



The signature of the Ricci curvature of left-invariant Riemannian metrics on nilpotent Lie groups



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ARTICLE INFO

Article history:

Received 8 July 2015

Available online xxxx

Communicated by D.V. Alekseevsky

MSC:

53C25

53D05

17B30

Keywords:

Signature Ricci curvature

Left invariant metrics

Nilpotent Lie algebras

ABSTRACT

Let (G, h) be a nilpotent Lie group endowed with a left invariant Riemannian metric, \mathfrak{g} its Euclidean Lie algebra and $Z(\mathfrak{g})$ the center of \mathfrak{g} . By using an orthonormal basis adapted to the splitting $\mathfrak{g} = (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus O^+ \oplus (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \oplus O^-$, where O^+ (resp. O^-) is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $[\mathfrak{g}, \mathfrak{g}]$ (resp. is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ in $[\mathfrak{g}, \mathfrak{g}]^\perp$), we show that the signature of the Ricci operator of (G, h) is determined by the dimensions of the vector spaces $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$, $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ and the signature of a symmetric matrix of order $\dim[\mathfrak{g}, \mathfrak{g}] - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])$. This permits to associate to G a subset $\mathbf{Sign}(\mathfrak{g})$ of \mathbb{N}^3 depending only on the Lie algebra structure, easy to compute and such that, for any left invariant Riemannian metric on G , the signature of its Ricci operator belongs to $\mathbf{Sign}(\mathfrak{g})$. We show also that for any nilpotent Lie group of dimension less or equal to 6, $\mathbf{Sign}(\mathfrak{g})$ is actually the set of signatures of the Ricci operators of all left invariant Riemannian metrics on G . We give also some general results which support the conjecture that the last result is true in any dimension.

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1. Introduction

It is a well established fact that there are deep relations between the topology and the geometry of a manifold on one side, and the curvature of a given Riemannian metric on this manifold on the other side. There is a long list of theorems supporting this fact (see for instance [1]) and many of them involve the Ricci curvature. It is a symmetric bilinear tensor and hence has a signature. In the case of a homogeneous Riemannian manifold this signature is the same in any point of the manifold. The determination of the possible signatures of the Ricci operators of G -invariant metrics on a G -homogeneous space can be useful in

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¹ The first author would like to thank Agence Universitaire pour la francophonie for supporting his many stays at Cadi-Ayyad University where this work has been done.

many geometrical problems, for instance, in the study of the Ricci flow. This has led naturally to the study of the following open problem that constitutes the topic of this paper.

Problem 1. *For a connected Lie group G , determine all the signatures of the Ricci operators for all left-invariant Riemannian metrics on G .*

This problem has been studied mainly in the low dimensions. In [9] and [4,5], Problem 1 has been solved, respectively, in the case of 3-dimensional Lie groups and 4-dimensional Lie groups. For Lie groups of dimension 5 there are only partial results. In [6], A.G. Kremlev, solved Problem 1 in the case of five-dimensional nilpotent Lie groups. In this paper, we study Problem 1 when G is nilpotent. We show that, associated to any nilpotent Lie group G , there is a subset $\mathbf{Sign}(\mathfrak{g})$ of \mathbb{N}^3 depending only on the Lie algebra \mathfrak{g} of G , easy to compute and such that, for any left invariant Riemannian metric on G , the signature of its Ricci operator belongs to $\mathbf{Sign}(\mathfrak{g})$. In the case where $\dim G \leq 6$, $\mathbf{Sign}(\mathfrak{g})$ is actually the set of signatures of the Ricci operators of all left invariant Riemannian metrics on G . We give also some general results which support the conjecture that the last result is true in any dimension.

Now, we introduce $\mathbf{Sign}(\mathfrak{g})$ and state our main results. Throughout this paper, we will use the following convention. The signature of a symmetric operator A on an Euclidean vector space V is the sequence (s^-, s^0, s^+) where $s^+ = \sum_{\lambda_i > 0} \dim \ker(A - \lambda_i I_V)$, $s^- = \sum_{\lambda_i < 0} \dim \ker(A - \lambda_i I_V)$ and $s^0 = \dim \ker A$, where $\lambda_1, \dots, \lambda_r$ are the eigenvalues of A .

Let \mathfrak{g} be a nilpotent n -dimensional Lie algebra, $Z(\mathfrak{g})$ its center and $[\mathfrak{g}, \mathfrak{g}]$ its derived ideal. Put $d = \dim[\mathfrak{g}, \mathfrak{g}]$, $k = \dim Z(\mathfrak{g})$ and $\ell = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])$. We associate to \mathfrak{g} the subset of \mathbb{N}^3

$$\begin{aligned} \mathbf{Sign}(\mathfrak{g}) &= \{(n - d - p + m^-, p + m^0, \ell + m^+) \\ &: \max(k - d, 0) \leq p \leq k - \ell, m^- + m^0 + m^+ = d - \ell\}. \end{aligned} \tag{1}$$

For instance, if \mathfrak{g} is 2-step nilpotent then $[\mathfrak{g}, \mathfrak{g}] \subset Z(\mathfrak{g})$ and hence $\mathbf{Sign}(\mathfrak{g}) = \{(n - k, k - d, d)\}$. If \mathfrak{g} is a filiform nilpotent Lie algebra then $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$, $\dim Z(\mathfrak{g}) = 1$, $\dim[\mathfrak{g}, \mathfrak{g}] = n - 2$ and hence

$$\mathbf{Sign}(\mathfrak{g}) = \{(2 + m^-, m^0, 1 + m^+), m^- + m^0 + m^+ = n - 3\}.$$

The signature of the Ricci operator of a left invariant Riemannian metric on Lie group of dimension n belongs to $\{(n^-, n^0, n^+) : n^- + n^0 + n^+ = n\}$ whose cardinal is $\frac{(n+1)(n+2)}{2}$. Our first main result reduces drastically the set of possibilities for a nilpotent Lie group.

Theorem 1. *Let (G, h) be a nilpotent Lie group endowed with a left invariant Riemannian metric and \mathfrak{g} its Lie algebra. Then the signature of the Ricci operator of (G, h) belongs to $\mathbf{Sign}(\mathfrak{g})$.*

As an immediate consequence of this result, if G is 2-step nilpotent then any left invariant Riemannian metric on G has the signature of its Ricci operator equal to $(\dim \mathfrak{g} - \dim Z(\mathfrak{g}), \dim Z(\mathfrak{g}) - \dim[\mathfrak{g}, \mathfrak{g}], \dim[\mathfrak{g}, \mathfrak{g}])$. On the other hand, Theorem 1 has the following corollary which gives a new proof to a result proved first in [11].

Corollary 1. *Let (G, h) be a noncommutative nilpotent Lie group endowed with a left invariant Riemannian metric and \mathfrak{g} its Lie algebra. Then the Ricci operator of (G, h) has at least two negative eigenvalues.*

Theorem 1 gives a candidate to be the set of the signatures of the Ricci operators of all left invariant Riemannian metrics on a nilpotent Lie group. Indeed, our second main result together with Theorem 1 solve Problem 1 completely for nilpotent Lie groups up to dimension 6.

Theorem 2. *Let G be a nilpotent Lie group of dimension ≤ 6 and \mathfrak{g} its Lie algebra. Then, for any $(s^-, s^0, s^+) \in \mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+) .*

Our third main result involves the notion of nice basis. Recall that a basis (X_1, \dots, X_n) of a nilpotent Lie algebra \mathfrak{g} is called nice if:

1. For any i, j with $i \neq j$, $[X_i, X_j] = 0$ or there exists k such that $[X_i, X_j] = C_{ij}^k X_k$ with $C_{ij}^k \neq 0$.
2. If $[X_i, X_j] = C_{ij}^k X_k$ and $[X_s, X_r] = C_{sr}^k X_k$ with $C_{ij}^k \neq 0$ and $C_{sr}^k \neq 0$ then $\{i, j\} \cap \{s, r\} = \emptyset$.

This notion appeared first in [8]. One of the most important property of a nice basis \mathbb{B} is that any Euclidean inner product on \mathfrak{g} for which \mathbb{B} is orthogonal has its Ricci curvature diagonal in \mathbb{B} . The proof of [Theorem 2](#) is based mainly on the fact that all the nilpotent Lie algebras of dimension less or equal to 6 have a nice basis except one. It is also known (see [7]) that any filiform Lie algebra has a nice basis.

Theorem 3. *Let G be a nilpotent Lie group such that its Lie algebra \mathfrak{g} admits a nice basis and $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ with $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) = 1$. Then, for any $(s^-, s^0, s^+) \in \mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+) .*

This theorem together with [Theorem 1](#) solve [Problem 1](#) for a large class of nilpotent Lie groups. Indeed, in the list of indecomposable seven-dimensional nilpotent Lie algebras given in [10] there are more than 35 ones satisfying the hypothesis of [Theorem 3](#). On the other hand, we will point out the difficulty one can face when trying to generalize [Theorem 3](#) when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. We will also give a method using the inverse function theorem to overcome this difficulty. Although, we have not succeeded yet to show that this method works in the general case, we will use it successfully in the proof of [Theorem 2](#). We will refer to this method as inverse function theorem trick.

The results above, the tools we will use to establish them and the examples we will give support the following conjecture.

Conjecture 1. *Let G be a nilpotent Lie group and \mathfrak{g} its Lie algebra. Then, for any $(s^-, s^0, s^+) \in \mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+) .*

The paper is organized as follows. In Sections 2–3, we prove a key lemma (see [Lemma 1](#)) which implies that, for any nilpotent Lie group (G, h) endowed with a left invariant Riemannian metric, by using an orthonormal basis adapted to the splitting of an Euclidean Lie algebra

$$\mathfrak{g} = (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus O^+ \oplus (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \oplus O^-,$$

where O^+ (resp. O^-) is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $[\mathfrak{g}, \mathfrak{g}]$ (resp. is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ in $[\mathfrak{g}, \mathfrak{g}]^\perp$), the signature of the Ricci operator of (G, h) is determined by the dimensions of the vector spaces $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$, $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ and the signature of a symmetric matrix of order $\dim[\mathfrak{g}, \mathfrak{g}] - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])$. Thereafter, we give a proof of [Theorem 1](#), its corollary and [Theorem 3](#). At the end of Section 3, we outline the inverse function theorem trick that we will use in the proof of [Theorem 2](#). Section 3 is devoted to a proof of [Theorem 2](#). We summarize at the end of the paper in a table all the realizable signatures of Ricci operators on nilpotent Lie groups up to dimension 6. It reduces, according to [Theorems 1 and 2](#), to computing $\mathbf{Sign}(\mathfrak{g})$ for any nilpotent Lie algebra of dimension less or equal to 6. Since we will use the classification of 5-dimensional and 6-dimensional Lie nilpotent algebras given by Willem A. de Graaf in [3], we give here the lists of these Lie algebras from [3] ([Tables 1 and 2](#)).

Table 1
 $\epsilon \in \{-1, 0, 1\}$: List of six-dimensional nilpotent Lie algebras.

Lie algebra \mathfrak{g}	Nonzero commutators
$L_{6,2} = L_{5,2} \oplus \mathbb{R}$	$[e_1, e_2] = e_3$
$L_{6,3} = L_{5,3} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$L_{6,4} = L_{5,4} \oplus \mathbb{R}$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$
$L_{6,5} = L_{5,5} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$
$L_{6,6} = L_{5,6} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$
$L_{6,7} = L_{5,7} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$
$L_{6,8} = L_{5,8} \oplus \mathbb{R}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$
$L_{6,9} = L_{5,9} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$
$L_{6,10}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_6, [e_4, e_5] = e_6$
$L_{6,11}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_6, [e_2, e_5] = e_6$
$L_{6,12}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_5] = e_6$
$L_{6,13}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5, [e_1, e_5] = e_6, [e_3, e_4] = e_6$
$L_{6,14}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5, [e_2, e_5] = e_6, [e_3, e_4] = -e_6$
$L_{6,15}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5, [e_2, e_4] = e_6, [e_1, e_5] = e_6$
$L_{6,16}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_5] = e_6, [e_3, e_4] = -e_6$
$L_{6,17}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_2, e_3] = e_6$
$L_{6,18}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6$
$L_{6,19}(\epsilon)$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_4] = e_6, [e_3, e_5] = \epsilon e_6$
$L_{6,20}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_5] = e_6, [e_2, e_4] = e_6$
$L_{6,21}(\epsilon)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_5] = \epsilon e_6$
$L_{6,22}(\epsilon)$	$[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = \epsilon e_6, [e_3, e_4] = e_5$
$L_{6,23}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_4] = e_5$
$L_{6,24}(\epsilon)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = \epsilon e_6, [e_2, e_3] = e_6, [e_2, e_4] = e_5$
$L_{6,25}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6$
$L_{6,26}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_6$

Table 2
 List of five-dimensional nilpotent Lie algebras.

Lie algebra \mathfrak{g}	Nonzero commutators
$L_{5,2} = L_{3,2} \oplus \mathbb{R}^2$	$[e_1, e_2] = e_3$
$L_{5,3} = L_{4,3} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$L_{5,4}$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$
$L_{5,5}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$
$L_{5,6}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$
$L_{5,7}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$
$L_{5,8}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$
$L_{5,9}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$

2. Reduction of the Ricci operator of a Riemannian Lie group and Ricci signature underestimate

In this section, we prove a key lemma that will play a crucial role in the proofs of our main results.

A Lie group G together with a left-invariant Riemannian metric h is called a *Riemannian Lie group*. The metric h defines a symmetric positive definite inner product $\langle \cdot, \cdot \rangle = h(e)$ on the Lie algebra \mathfrak{g} of G , and conversely, any symmetric definite positive inner product on \mathfrak{g} gives rise to a unique left-invariant Riemannian metric on G .

We will refer to a Lie algebra endowed with a symmetric positive definite inner product as an *Euclidean Lie algebra*.

The Levi-Civita connection of (G, h) defines a product $L : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called *Levi-Civita product* given by Koszul's formula

$$2\langle L_u v, w \rangle = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle. \tag{2}$$

For any $u, v \in \mathfrak{g}$, $L_u : \mathfrak{g} \rightarrow \mathfrak{g}$ is skew-symmetric and $[u, v] = L_u v - L_v u$. The curvature on \mathfrak{g} is given by

$$K(u, v) = L_{[u, v]} - [L_u, L_v].$$

The Ricci curvature on \mathfrak{g} is defined by $\text{ric}(u, v) = \text{tr}(w \rightarrow K(u, w)v)$. The mean curvature vector on \mathfrak{g} is the vector H defined by the following relation $\langle H, u \rangle = \text{tr}(\text{ad}_u)$, where $\text{ad}_u : \mathfrak{g} \rightarrow \mathfrak{g}, v \mapsto [u, v]$. It is well-known that ric is given by

$$\text{ric}(u, v) = -\frac{1}{2}\text{tr}(\text{ad}_u \circ \text{ad}_v) - \frac{1}{2}\text{tr}(\text{ad}_u \circ \text{ad}_v^*) - \frac{1}{4}\text{tr}(J_u \circ J_v) - \frac{1}{2}\langle \text{ad}_H u, v \rangle - \frac{1}{2}\langle \text{ad}_H v, u \rangle, \quad (3)$$

where J_u is the skew-adjoint endomorphism given by $J_u v = \text{ad}_v^* u$ (ad_u^* is the adjoint of ad_u with respect to $\langle \cdot, \cdot \rangle$). The Ricci operator is the auto-adjoint endomorphism $\text{Ric} : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\langle \text{Ric}(u), v \rangle = \text{ric}(u, v)$. The signature of Ric is called *Ricci signature* of (G, h) or $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.

We consider now the Lie subalgebra of left invariant Killing vector fields on G given by

$$K(\langle \cdot, \cdot \rangle) = \{u \in \mathfrak{g}, \text{ad}_u + \text{ad}_u^* = 0\}.$$

It contains obviously the center $Z(\mathfrak{g})$ of \mathfrak{g} . Put $K^+(\langle \cdot, \cdot \rangle) = K(\langle \cdot, \cdot \rangle) \cap [\mathfrak{g}, \mathfrak{g}]$ and $K^-(\langle \cdot, \cdot \rangle) = K(\langle \cdot, \cdot \rangle) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$. Denote by O^+ (resp. O^-) the orthogonal of $K^+(\langle \cdot, \cdot \rangle)$ in $[\mathfrak{g}, \mathfrak{g}]$ (resp. the orthogonal of $K^-(\langle \cdot, \cdot \rangle)$ in $[\mathfrak{g}, \mathfrak{g}]^\perp$). Then

$$\mathfrak{g} = K^+(\langle \cdot, \cdot \rangle) \oplus O^+ \oplus K^-(\langle \cdot, \cdot \rangle) \oplus O^-. \quad (4)$$

We call this splitting *characteristic splitting* of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and any basis of \mathfrak{g} of the form $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3 \cup \mathbb{B}_4$ where $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3$ and \mathbb{B}_4 are, respectively, bases of $K^+(\langle \cdot, \cdot \rangle), O^+, K^-(\langle \cdot, \cdot \rangle), O^-$ is called *characteristic basis*.

Lemma 1. *With the hypothesis and the notations above, let $n_1 = \dim K^+(\langle \cdot, \cdot \rangle)$, $n_2 = \dim O^+$, $n_3 = \dim K^-(\langle \cdot, \cdot \rangle)$ and $n_4 = \dim O^-$. Then we have:*

- (i) $K^-(\langle \cdot, \cdot \rangle) \subset \ker \text{Ric}$ and if $K^+(\langle \cdot, \cdot \rangle) \neq \{0\}$ then the restriction of ric to $K^+(\langle \cdot, \cdot \rangle)$ is positive definite.
- (ii) If $O^- \neq \{0\}$ then the restriction of ric to O^- is negative definite and $\text{ric}(K^+(\langle \cdot, \cdot \rangle), O^-) = 0$.
- (iii) For any characteristic basis \mathbb{B} of \mathfrak{g} , the matrix of the Ricci tensor in \mathbb{B} is given by

$$\text{Mat}(\text{ric}, \mathbb{B}) = \frac{1}{2} \begin{bmatrix} Z & V & 0 & 0 \\ V^t & X & 0 & W \\ 0 & 0 & 0 & 0 \\ 0 & W^t & 0 & Y \end{bmatrix},$$

where Z, X, Y are square matrices of order n_1, n_2, n_4 , respectively, and the Ricci signature of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is given by

$$(s^-, s^0, s^+) = (\dim[\mathfrak{g}, \mathfrak{g}]^\perp - \dim K^-(\langle \cdot, \cdot \rangle) + m^-, \dim K^-(\langle \cdot, \cdot \rangle) + m^0, \dim K^+(\langle \cdot, \cdot \rangle) + m^+), \quad (5)$$

where (m^-, m^0, m^+) is the signature of the symmetric matrix

$$R(\text{ric}, \mathbb{B}) = X - V^t Z^{-1} V - W Y^{-1} W^t. \quad (6)$$

Proof. First remark that, for any $u \in \mathfrak{g}$, J_u is skew-symmetric and $J_u = 0$ iff $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$. With this remark in mind, by using (3), we get for any $u \in K^+(\langle \cdot, \cdot \rangle)$, $\text{ric}(u, u) = -\frac{1}{4}\text{tr}(J_u^2) \geq 0$ and $\text{ric}(u, u) = 0$ if and only if $J_u = 0$. This shows that the restriction of ric to $K^+(\langle \cdot, \cdot \rangle)$ is definite positive. On the other hand, for any $u \in O^-$, by using (3) we get $\text{ric}(u, u) = -\frac{1}{4}\text{tr}((\text{ad}_u + \text{ad}_u^*)^2) \leq 0$ and $\text{ric}(u, u) = 0$ iff $u \in K(\langle \cdot, \cdot \rangle)$. This shows that the restriction of ric to O^- is negative definite. We have also, for any $u \in K^-(\langle \cdot, \cdot \rangle)$ and any

$v \in \mathfrak{g}$, $\text{ric}(u, v) = 0$. Finally, for any $u \in K^+(\langle \cdot, \cdot \rangle)$ and any $v \in O^-$, $\text{ric}(u, v) = 0$ this completes the proof of (i)–(ii).

In any characteristic basis \mathbb{B} of \mathfrak{g} , according to the results shown in (i)–(ii), the matrix $R(\text{ric}, \mathbb{B})$ has the desired form. Put

$$Q = \begin{bmatrix} I_{n_1} & -Z^{-1}V & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & -Y^{-1}W^t & 0 & I_{n_4} \end{bmatrix}.$$

We can check easily that

$$Q^t \text{Mat}(\text{ric}, \mathbb{B}) Q = \frac{1}{2} \begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & R(\text{ric}, \mathbb{B}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y \end{bmatrix}.$$

This formula combined with the results in (i)–(ii) give the desired formula for the signature of ric . \square

Definition 1. Let (G, h) be a Riemannian Lie group and $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ its associated Euclidean Lie algebra.

- We call $(r^-, r^0, r^+) = (\dim[\mathfrak{g}, \mathfrak{g}]^\perp - \dim K^-(\langle \cdot, \cdot \rangle), \dim K^-(\langle \cdot, \cdot \rangle), \dim K^+(\langle \cdot, \cdot \rangle))$ the Ricci signature underestimate of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$.
- For any characteristic basis \mathbb{B} of \mathfrak{g} , we call $R(\text{ric}, \mathbb{B})$ defined by (6) *reduced matrix* of the Ricci curvature in \mathbb{B} . It is a symmetric $(s \times s)$ -matrix with $s = \dim[\mathfrak{g}, \mathfrak{g}] - \dim K^+(\langle \cdot, \cdot \rangle)$.

Note that the order of $R(\text{ric}, \mathbb{B})$ is zero iff $[\mathfrak{g}, \mathfrak{g}] \subset K(\langle \cdot, \cdot \rangle)$. In this case $K(\langle \cdot, \cdot \rangle) = [\mathfrak{g}, \mathfrak{g}] \oplus K^-(\langle \cdot, \cdot \rangle)$ and we get:

Corollary 2. Let (G, h) be a Riemannian Lie group such that $[\mathfrak{g}, \mathfrak{g}] \subset K(\langle \cdot, \cdot \rangle)$. Then the signature of the Ricci curvature of h is given by

$$(s^-, s^0, s^+) = (\dim \mathfrak{g} - \dim K(\langle \cdot, \cdot \rangle), \dim K(\langle \cdot, \cdot \rangle) - \dim[\mathfrak{g}, \mathfrak{g}], \dim[\mathfrak{g}, \mathfrak{g}]).$$

Remark 1. The case where the Riemannian metric is bi-invariant ($\mathfrak{g} = K(\langle \cdot, \cdot \rangle)$) is a particular case of the situation in Corollary 2 and in this case $Z(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]^\perp$ and hence the signature is given by

$$(s^-, s^0, s^+) = (0, \dim Z(\mathfrak{g}), \dim[\mathfrak{g}, \mathfrak{g}]).$$

3. Ricci signature underestimates in nilpotent Riemannian Lie groups and a proof of Theorems 1 and 3

In this section, we will show that Lemma 1 turn out to be very useful in the case of nilpotent Riemannian Lie groups that permits us to prove Theorems 1 and 3.

3.1. Preliminaries

Let (G, h) be a nilpotent Riemannian Lie group. The formula (3) becomes in this case quite simple

$$\text{ric}(u, v) = -\frac{1}{2} \text{tr}(\text{ad}_u \circ \text{ad}_v^*) - \frac{1}{4} \text{tr}(J_u \circ J_v) = -\frac{1}{2} \langle \text{ad}_u, \text{ad}_v \rangle_1 + \frac{1}{4} \langle J_u, J_v \rangle_1,$$

where $\langle \cdot, \cdot \rangle_1$ is the Euclidean product on $\text{End}(\mathfrak{g})$ associated to $\langle \cdot, \cdot \rangle$. In particular, if (e_1, \dots, e_n) is an orthonormal basis of \mathfrak{g} then

$$\text{ric}(u, v) = -\frac{1}{2} \sum_{i,j} \langle [u, e_i], e_j \rangle \langle [v, e_i], e_j \rangle + \frac{1}{2} \sum_{i < j} \langle [e_i, e_j], u \rangle \langle [e_i, e_j], v \rangle. \tag{7}$$

Moreover, since a skew-symmetric nilpotent endomorphism must vanishes then $K(\langle \cdot, \cdot \rangle) = Z(\mathfrak{g})$. This simple fact combined with the result of Lemma 1 will have surprising consequences. Note first that, as a particular case of Corollary 2, we get the following result which first appeared in [2] and which solves Problem 1 for 2-step nilpotent Lie groups.

Corollary 3. *Let G be a 2-step nilpotent Lie group. Then, for any left-invariant Riemannian metric on G , the signature of its Ricci curvature is given by*

$$(s^-, s^0, s^+) = (\dim \mathfrak{g} - \dim Z(\mathfrak{g}), \dim Z(\mathfrak{g}) - \dim[\mathfrak{g}, \mathfrak{g}], \dim[\mathfrak{g}, \mathfrak{g}]).$$

3.2. Proof of Theorem 1 and Corollary 1

Proof. Let (G, h) be a nilpotent Riemannian Lie group. We distinguish two cases.

- $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. In this case, it is obvious that the Ricci signature underestimate of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is given by

$$(r^-, r^0, r^+) = (\dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}], 0, \dim Z(\mathfrak{g})).$$

On the other hand, by using (1), one can see easily that

$$\mathbf{Sign}(\mathfrak{g}) = \{(r^- + m^-, r^0 + m^0, r^+ + m^+), m^- + m^0 + m^+ = \dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g})\}.$$

According to Lemma 1, the Ricci signature of h belongs to $\mathbf{Sign}(\mathfrak{g})$ and we obtain the result in this case. Corollary 1 follows from the fact that $r^- = \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}] \geq 2$. In a nilpotent Lie algebra the derived ideal is always of codimension greater than 2.

- $Z(\mathfrak{g}) \not\subset [\mathfrak{g}, \mathfrak{g}]$. Choose a complement I of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $Z(\mathfrak{g})$ and a complement U of $[\mathfrak{g}, \mathfrak{g}] \oplus I$ in \mathfrak{g} . Thus $\mathfrak{g} = \mathfrak{g}_1 \oplus I$ where $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \oplus U$ is an ideal of \mathfrak{g} and I is a central ideal. Moreover, $Z(\mathfrak{g}_1) = Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1]$. By using the same notations as in (1), we get that the Ricci signature underestimate of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is given by

$$(r^-, r^0, r^+) = (n - d - p, p, \ell), \quad p = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp).$$

We have obviously $p \leq \dim I = \dim Z(\mathfrak{g}) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])$ and

$$p = \dim Z(\mathfrak{g}) + \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}] - \dim(Z(\mathfrak{g}) + [\mathfrak{g}, \mathfrak{g}]^\perp) \geq \dim Z(\mathfrak{g}) - \dim[\mathfrak{g}, \mathfrak{g}].$$

According to Lemma 1, the Ricci signature of h belongs to $\mathbf{Sign}(\mathfrak{g})$ and we obtain the result in this case. Corollary 1 follows from the fact that

$$\begin{aligned} r^- &= \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}] - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \\ &= \dim \mathfrak{g}_1 - \dim[\mathfrak{g}_1, \mathfrak{g}_1] + \dim I - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \\ &= \dim \mathfrak{g}_1 - \dim[\mathfrak{g}_1, \mathfrak{g}_1] + \dim Z(\mathfrak{g}) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \\ &\geq \dim \mathfrak{g}_1 - \dim[\mathfrak{g}_1, \mathfrak{g}_1] \geq 2. \quad \square \end{aligned}$$

3.3. Proof of Theorem 3

Proof. We have obviously $\text{Sign}(\mathfrak{g}) = \{(n - d + m^-, m^0, d - 1 + m^+) : m^- + m^0 + m^+ = 1\}$, where $d = \dim[\mathfrak{g}, \mathfrak{g}]$ and $n = \dim \mathfrak{g}$. Note first that we can choose a nice basis $\mathbb{B} = (X_i)_{i=1}^n$ where $Z(\mathfrak{g}) = \text{span}\{X_i\}_{i=1}^{d-1}$ and $[\mathfrak{g}, \mathfrak{g}] = \text{span}\{X_i\}_{i=1}^d$. Indeed, suppose that $\mathbb{B} = (X_i)_{i=1}^n$ with $[\mathfrak{g}, \mathfrak{g}] = \text{span}\{X_i\}_{i=1}^d$. Let $z = \sum_{i=1}^d a_i X_i \in Z(\mathfrak{g})$. Suppose that there exists $a_i \neq 0$ and $X_i \notin Z(\mathfrak{g})$. Then there exists $\ell \in \{1, \dots, n\}$ such that $[X_\ell, X_i] \neq 0$. So we get $\sum_{j=1}^d a_j [X_\ell, X_j] = 0$. From the properties of a nice basis we deduce that $\{[X_\ell, X_j], j = 1, \dots, d, [X_\ell, X_j] \neq 0\}$ is a linearly independent family and hence $a_i = 0$. This shows that $\{X_i, X_i \in Z(\mathfrak{g})\}$ is basis of $Z(\mathfrak{g})$.

We consider the Euclidean product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} for which \mathbb{B} is orthogonal and $a_i = \langle X_i, X_i \rangle$. It is obvious that \mathbb{B} is a characteristic basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and it is also nice of \mathfrak{g} so $M(\text{ric}, \mathbb{B})$ is diagonal and hence $R(\text{ric}, \mathbb{B})$ is also diagonal. According to Lemma 1, the reduced matrix has order 1 and is given by $R(\text{ric}, \mathbb{B}) = (2\text{ric}(X_d, X_d))$. Moreover, the Ricci signature of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is given by $(n - d + m^-, m^0, d - 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. To complete the proof, we will show that we can choose suitable a_i so that $\text{ric}(X_d, X_d)$ can be either zero, positive or negative.

Denote by C_{ij}^k the structure constants of the Lie bracket in \mathbb{B} . The basis $(e_i)_{i=1}^n = (\frac{1}{\sqrt{a_i}} X_i)_{i=1}^n$ is an orthonormal basis of \mathfrak{g} and from (7)

$$2\text{ric}(X_d, X_d) = \sum_{i < j} \frac{(C_{ij}^d)^2 a_d^2}{a_i a_j} - \sum_{i, j} \frac{(C_{id}^j)^2 a_j}{a_i} \tag{8}$$

Note that for any (i, j) , such that $[X_i, X_j] = C_{ij}^d X_d$ with $C_{ij}^d \neq 0$, $i \neq d$ and $j \neq d$. Indeed, if $i = d$, we have $[X_d, X_j] = C_{dj}^d X_d$ and hence X_d is an eigenvector of ad_{X_j} with the real non zero eigenvalue $-C_{di}^d$ which is impossible since ad_{X_j} is nilpotent. We have also that if $[X_d, X_i] = C_{di}^j X_j$ with $C_{di}^j \neq 0$ then $i \neq d$ and $j \neq d$. So

$$\text{ric}(X_d, X_d) = \alpha a_d^2 - \beta.$$

Now since $X_d \in [\mathfrak{g}, \mathfrak{g}] \setminus Z(\mathfrak{g})$, $\alpha > 0$, $\beta > 0$ and both α and β depend only on a_i with $i \neq d$. So we can choose a_d such that $\text{ric}(X_d, X_d) = 0, > 0$ or < 0 . This completes the proof. \square

One can ask naturally if this theorem is still true when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. By looking to the proof given here, one can conjecture that the answer is true, it suffices to solve some systems of polynomial equations. This can be very difficult. To be precise, we will point out the difficulty one can face when trying to generalize Theorem 3 when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. We will also give a method to overcome this difficulty. Although, we have not succeeded yet to show that this method works in the general case, we will use it successfully in the proof of Theorem 2.

3.4. Inverse function theorem trick

Suppose that \mathfrak{g} is a nilpotent Lie algebra having a nice basis \mathbb{B} and satisfying $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. Write $\mathbb{B} = (X_i)_{i=1}^n$ where $(X_i)_{i=1}^\ell$ is a basis of $Z(\mathfrak{g})$ and $(X_i)_{i=1}^d$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. We have obviously

$$\text{Sign}(\mathfrak{g}) = \{(n - d + m^-, m^0, \ell + m^+) : m^- + m^0 + m^+ = d - \ell\}.$$

We consider the Euclidean product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} for which \mathbb{B} is orthogonal and $a_i = \langle X_i, X_i \rangle$. It is clear that \mathbb{B} is a characteristic basis of $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ and it is also nice so $M(\text{ric}, \mathbb{B})$ is diagonal. According to Lemma 1, the reduced matrix has order $d - \ell$ and is diagonal and given by

$$R(\text{ric}, \mathbb{B}) = \text{diag}(2\text{ric}(X_{\ell+1}, X_{\ell+1}), \dots, 2\text{ric}(X_d, X_d)).$$

Moreover, the signature is given by $(n - d + m^-, m^0, \ell + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. According to (8), for any $i = \ell + 1, \dots, d$, we can write in a unique way

$$2\text{ric}(X_i, X_i) = \frac{F_{i-\ell}(a_1, \dots, a_n)}{a_{n_1} \dots a_{n_i}},$$

where $F_{i-\ell}$ is a homogeneous polynomial on (a_1, \dots, a_n) . So to generalize Theorem 3 when $d - \ell \geq 2$, it suffices to find suitable values of (a_1, \dots, a_n) such that $(F_i(a_1, \dots, a_n))_{i=1}^{d-\ell}$ have all the possible signs. It is very difficult in the general case. We give now a situation where we can conclude.

Suppose that there exists $(\alpha_1, \dots, \alpha_n)$ such that $F_j(\alpha_1, \dots, \alpha_n) = 0$ for $j = 1, \dots, d - \ell$ and define

$$F : \{(x_1, \dots, x_{d-\ell}) \in \mathbb{R}^{d-\ell}, x_i > 0\} \longrightarrow \mathbb{R}^{d-\ell}$$

by

$$F(x_1, \dots, x_{d-\ell}) = (F_1(\alpha_1, \dots, \alpha_\ell, x_1, \dots, x_{d-\ell}, \alpha_{d+1}, \dots, \alpha_n), \dots, F_{d-\ell}(\alpha_1, \dots, \alpha_\ell, x_1, \dots, x_{d-\ell}, \alpha_{d+1}, \dots, \alpha_n)).$$

We have $F(\alpha_{\ell+1}, \dots, \alpha_d) = 0$ and if the differential $DF(\alpha_{\ell+1}, \dots, \alpha_d)$ is invertible we can apply the inverse function theorem and hence F realizes a diffeomorphism from an open set centred in $(\alpha_{\ell+1}, \dots, \alpha_d)$ into an open ball centred in $(0, \dots, 0)$. So, for a suitable choice of a_i , $R(\text{ric}, \mathbb{B})$ can have all the possible signatures.

So far we have shown that Theorem 3 is true when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$ if there exists $(\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > 0, \dots, \alpha_n > 0$ satisfying $F_j(\alpha_1, \dots, \alpha_n) = 0$ for $j = 1, \dots, d - \ell$ and $\det DF(\alpha_{\ell+1}, \dots, \alpha_d) \neq 0$.

Definition 2. We call nice a nilpotent Lie algebra \mathfrak{g} with $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and having a nice basis for which there exists $(\alpha_1, \dots, \alpha_n)$ with $\alpha_1 > 0, \dots, \alpha_n > 0$ satisfying $F_j(\alpha_1, \dots, \alpha_n) = 0$ for $j = 1, \dots, d - \ell$ and $\det DF(\alpha_{\ell+1}, \dots, \alpha_d) \neq 0$.

So, according to our study above, we have the following result.

Theorem 4. Let G be a nilpotent Lie group such that its Lie algebra \mathfrak{g} is nice. Then for any $(s^-, s^0, s^+) \in \text{Sign}(\mathfrak{g})$ there exists a left invariant Riemannian metric on G such that its Ricci signature is (s^-, s^0, s^+) .

Remark 2. It is seems reasonable to conjecture that any nilpotent Lie algebra \mathfrak{g} with $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and having a nice basis is actually nice.

We give now two examples of nice nilpotent Lie algebras.

Example 1.

1. We consider the 7-dimensional nilpotent Lie algebra labelled (12457L1) in [10] given by

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = -e_6, [e_1, e_6] = e_7, [e_2, e_3] = e_5, [e_2, e_5] = -e_6, [e_3, e_5] = -e_7.$$

We have $Z(\mathfrak{g}) = \{e_7\} \subset [\mathfrak{g}, \mathfrak{g}] = \{e_3, e_4, e_5, e_6, e_7\}$ and $\mathbb{B} = (e_7, e_3, e_4, e_5, e_6, e_1, e_2)$ is a nice basis. Let compute $2\text{ric}(e_i, e_i)$ for $i = 3, \dots, 6$ for the metric for which \mathbb{B} is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (8), we get

$$2\text{ric}(e_3, e_3) = \frac{a_3^2}{a_1 a_2} - \frac{a_4}{a_1} - \frac{a_5}{a_2} - \frac{a_7}{a_5} = \frac{a_3^2 a_5 - a_2 a_4 a_5 - a_1 a_5^2 - a_1 a_2 a_7}{a_1 a_2 a_5} = \frac{F_1(a_1, \dots, a_7)}{a_1 a_2 a_5},$$

$$\begin{aligned} 2\text{ric}(e_4, e_4) &= \frac{a_4^2}{a_1 a_3} - \frac{a_6}{a_1} = \frac{a_4^2 - a_3 a_6}{a_1 a_3} = \frac{F_2(a_1, \dots, a_7)}{a_1 a_3}, \\ 2\text{ric}(e_5, e_5) &= \frac{a_5^2}{a_2 a_3} - \frac{a_6}{a_2} - \frac{a_7}{a_3} = \frac{a_5^2 - a_3 a_6 - a_2 a_7}{a_2 a_3} = \frac{F_3(a_1, \dots, a_7)}{a_2 a_3}, \\ 2\text{ric}(e_6, e_6) &= \frac{a_6^2}{a_2 a_5} + \frac{a_6^2}{a_1 a_4} - \frac{a_7}{a_1} = \frac{(a_1 a_4 + a_2 a_5) a_6^2 - a_2 a_4 a_5 a_7}{a_1 a_2 a_4 a_5} = \frac{F_4(a_1, \dots, a_7)}{a_1 a_2 a_4 a_5}. \end{aligned}$$

The sequence $(\frac{7}{240}, \frac{1127}{1200}, 1, 1, \frac{7}{5}, 1, \frac{1152}{1127})$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_7) = 0$ for $i = 1, \dots, 4$ and satisfies $\det DF(\alpha_3, \alpha_4, \alpha_5, \alpha_6) \neq 0$ and hence this Lie algebra is nice.

2. We consider the \mathbb{N} -graded filiform n -dimensional Lie algebra $\mathfrak{m}_0(n) = \text{span}\{X_1, \dots, X_n\}$ with the non vanishing Lie brackets $[X_1, X_i] = X_{i+1}$, $i = 2, \dots, n - 1$. We have

$$\text{Sign}(\mathfrak{m}_0(n)) = \{(2 + m^-, m^0, 1 + m^+), m^- + m^0 + m^+ = n - 3\}.$$

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product on $\mathfrak{m}_0(n)$ for which $(X_i)_{i=1}^n$ is an orthogonal basis with $\langle X_i, X_i \rangle = a_i$. The basis $\mathbb{B} = (X_n, X_3, \dots, X_{n-1}, X_1, X_2)$ is a characteristic basis of $\langle \cdot, \cdot \rangle$ and $R(\text{ric}, \mathbb{B}) = \text{diag}(2\text{ric}(X_i, X_i))_{i=3}^{n-1}$. By using (8), we get for any $i = 3, \dots, n - 1$

$$2\text{ric}(X_i, X_i) = \frac{a_i^2}{a_1 a_{i-1}} - \frac{a_{i+1}}{a_1} = \frac{a_i^2 - a_{i-1} a_{i+1}}{a_1 a_{i-1}} = \frac{F_{i-2}(a_1, \dots, a_n)}{a_1 a_{i-1}}.$$

It is obvious that $F_i(1, \dots, 1) = 0$ and $\det DF(1, \dots, 1) \neq 0$ and hence $\mathfrak{m}_0(n)$ is nice.

4. Proof of Theorem 2

Proof. The proof goes as follows. There are, up to an isomorphism, 44 non abelian nilpotent Lie algebras of dimension less or equal to 6: 1 of dimension 3, 2 of dimension 4, 8 of dimension 5 and 33 of dimension 6 (see Tables 1 and 2). Among these Lie algebras, 12 are 2-step nilpotent and we can apply Corollary 3, 10 satisfy the hypothesis of Theorem 3 and 15 are nice in the sense of Definition 2 and we can apply Theorem 4. At the end, we are left with 7 Lie algebras needing each of them a special treatment.

The Lie algebras $L_{3,2}, L_{4,2}, L_{5,2}, L_{5,4}, L_{5,8}, L_{6,2}, L_{6,4}, L_{6,8}, L_{6,22}(\epsilon), L_{6,26}$ are obviously 2-step nilpotent and we can apply Corollary 3.

The Lie algebras $L_{4,3}, L_{5,5}, L_{5,9}, L_{6,10}, L_{6,19}(0), L_{6,23}, L_{6,24}(\epsilon)$ and $L_{6,25}$ satisfy clearly the hypothesis of Theorem 3.

We will show now that the Lie algebras $L_{5,6}, L_{5,7}, L_{6,12}, L_{6,13}, L_{6,14}, L_{6,15}, L_{6,16}, L_{6,17}, L_{6,18}, L_{6,19}(\epsilon \neq 0), L_{6,20}, L_{6,21}(0)$ and $L_{6,21}(\epsilon \neq 0)$ are nice in the sense of Definition 2 so that we can apply Theorem 3. Since the computations are straightforward, we will give for any Lie algebra among these Lie algebras, a nice basis \mathbb{B} , the reduced matrix in \mathbb{B} associated to the Euclidean product for which \mathbb{B} is diagonal with $a_i = \langle e_i, e_i \rangle$ and the value $(\alpha_1, \dots, \alpha_n)$ appearing in Definition 2. Note that $\mathbb{B}_0 = (e_1, \dots, e_n)$ is the basis appearing in Tables 1 and 2.

$$\begin{aligned} L_{5,6} &: \left[(e_5, e_3, e_4, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2}, \frac{a_4^2 - a_3 a_5}{a_1 a_3} \right), \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1 \right) \right]. \\ L_{5,7} &: \left[(e_5, e_3, e_4, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}, \frac{a_4^2 - a_3 a_5}{a_1 a_3} \right), (1, 1, 1, 1, 1) \right]. \\ L_{6,12} &: \left[(e_6, e_3, e_4, e_1, e_2, e_5), \text{diag} \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}, \frac{a_4^2 - a_3 a_6}{a_1 a_3} \right), (1, 1, 1, 1, 1, 1) \right]. \end{aligned}$$

$$L_{6,13} : \left[(e_6, e_3, e_5, e_1, e_2, e_4), \text{diag} \left(\frac{a_4 a_3^2 - a_2 a_4 a_5 - a_1 a_2 a_6}{a_1 a_2 a_4}, \frac{(a_2 a_4 + a_1 a_3) a_5^2 - a_2 a_3 a_4 a_6}{a_1 a_2 a_3 a_4} \right), \right. \\ \left. (1, 2, 2, 1, 1, 1) \right].$$

$$L_{6,14} : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_4 a_3^2 - a_2 a_4^2 - a_1 a_4 a_5 - a_1 a_2 a_6}{a_1 a_2 a_4}, \frac{a_4^2 - a_3 a_5 - a_1 a_6}{a_1 a_3}, \right. \right. \\ \left. \left. \frac{(a_2 a_3 + a_1 a_4) a_5^2 - a_1 a_3 a_4 a_6}{a_1 a_2 a_3 a_4} \right), \left(\frac{27}{200}, \frac{3}{40}, 1, 3, 5, \frac{800}{27} \right) \right].$$

$$L_{6,15} : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2}, \frac{a_2 a_4^2 - a_2 a_3 a_5 - a_1 a_3 a_6}{a_1 a_2 a_3}, \right. \right. \\ \left. \left. \frac{(a_2 a_3 + a_1 a_4) a_5^2 - a_2 a_3 a_4 a_6}{a_1 a_2 a_3 a_4} \right), \left(\frac{4}{3}, \frac{4}{3}, 2, 1, 1, 1 \right) \right].$$

$$L_{6,16} : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_4 a_3^2 - a_2 a_4^2 - a_1 a_2 a_6}{a_1 a_2 a_4}, \frac{a_4^2 - a_3 a_5 - a_1 a_6}{a_1 a_2 a_3}, \frac{a_2 a_5^2 - a_1 a_4 a_6}{a_1 a_2 a_4} \right), \right. \\ \left. \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, 1, 1 \right) \right].$$

$$L_{6,17} : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4 - a_1 a_6}{a_1 a_2}, \frac{a_4^2 - a_3 a_5}{a_1 a_3}, \frac{a_5^2 - a_4 a_6}{a_1 a_4} \right), \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1 \right) \right].$$

$$L_{6,18} : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}, \frac{a_4^2 - a_3 a_5}{a_1 a_3}, \frac{a_5^2 - a_4 a_6}{a_1 a_4} \right), (1, 1, 1, 1, 1, 1) \right].$$

$$L_{6,19}(\epsilon \neq 0) : \left[(e_6, e_4, e_5, e_1, e_2, e_3), \text{diag} \left(\frac{a_4^2 - a_1 a_6}{a_1 a_2}, \frac{a_5^2 - a_1 a_6}{a_1 a_3} \right), (1, 1, 1, 1, 1, 1) \right].$$

$$L_{6,20} : \left[(e_6, e_4, e_5, e_1, e_2, e_3), \text{diag} \left(\frac{a_4^2 - a_1 a_6}{a_1 a_2}, \frac{a_5^2 - a_3 a_6}{a_1 a_3} \right), (1, 1, 1, 1, 1, 1) \right].$$

$$L_{6,21}(0) : \left[(e_5, e_6, e_3, e_4, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2}, \frac{a_4^2 - a_3 a_6}{a_1 a_3} \right), (2, \sqrt{2}, 2, \sqrt{2}, 1, 1) \right].$$

$$L_{6,21}(\epsilon \neq 0) : \left[(e_6, e_3, e_4, e_5, e_1, e_2), \text{diag} \left(\frac{a_3^2 - a_1 a_4 - a_2 a_5}{a_1 a_2}, \frac{a_4^2 - a_3 a_6}{a_1 a_3}, \frac{a_5^2 - a_3 a_6}{a_2 a_3} \right), \left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1 \right) \right].$$

To complete the proof, we treat now the seven remaining Lie algebras using a case by case approach.

• **The Lie algebra $L_{6,11}$.**

This is the only Lie algebra in the list which has no nice basis. Its center is contained in its derived ideal. We have $\mathfrak{g} = L_{6,11} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_6, [e_2, e_5] = e_6$$

and $\text{Sign}(\mathfrak{g}) = \{(3 + m^-, m^0, 1 + m^+), m^- + m^0 + m^+ = 2\}$. We consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,11}$ such that $\mathbb{B} = (e_6, e_3, e_4, e_1, e_2, e_5)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$. It is obvious that \mathbb{B} is an orthogonal characteristic basis and, according to Lemma 1, the signature of $\langle \cdot, \cdot \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of the characteristic matrix $R(\text{ric}, \mathbb{B})$. Now a direct computation using (7) and (6) gives

$$R(\text{ric}, \mathbb{B}) = \begin{pmatrix} \frac{a_3^2 - a_2 a_4}{a_1 a_2} & 0 \\ 0 & \frac{a_4^2 - a_3 a_6}{a_1 a_3} \end{pmatrix}.$$

If we take $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ we get $R(\text{ric}, \mathbb{B}) = 0$ and we can use the inverse function theorem trick. So, for a suitable choice of a_i , $R(\text{ric}, \mathbb{B})$ can have all the possible signatures which prove the theorem for $L_{6,11}$.

• **The Lie algebra $L_{5,3}$.**

We have $\mathfrak{g} = L_{5,3} = \text{span}\{e_1, \dots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4,$$

and $\text{Sign}(\mathfrak{g}) = \{(2, 1, 2), (2, 2, 1), (3, 0, 2), (3, 1, 1), (4, 0, 1)\}$. In this case, the parameter p in (1) has two values $p = 0$ or 1 , so to realize the signatures in $\text{Sign}(\mathfrak{g})$, we will consider two types of Euclidean inner products on $L_{5,3}$. The first ones are those satisfying $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$ and the second ones are those satisfying $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 0$.

For the first type, consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{5,3}$ for which $\mathbb{B} = (e_4, e_3, e_5, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$ and it is also nice. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(2 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Now a direct computation using (8) gives $R(\text{ric}, \mathbb{B}) = (2\text{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}\right)$ and, for suitable values of the a_i , the Ricci signatures of $\langle \cdot, \cdot \rangle$ are $(2, 1, 2)$, $(2, 2, 1)$ or $(3, 1, 1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5) = (e_4, e_3, e_5 + e_3 + e_1, e_1, e_2)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_3] = -f_1, [f_2, f_4] = -f_1, [f_3, f_4] = -f_1, [f_3, f_5] = f_2, [f_4, f_5] = f_2.$$

Consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{5,3}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We have chosen \mathbb{B} and $\langle \cdot, \cdot \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way a (1×1) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = a_4 = a_5 = 1$ and $a_3 = 2$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \left(\frac{12a_2^4 + 6a_2^3 + 9a_2^2 - a_2 - 3}{4(2a_2^2 + a_2 + 2)}\right).$$

It is clear that we can choose a_2 such that the signature of $\langle \cdot, \cdot \rangle$ is $(3, 0, 2)$ or $(4, 0, 1)$. This completes the proof for $L_{5,3}$.

• **The Lie algebra $L_{6,3}$.**

The treatment is similar to $L_{5,3}$ with a slight difference, the parameter p takes 1 or 2. We have $\mathfrak{g} = L_{6,3} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4,$$

and $\text{Sign}(\mathfrak{g}) = \{(2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1)\}$.

For the first type, consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,3}$ for which $\mathbb{B} = (e_4, e_3, e_5, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 2$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$ and it is also nice. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(2 + m^-, 2 + m^0, 1 + m^+)$ where (m^-, m^0, m^+)

is the signature of $R(\text{ric}, \mathbb{B})$. Now a direct computation using (8) gives $R(\text{ric}, \mathbb{B}) = (2\text{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}\right)$ and, for suitable values of the a_i , the Ricci signatures of \langle , \rangle are $(2, 2, 2)$, $(2, 3, 1)$ or $(3, 2, 1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_4, e_3, e_5, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_4] = -f_1, [f_2, f_6] = -f_1, [f_4, f_5] = f_2, [f_4, f_6] = f_1, [f_5, f_6] = -f_2.$$

Consider the Euclidean inner product \langle , \rangle on $L_{6,3}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We have chosen \mathbb{B} and \langle , \rangle such that $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then \mathbb{B} is a characteristic basis for \langle , \rangle . Then according to Lemma 1 the Ricci signature of \langle , \rangle is $(3 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way a (1×1) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = a_3 = a_4 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \left(\frac{-4a_2^5 + 2a_2^3 + 3a_2 - 2}{1 - a_2 - 2a_2^3}\right).$$

It is clear that we can choose a_2 such that the signature of \langle , \rangle is $(3, 1, 2)$ or $(4, 1, 1)$. This completes the proof for $L_{6,3}$.

• **The Lie algebra $L_{6,5}$.**

The treatment is similar to $L_{5,3}$. We have $\mathfrak{g} = L_{6,5} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$$

and $\text{Sign}(\mathfrak{g}) = \{(3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)\}$.

For the first type, consider the Euclidean inner product \langle , \rangle on $L_{6,5}$ for which $\mathbb{B} = (e_5, e_3, e_6, e_1, e_2, e_4)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then \mathbb{B} is a characteristic basis for \langle , \rangle and it is also nice. Then according to Lemma 1 the Ricci signature of \langle , \rangle is $(3 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Now a direct computation using (8) gives $R(\text{ric}, \mathbb{B}) = (2\text{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2 a_5}{a_1 a_2}\right)$ and, for suitable values of the a_i , the Ricci signatures of \langle , \rangle are $(3, 1, 2)$, $(3, 2, 1)$ or $(4, 1, 1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_1, e_2, e_4, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_3] = -f_1, [f_2, f_6] = -f_1, [f_3, f_4] = f_2, [f_3, f_6] = f_1, [f_4, f_5] = f_1, [f_4, f_6] = -f_2.$$

Consider the Euclidean inner product \langle , \rangle on $L_{6,5}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We have chosen \mathbb{B} and \langle , \rangle such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$. Then \mathbb{B} is a characteristic basis for \langle , \rangle . Then according to Lemma 1 the Ricci signature of \langle , \rangle is $(4 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way a (1×1) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = a_3 = a_4 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \left(\frac{4a_2^6 + 6a_2^5 + 6a_2^4 - a_2^3 - 3a_2^2 - 3a_2 - 1}{a_2(2a_2^3 + 3a_2^3 + 2a_2 + 1)}\right).$$

It is clear that we can choose $a_2 > 0$ such that the signature of \langle , \rangle is $(4, 0, 2)$ or $(5, 0, 1)$. This completes the proof for $L_{6,5}$.

• **The Lie algebra $L_{6,9}$.**

The treatment is similar to $L_{5,3}$ and $L_{6,5}$. We have $\mathfrak{g} = L_{6,9} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5,$$

and $\text{Sign}(\mathfrak{g}) = \{(2, 1, 3), (2, 2, 2), (3, 0, 3), (3, 1, 2), (4, 0, 2)\}$.

For the first type, consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,9}$ for which $\mathbb{B} = (e_5, e_4, e_3, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$ and it is also nice. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(2 + m^-, 1 + m^0, 2 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Now a direct computation using (8) gives $R(\text{ric}, \mathbb{B}) = (2\text{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2(a_4 + a_5)}{a_1 a_2}\right)$ and, for suitable values of the a_i , the Ricci signatures of $\langle \cdot, \cdot \rangle$ are $(2, 1, 3)$, $(2, 2, 2)$ or $(3, 1, 2)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_4, e_3, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_3, f_4] = -f_2, [f_3, f_5] = -f_1, [f_3, f_6] = -f_2, [f_4, f_5] = f_3, [f_4, f_6] = f_2, [f_5, f_6] = f_1 - f_3.$$

Consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,9}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We have chosen \mathbb{B} and $\langle \cdot, \cdot \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(3 + m^-, m^0, 2 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way a (1×1) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = a_2 = a_3 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \left(\frac{12 - a_4 - 35a_4^2}{2(8a_4 + 3)a_4}\right).$$

It is clear that we can choose a_4 such that the signature of $\langle \cdot, \cdot \rangle$ is $(3, 0, 3)$ or $(4, 0, 2)$. This completes the proof for $L_{6,9}$.

• **The Lie algebra $L_{6,6}$.**

The situation here is different from the precedent cases. We still have two types of Euclidean products but the order of the reduced matrix of the Ricci curvature is 2. We have $\mathfrak{g} = L_{6,6} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$$

and $\text{Sign}(\mathfrak{g}) = \{(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)\}$.

For the first type, consider the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,6}$ for which $\mathbb{B} = (e_5, e_3, e_4, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$ and it is also nice. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(2 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B}) = \text{diag}(2\text{ric}(e_3, e_3), 2\text{ric}(e_4, e_4))$. Now a direct computation using (7) gives

$$2\text{ric}(e_3, e_3) = \frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2} \quad \text{and} \quad 2\text{ric}(e_4, e_4) = \frac{a_4^2 - a_3 a_5}{a_1 a_3}.$$

If we take $a_1 = 6, a_2 = 5, a_3 = 4, a_4 = 2, a_5 = a_6 = 1$, we get $R(\text{ric}, \mathbb{B}) = 0$ and we can apply the inversion theorem trick to get that for a suitable choice of the a_i the Ricci signature of \langle , \rangle is $(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1)$ or $(4, 1, 1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6)$ and the Euclidean inner product \langle , \rangle on $L_{6,6}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We choose \mathbb{B} and \langle , \rangle such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$.

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_4] = -f_3, [f_2, f_5] = -f_1, [f_3, f_4] = -f_1, [f_4, f_5] = f_2, [f_4, f_6] = f_3, [f_5, f_6] = f_1.$$

Then \mathbb{B} is a characteristic basis for \langle , \rangle and is not nice. Then, according to Lemma 1, the Ricci signature of \langle , \rangle is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Now, a direct computation using (7) and (6) gives

$$R(\text{ric}, \mathbb{B}) = \text{diag} \left(\frac{a_2^2}{a_4 a_5}, \frac{-a_1 a_2 a_6 + a_3^2 (a_2 + a_6)}{a_2 a_4 a_6} \right).$$

Thus, for suitable values of a_i , the signatures $(3, 0, 3)$ and $(4, 0, 2)$ are realizable as the Ricci signature of \langle , \rangle .

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing Lie brackets are

$$[f_2, f_4] = -f_3, [f_2, f_5] = -f_1, [f_2, f_6] = -f_3, [f_3, f_4] = -f_1, [f_3, f_6] = -f_1, [f_4, f_5] = f_2, [f_4, f_6] = f_3, [f_5, f_6] = -f_2 + f_1.$$

Then \mathbb{B} is a characteristic for \langle , \rangle . According to Lemma 1, the Ricci signature of \langle , \rangle is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way a (2×2) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = 3, a_2 = a_4 = a_5 = 2 = a_6 = 1$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \text{diag} \left(-\frac{18 + 66a_3 + 121a_3^2 + 120a_3^3 + 73a_3^4 + 24a_3^5}{18 + 36a_3 + 34a_3^2 + 22a_3^3 + 6a_3^4}, \frac{-57 + 8a_3^2}{8} \right)$$

It is clear that for suitable values of a_3 , the signature $(5, 0, 1)$ is realizable as the Ricci signature of \langle , \rangle . This completes the proof for $L_{6,6}$.

• **The Lie algebra $L_{6,7}$.**

The treatment is similar to $L_{6,6}$. We have $\mathfrak{g} = L_{6,7} = \text{span}\{e_1, \dots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$$

and $\text{Sign}(\mathfrak{g}) = \{(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)\}$.

For the first type, consider the Euclidean inner product \langle , \rangle on $L_{6,7}$ for which $\mathbb{B} = (e_5, e_3, e_4, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then \mathbb{B} is a characteristic basis for \langle , \rangle and it is also nice. Then according to Lemma 1 the Ricci signature of \langle , \rangle is $(2 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B}) = \text{diag}(2\text{ric}(e_3, e_3), 2\text{ric}(e_4, e_4))$. Now a direct computation using (7) gives

$$2\text{ric}(e_3, e_3) = \frac{a_3^2 - a_2 a_4}{a_1 a_2} \quad \text{and} \quad 2\text{ric}(e_4, e_4) = \frac{a_4^2 - a_3 a_5}{a_1 a_3}.$$

Table 3
Realizable Ricci signatures on nilpotent Lie groups of dimension ≤ 6 .

Lie algebra \mathfrak{g}	Realizable Ricci signatures
$L_{3,2}$	(2, 0, 1)
$L_{4,2}$	(2, 1, 1)
$L_{4,3}$	(2, 1, 1), (2, 0, 2), (3, 0, 1)
$L_{5,2}$	(2, 2, 1)
$L_{5,3}$	(2, 1, 2), (2, 2, 1), (3, 0, 2), (3, 1, 1), (4, 0, 1)
$L_{5,4}$	(4, 0, 1)
$L_{5,5}$	(3, 0, 2), (3, 1, 1), (4, 0, 1)
$L_{5,6}, L_{5,7}$	(2, 0, 3), (2, 1, 2), (2, 2, 1), (3, 0, 2), (3, 1, 1), (4, 0, 1)
$L_{5,8}$	(3, 0, 2)
$L_{5,9}$	(2, 0, 3), (2, 1, 2), (3, 0, 2)
$L_{6,2}$	(2, 3, 1)
$L_{6,3}$	(2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1)
$L_{6,4}$	(4, 1, 1)
$L_{6,5}$	(3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)
$L_{6,6}, L_{6,7}$	(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)
$L_{6,8}$	(3, 1, 2)
$L_{6,9}$	(2, 1, 3), (2, 2, 2), (3, 0, 3), (3, 1, 2), (4, 0, 2)
$L_{6,10}$	(4, 0, 2), (4, 1, 1), (5, 0, 1)
$L_{6,11}, L_{6,12}, L_{6,13}, L_{6,20},$ $L_{6,19}(\epsilon), \epsilon \in \{-1, 1\}$	(3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)
$L_{6,14}, L_{6,15}, L_{6,16}, L_{6,17},$ $L_{6,18}, L_{6,21}(\epsilon), \epsilon \in \{-1, 1\}$	(2, 0, 4), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)
$L_{6,19}(0), L_{6,23}, L_{6,25},$ $L_{6,24}(\epsilon), \epsilon \in \{-1, 0, 1\}$	(3, 0, 3), (3, 1, 2), (4, 0, 2)
$L_{6,21}(0)$	(2, 0, 4), (2, 1, 3), (2, 2, 2), (3, 0, 3), (3, 1, 2), (4, 0, 2)
$L_{6,22}(\epsilon), \epsilon \in \{-1, 0, 1\}$	(4, 0, 2)
$L_{6,26}$	(3, 0, 3)

If we take $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ we get $R(\text{ric}, \mathbb{B}) = 0$ and we can apply the inversion theorem trick to get that for a suitable choice of the a_i the Ricci signature of $\langle \cdot, \cdot \rangle$ is (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1) or (4, 1, 1).

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6)$ and the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $L_{6,7}$ for which \mathbb{B} is orthogonal and $a_i = \langle f_i, f_i \rangle$. We choose \mathbb{B} and $\langle \cdot, \cdot \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$.

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_4] = -f_3, [f_3, f_4] = -f_1, [f_4, f_5] = f_2, [f_4, f_6] = f_3.$$

Then \mathbb{B} is a characteristic basis for $\langle \cdot, \cdot \rangle$ and is not nice. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Now a direct computation using (7) and (6) gives

$$R(\text{ric}, \mathbb{B}) = \text{diag} \left(\frac{a_2^2}{a_4 a_5}, \frac{a_2 a_3^2 + (-a_1 a_2 + a_3^2) a_6}{a_2 a_4 a_6} \right).$$

Thus for suitable values of a_i , the signatures (3, 0, 3) and (4, 0, 2) are realizable as the Ricci signature of $\langle \cdot, \cdot \rangle$.

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing brackets are

$$[f_2, f_4] = -f_3, [f_2, f_6] = -f_3, [f_3, f_4] = -f_1, [f_3, f_6] = -f_1, [f_4, f_5] = f_2, [f_4, f_6] = f_3, [f_5, f_6] = -f_2$$

Then \mathbb{B} is a characteristic for $\langle \cdot, \cdot \rangle$. Then according to Lemma 1 the Ricci signature of $\langle \cdot, \cdot \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $R(\text{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because \mathbb{B} is not a nice basis and the computation of $R(\text{ric}, \mathbb{B})$, which is by the way

a (2×2) -matrix, is complicated according to formula (6). We don't need to give the general expression of $R(\text{ric}, \mathbb{B})$, its value when $a_1 = 2, a_2 = a_3 = a_4 = a_6 = 1$ will suffice to our purpose. We get

$$R(\text{ric}, \mathbb{B}) = \text{diag} \left(\frac{8 + 17a_5 - 12a_5^2}{4(1 + 3a_5)a_5}, -2 \right)$$

It is clear that we can choose a_5 such that the signature of $\langle \cdot, \cdot \rangle$ is $(5, 0, 1)$. This completes the proof for $L_{6,7}$. \square

We end this work by giving all the realizable Ricci signatures on nilpotent Lie groups up to dimension 6 (Table 3).

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