phi_i, \phi_j]_\pi = \mathcal{D}_{\phi_i}\phi_j - \mathcal{D}_{\phi_j}\phi_i = 0$ which yields

$$\begin{aligned}
Y_u \cdot \pi(\phi_i, \phi_j) &= \mathcal{L}_{\partial/\partial x^i}\phi_j\,(Y_u) - \mathcal{L}_{\partial/\partial x^j}\phi_i\,(Y_u)\\
&= -\phi_j\left(\left[\frac{\partial}{\partial x^i}, Y_u\right]\right) + \phi_i\left(\left[\frac{\partial}{\partial x^j}, Y_u\right]\right)\\
&= -\mathcal{L}_{Y_u}\phi_j\left(\frac{\partial}{\partial x^i}\right) + Y_u \cdot \pi(\phi_i, \phi_j) + \mathcal{L}_{Y_u}\phi_i\left(\frac{\partial}{\partial x^j}\right) - Y_u \cdot \pi(\phi_j, \phi_i)\\
&= -\pi(\phi_i, \mathcal{L}_{Y_u}\phi_j) - \pi(\mathcal{L}_{Y_u}\phi_i, \phi_j) + 2Y_u \cdot \pi(\phi_i, \phi_j),
\end{aligned}$$

hence $\mathcal{L}_{Y_u}\pi\ (\phi_i, \phi_j) = 0$; in addition, we have

$$\mathcal{L}_{Y_u}\pi\ (\phi_i, dy^v) = -\pi(\phi_i, \mathcal{L}_{Y_u} dy^v) = -\pi(\phi_i, d(Y_u(y^v))) = 0,$$

and it is clear that we also have $\mathcal{L}_{Y_u}\pi\ (dy^v, dy^w) = 0$. It follows that $\mathcal{L}_{Y_u}\pi = 0$, which means that Y_u is Poisson. Finally,

$$[X_i, X_j] = H_{\pi(dx^i, dx^j)} = -\sum_{k=1}^{2r} \frac{\partial \pi^{ij}}{\partial x^k} X_k, \qquad [X_i, Y_u] = H_{Y_u(x^i)} = \sum_{j=1}^{2r} \frac{\partial A_u^i}{\partial x^j} X_j,$$

and the last equality of (21) follows by direct computation. □

We now can give the expression of the metacurvature in the coframe $\mathbf{F}^*$.

Theorem 3.8. *With the same notations as above, we have*

(a) *For any $u = 1, \ldots, d-2r$, $\mathcal{M}(dy^u, \cdot, \cdot) = 0$.*
(b) *For any $i, j, k = 1, \ldots, 2r$,*

$$\begin{aligned}
\mathcal{M}(\phi_i, \phi_j, \phi_k) = &-\sum_{l<m} \frac{\partial^3 \pi^{lm}}{\partial x^i \partial x^j \partial x^k} \phi_l \wedge \phi_m + \sum_{l,u} \frac{\partial^3 A_u^l}{\partial x^i \partial x^j \partial x^k} \phi_l \wedge dy^u\\
&+ \sum_{u<v,\, l} \frac{\partial^2}{\partial x^i \partial x^j}\left(\bar{\pi}_{kl}\left(\frac{\partial A_u^l}{\partial y^v} - \frac{\partial A_v^l}{\partial y^u} + \sum_m A_u^m \frac{\partial A_v^l}{\partial x^m} - A_v^m \frac{\partial A_u^l}{\partial x^m}\right)\right) dy^u \wedge dy^v.
\end{aligned} \tag{23}$$

Proof. Part (a) is immediate from (13) and (9).

For (b), on the one hand, we have by (15),

$$\mathcal{M}(\phi_i, \phi_j, \phi_k) = -\mathcal{D}_{\phi_i}\mathcal{D}_{\phi_j} d\phi_k \quad \text{for all } i, j, k.$$

On the other hand, using Lemma 3.7 gives

$$d\phi_i = \sum_{j<k} \frac{\partial \pi^{jk}}{\partial x^i} \phi_j \wedge \phi_k - \sum_{j,u} \frac{\partial A_u^j}{\partial x^i} \phi_j \wedge dy^u - \sum_{u<v,j} \bar{\pi}_{ij} \left(\frac{\partial A_u^j}{\partial y^v} - \frac{\partial A_v^j}{\partial y^u} + \sum_k A_u^k \frac{\partial A_v^j}{\partial x^k} - A_v^k \frac{\partial A_u^j}{\partial x^k} \right) dy^u \wedge dy^v, \quad (24)$$

and the desired formula follows. □

Likewise, we get the following expression for the tensor **T**.

Theorem 3.9. (a) *For any* $u = 1, \ldots, d-2r$, $\mathbf{T}(dy^u, \cdot) = 0$.
(b) *For any* $i, j, k = 1, \ldots, 2r$,

$$\mathbf{T}(\phi_i, \phi_j) = -\sum_{k<l} \frac{\partial^2 \pi^{kl}}{\partial x^i \partial x^j} \phi_k \wedge \phi_l + \sum_{k,u} \frac{\partial^2 A_u^k}{\partial x^i \partial x^j} \phi_k \wedge dy^u + \sum_{u<v,k} \frac{\partial}{\partial x^i} \left(\bar{\pi}_{jk} \left(\frac{\partial A_u^k}{\partial y^v} - \frac{\partial A_v^k}{\partial y^u} + \sum_l A_u^l \frac{\partial A_v^k}{\partial x^l} - A_v^l \frac{\partial A_u^k}{\partial x^l} \right) \right) dy^u \wedge dy^v. \quad (25)$$

3.1. The symplectic case

If the Poisson tensor π is invertible, i.e., $\pi = \omega^{-1}$ where ω is a symplectic 2-form, then the flat and torsion-free contravariant connection $\mathcal{D}$ (which is, in this case, an $\mathcal{F}$-connection since the kernel of the anchor map reduces to zero) is related to a flat, torsion-free, covariant connection ∇ on M via $\pi_\sharp(\mathcal{D}_\alpha \beta) = \nabla_{\pi_\sharp(\alpha)} \pi_\sharp(\beta)$. In that case, a flat coordinate system is one with respect to whom ∇ is given trivially by partial derivatives (Remark 3.6).

Corollary 3.10. *If* $\pi = \omega^{-1}$*, then the components of* $\mathcal{M}$ *and* **T** *w.r.t. any flat coordinate system* $(x^1, \ldots, x^d)$ *are given respectively by*

$$\mathcal{M}_{lm}^{ijk} = -\sum_{a,b,c,d,e} \pi^{ai} \pi^{bj} \pi^{ck} \omega_{dl} \omega_{em} \frac{\partial^3 \pi^{de}}{\partial x^a \partial x^b \partial x^c}, \quad (26)$$

$$\mathbf{T}_{kl}^{ij} = -\sum_{a,b,c,d} \pi^{ai} \pi^{bj} \omega_{ck} \omega_{dl} \frac{\partial^2 \pi^{cd}}{\partial x^a \partial x^b}. \quad (27)$$

Remark 3.11. Formula (26) has already been established by Hawkins, see [2, Theorem 2.4].

Proof. Since the kernel of $\pi_\sharp$ reduces to zero, then by Theorems 3.8 and 3.9

$$\mathcal{M}(\phi_i, \phi_j, \phi_k) = -\sum_{l<m} \frac{\partial^3 \pi^{lm}}{\partial x^i \partial x^j \partial x^k} \phi_l \wedge \phi_m$$

and

$$\mathbf{T}(\phi_i, \phi_j) = -\sum_{k<l} \frac{\partial^2 \pi^{kl}}{\partial x^i \partial x^j} \phi_k \wedge \phi_l$$

and by (22) we have $\phi_i = \sum_j \bar{\pi}_{ij} dx^j$, and the desired formulas follow. □

This means that for a symplectic manifold, $\mathcal{M}$ (resp. **T**) vanishes if and only if π is polynomial of degree at most 2 (resp. 1) in the affine structure defined by ∇.

Example 3.12. Let (M, ω) be a symplectic manifold. If $\mathcal{D}$ is a flat, torsion-free, Poisson connection on M w.r.t. $\pi = \omega^{-1}$, then **T** vanishes identically (and hence so does $\mathcal{M}$). In fact, the condition $\mathcal{D}\pi = 0$ is equivalent to saying that the components of π w.r.t. any flat coordinate system are constant.

Example 3.13. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $r \in \wedge^2 \mathfrak{g}$ be a solution of the classical Yang–Baxter equation. For any tensor τ on $\mathfrak{g}$, denote by τ^+ the corresponding left-invariant tensor field on G. Following [4, p. 71], the formula

$$\mathcal{D}_{a^+}^r b^+ = -(\mathrm{ad}_{r(a)}^* b)^+,$$

where $a, b \in \mathfrak{g}^*$, defines a left-invariant, flat, torsion-free contravariant connection $\mathcal{D}^r$ on (G, r^+), which is an $\mathcal{F}$-connection with vanishing $\mathbf{T}$ by (c′) from the introduction. It is well known (see, e.g., [12]) that if r is invertible, then the left-invariant symplectic form ω^+ inverse of r^+ defines a left-invariant, flat, torsion-free (covariant) connection ∇ on G via

$$\omega^+(\nabla_{u^+}v^+, w^+) = -\omega^+(v^+, [u^+, w^+]), \quad u, v, w \in \mathfrak{g}.$$

In that case, it is easily seen that $\mathcal{D}^r$ and ∇ are related by: $r^+_\sharp(\mathcal{D}^r_{a^+}b^+) = \nabla_{r(a)^+}r(b)^+$ where $r^+_\sharp$ is the anchor map associated to r^+. Accordingly, r^+ is polynomial of degree at most one with respect to the affine structure defined by ∇ since $\mathbf{T} = 0$ and r is invertible, recovering thus a result of the second author and Medina (cf. [13, Theorem 1.1-(1)]).

3.2. The Riemannian case

Let $\mathcal{D}$ be the metric contravariant connection associated to a Poisson tensor π and a Riemannian metric g on a smooth manifold M. Thanks to the metric g, the cotangent bundle splits orthogonally into

$$T^*M = \operatorname{Ker}\pi_\sharp \oplus (\operatorname{Ker}\pi_\sharp)^\perp.$$

Lemma 3.14. *Let $U \subseteq M$ be an open set on which the rank of π is constant. Assume that $\mathcal{D}$ is an $\mathcal{F}$-connection on U. Then $(\operatorname{Ker}\pi_{\sharp|_U})^\perp$ is stable under $\mathcal{D}$.*

Thus if $\mathcal{D}$ is flat and is an $\mathcal{F}^{\mathrm{reg}}$-connection, then by Corollary 3.5 there exists around any $x \in M^{\mathrm{reg}}$ an $\mathcal{S}$-foliated chart with leafwise coordinates $\{x^i\}_{i=1}^{2r}$ and transverse coordinates $\{y^u\}_{u=1}^{d-2r}$ such that $\{\phi_i = \varpi^\perp(\partial/\partial x^i);\ dy^u\}$ is a flat coframe of M near x, where we have denoted by $\varpi^\perp : T\mathcal{S} \to (\operatorname{Ker}\pi_\sharp)^\perp$ the inverse of $\pi_\sharp : (\operatorname{Ker}\pi_\sharp)^\perp \to T\mathcal{S}$. In this case, the functions A^i_u defined by (19) can be computed by means of the metric; indeed, using (22) and the fact that $\langle\phi_i, dy^u\rangle = 0$, one has $-A^i_u = \sum_v g^{iv}g_{uv}$ where $g^{iv} = \langle dx^i, dy^v\rangle$ and (g_{uv}) is the inverse matrix of the one whose coefficients are $g^{uv} = \langle dy^u, dy^v\rangle$.

4. Proof of Theorem 1.1

Let (x^i, y^u), with $i = 1, \ldots, 2r$ and $u = 1, \ldots, d-2r$, be a flat coordinate system around x_0, choose $\mathcal{H}$ as in Lemma 3.2, and let $\mathbf{F}^* = \{\phi_i, dy^u\}$ be the corresponding flat coframe and $\{X_i, Y_u\}$ its dual frame. We shall construct a family of vector fields $\{Z_1, \ldots, Z_{2r}\}$ on a neighborhood U of x_0 which span $T\mathcal{S}$ and commute with the X_i's and the Y_u's. In that case,

- The family $\{Z_1, \ldots, Z_{2r}\}$ will form a $2r$-dimensional real Lie algebra $\mathfrak{g}$, since by the Jacobi identity

  $$[[Z_i, Z_j], X_l] = [[Z_i, Z_j], Y_u] = 0 \quad \forall\, i, j, l,\ \forall\, u,$$

  so that $[Z_i, Z_j] = \sum_k c^k_{ij} Z_k$ with c^k_{ij} being constant; it is then clear that $\mathfrak{g}$ acts freely on U.
- The Poisson tensor π will be expressed as

  $$\pi = \frac{1}{2}\sum_{i,j} a^{ij} Z_i \wedge Z_j$$

  where the matrix $(a^{ij})_{1\le i,j\le 2r}$ is constant and invertible: since the X_i's and the Y_u's are Poisson (Lemma 3.7), then writing $\pi = \frac{1}{2}\sum_{i,j} a^{ij} Z_i \wedge Z_j$ where $a^{ij} \in \mathcal{C}^\infty(U)$, we get $X_k(a^{ij}) = Y_u(a^{ij}) = 0$.
- The connection $\mathcal{D}$ will be given on U by

  $$\mathcal{D}_\alpha\beta = \sum_{i,j} a^{ij}\alpha(Z_i)\,\mathcal{L}_{Z_j}\beta.$$

  In fact, this is true for any $\beta \in \mathbf{F}^*$ since $\mathcal{L}_{Z_i}\phi_j = \mathcal{L}_{Z_i}dy^u = 0$, and $\mathcal{D}_\alpha\beta - \sum_{i,j} a^{ij}\alpha(Z_i)\,\mathcal{L}_{Z_j}\beta$ is tensorial in β as $\pi_\sharp(\alpha) = \sum_{i,j} a^{ij}\alpha(Z_i)Z_j$.

We shall proceed in two steps. We first construct a family of vector fields which span $T\mathcal{S}$ and commute with the X_i's, and then construct from this the desired family.

To start, observe that by virtue of Theorem 3.9 and Lemma 3.7 we have

$$[X_i, X_j] = \sum_{k=1}^{2r}\lambda^k_{ij}X_k, \qquad [X_i, Y_u] = \sum_{j=1}^{2r}\mu^j_{iu}X_j, \qquad [Y_u, Y_v] = \sum_{i=1}^{2r}\nu^i_{uv}X_i,$$

where $\lambda^k_{ij},\ \mu^j_{iu},\ \nu^i_{uv}$ are Casimir functions. Let $\mathcal{T} \subseteq M$ be a smooth transversal to $T\mathcal{S}$ intersecting x_0; this is parametrized by the y^u's. Fixing $y \in \mathcal{T}$, the restrictions $X^y_1, \ldots, X^y_{2r}$ of $X_1, \ldots, X_{2r}$ to the symplectic leaf $\mathcal{S}_y$ passing through y form a Lie algebra $\mathfrak{g}_y$ which acts freely and transitively on $\mathcal{S}_y$. Therefore, according to [14], there exists a free transitive Lie algebra anti-homomorphism $\hat{\Gamma}_y : \mathfrak{g}_y \to \mathfrak{X}^1(\mathcal{S}_y)$ whose image is

$$\hat{\Gamma}_y(\mathfrak{g}_y) = \{T \in \mathfrak{X}^1(\mathcal{S}_y) : [T, X^y_i] = 0\ \forall\, i = 1, \ldots, 2r\},$$

and such that $\hat{\Gamma}_y(X_i^y)(y) = X_i(y)$ for all i. Setting for any i,

$$T_i(z) := \hat{\Gamma}_y(X_i^y)(z), \quad z \in \mathcal{S}_y$$

and varying y along $\mathcal{T}$, we get a family of linearly independent vector fields $\{T_1, \dots, T_{2r}\}$ which are tangent to $T\mathcal{S}$ and verify

$$[T_i, X_j] = 0 \quad \text{for all } i, j,$$

and such that $T_i(y) = X_i(y)$ for all i and all $y \in \mathcal{T}$. Note that $T_1, \dots, T_{2r}$ are smooth since the solutions of the system

$$[T, X_i] = 0, \quad i = 1, \dots, 2r$$

depend smoothly on the parameter $y \in \mathcal{T}$ and the initial values along $\mathcal{T}$. It is also worth noting that since the μ_{iu}^j's are Casimir, we have

$$[X_i, [T_j, Y_u]] = 0 \quad \text{for all } i, j \text{ and all } u,$$

so that

$$[T_i, Y_u] = \sum_{j=1}^{2r} \gamma_{iu}^j T_j,$$

where γ_{iu}^j are Casimir functions; in addition, since the ν_{uv}^i's are Casimir, we have

$$[T_i, [Y_u, Y_v]] = 0 \quad \text{for all } i \text{ and all } u, v$$

implying

$$\frac{\partial \gamma_{ju}^i}{\partial y_v} - \frac{\partial \gamma_{jv}^i}{\partial y_u} + \sum_{k=1}^{2r} \gamma_{ku}^i \gamma_{jv}^k - \gamma_{kv}^i \gamma_{ju}^k = 0 \qquad (*)$$

for all i, j and all u, v.
Now we would like to find an invertible matrix $\xi = (\xi_j^i)_{1 \le i,j \le 2r}$ where ξ_j^i are Casimir functions such that the vector fields

$$Z_i := \sum_{j=1}^{2r} \xi_i^j T_j, \quad i = 1, \dots, 2r$$

verify

$$[Z_i, Y_u] = 0 \quad \text{for all } i \text{ and all } u.$$

If such a matrix exists, the family $\{Z_1, \dots, Z_{2r}\}$ is clearly the desired one. Since the functions ξ_j^i are searched to be Casimir, the condition for the Z_i's to commute with the Y_u's can be rewritten as

$$\frac{\partial \xi_j^i}{\partial y^u} = \sum_{k=1}^{2r} \gamma_{ku}^i \xi_j^k \quad \forall\, i, j, \ \forall\, u,$$

or in matrix notation

$$\frac{\partial}{\partial y^u} \xi_j = \Gamma_u \xi_j,$$

where ξ_j is the jth column vector of ξ and $\Gamma_u := (\gamma_{ju}^i)_{1 \le i,j \le 2r}$. So we need to solve this system. Since the functions γ_{ju}^i are Casimir and ξ_j^i are searched to be Casimir, we only need to solve it on $\mathcal{T}$. According to Frobenius's Theorem, this system has solutions if and only if the following integrability condition

$$\Gamma_u \Gamma_v + \frac{\partial}{\partial y^v} \Gamma_u = \Gamma_v \Gamma_u + \frac{\partial}{\partial y^u} \Gamma_v$$

holds for all u, v, which is nothing else but $(*)$. It then suffices to take $\xi_j^i(x_0) = \delta_j^i$ (Kronecker delta) as initial conditions to conclude.

Finally, if $\mathcal{D}$ is the metric contravariant connection with respect to π and a Riemannian metric g, we choose $\mathcal{H} = (\operatorname{Ker} \pi_\sharp)^\perp$. In this case, we have

$$\mathcal{L}_{Z_i} g\,(\phi_j, \phi_k) = \mathcal{L}_{Z_i} g\,(\phi_j, dy^u) = \mathcal{L}_{Z_i} g\,(dy^u, dy^v) = 0$$

since $\mathcal{L}_{Z_i} \phi_j = \mathcal{L}_{Z_i} dy^u = 0$ and since $g(\phi_i, \phi_j)$ and $g(dy^u, dy^v)$ are Casimir functions. This shows that the vector fields Z_i are Killing. □

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