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Special bi-invariant linear connections on Lie groups and finite dimensional Poisson structures ☆

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ABSTRACT

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. We denote by ∇^0 the torsion free bi-invariant linear connection on G given by $\nabla^0_X Y = \frac{1}{2}[X,Y]$, for any left invariant vector fields X, Y. A Poisson structure on $\mathfrak{g}$ is a commutative and associative product on $\mathfrak{g}$ for which ad_u is a derivation, for any $u \in \mathfrak{g}$. A torsion free bi-invariant linear connections on G which have the same curvature as ∇^0 are called special. We show that there is a bijection between the space of special connections on G and the space of Poisson structures on $\mathfrak{g}$. We compute the holonomy Lie algebra of a special connection and we show that the Poisson structures associated to special connections which have the same holonomy Lie algebra as ∇^0 possess interesting properties. Finally, we study Poisson structures on a Lie algebra and we give a large class of examples which gives, of course, a large class of special connections.

1. Introduction

A Poisson algebra is both a Lie algebra and a commutative associative algebra which are compatible in a certain sense. Poisson algebras play important roles in many fields in mathematics and mathematical physics, such as the Poisson geometry, integrable systems, non-commutative (algebraic or differential) geometry, and so on. Finite dimensional Poisson algebras constitute an interesting topic in algebra and were studied by many authors (see for instance [13,19,21]).

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More precisely, a *Poisson algebra* is a Lie algebra $(\mathfrak{g}, [\ ,\ ])$ endowed with a commutative and associative product $\circ$ such that, for any $u, v, w \in \mathfrak{g}$,

$$[u, v \circ w] = [u, v] \circ w + v \circ [u, w]. \tag{1}$$

An algebra $(\mathrm{A}, .)$ is called *Poisson admissible* if $(\mathrm{A}, [\ ,\ ], \circ)$ is a Poisson algebra, where

$$[u, v] = u.v - v.u \quad \text{and} \quad u \circ v = \frac{1}{2}(u.v + v.u). \tag{2}$$

This paper aims to give some new insights on finite dimensional Poisson algebras based on an interesting geometric interpretation of these structures when the field is either $\mathbb{R}$ or $\mathbb{C}$ (see Theorem 2.1). Let us present briefly this geometric interpretation.

Let G be a Lie group with $\mathfrak{g} = T_eG$ its Lie algebra. The linear connection ∇^0 given by $\nabla^0_X Y = \frac{1}{2}[X, Y]$, where X, Y are left invariant vector fields, is torsion free, bi-invariant, complete and its curvature K^0 is given by $K^0(X, Y) = -\frac{1}{4}\mathrm{ad}_{[X,Y]}$. Moreover, $\nabla^0 K^0 = 0$ and the holonomy Lie algebra of ∇^0 at e is $\mathfrak{h}^0 = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$. The main fact (see Section 2) is that there is a bijection between the set of Poisson structures on $\mathfrak{g}$ and the space of bi-invariant torsion free linear connections on G which have the same curvature as ∇^0. We call such connections *special*. Moreover, we show that any special connection is semi-symmetric, i.e., its curvature tensor K satisfies $K.K = 0$ (see Proposition 2.1). In general, the holonomy Lie algebra $\mathfrak{h}$ of a bi-invariant linear connection is difficult to compute, however, we show that, for a special connection, $\mathfrak{h}$ contains $\mathfrak{h}^0$ and can be easily computed (see Lemma 2.2). A special connection whose holonomy Lie algebra coincides with $\mathfrak{h}^0$ will be called *strongly special*. So, according to the bijection above, to any real Poisson algebra corresponds a unique special connection on any associated Lie group. Poisson algebras whose corresponding special connection is strongly special are particularly interesting. We call such Poisson algebras *strong*. A Poisson algebra whose corresponding special connection is parallel is called *parallel*. With this interpretation in mind, we devote Section 3 to the study of the general properties of Poisson algebras and Poisson admissible algebras and we give some general methods to build new Poisson algebras from old ones (see Theorem 3.2). We show that any symmetric Leibniz algebra is a strong Poisson admissible algebra and the curvature of the corresponding special connection is parallel (see Theorem 3.1). By using the geometric interpretation of Poisson structures, we get a large class of Lie groups which carry a bi-invariant connection ∇ (different from ∇^0) which has the same curvature and the same linear holonomy as ∇^0 and moreover the curvature of ∇ is parallel. We get hence interesting examples of connections with parallel torsion and curvature. Such connections were studied by Nomizu [20]. Recall that symmetric Leibniz algebras constitute a subclass of Leibniz algebras introduced by Loday in [18]. At the end of Section 3, we show that there is no non-trivial Poisson structure on a semi-simple Lie algebra (see Theorem 3.3). This result generalizes a result by [13]. In Section 4, we show that an associative algebra is Poisson admissible if and only if the underline Lie algebra is 2-nilpotent and we give a description of associative Poisson admissible algebras which permit to build many examples. Section 5 is devoted to the study of symplectic Poisson algebras. It is well-known that if $(\mathfrak{g}, \omega)$ is a symplectic Lie algebra there is a product α^{a} on $\mathfrak{g}$ which is Lie-admissible and left symmetric. When the Lie algebra is real, α^{a} defines a left invariant flat torsion free linear connection ∇^{a} on any associated Lie group G. By using the general method to build a torsion free symplectic connection from any torsion free connection introduced in [4], we get from ∇^{a} a left invariant torsion free connection ∇^{s} for which the left invariant symplectic form associated to ω is parallel. To our knowledge this connection has never been considered before. From ∇^{s} we get a product α^{s} on $\mathfrak{g}$. We show that $(\mathfrak{g}, \alpha^{\mathrm{a}})$ is Poisson admissible iff $(\mathfrak{g}, \alpha^{\mathrm{s}})$ is Poisson admissible and this is equivalent to $\mathfrak{g}$ is 2-nilpotent Lie algebra and $[\mathrm{ad}_u, \mathrm{ad}^*_v] = 0$ for any $u, v \in \mathfrak{g}$ where ad^*_u is the adjoint of ad_u with respect to ω. A symplectic Lie algebra satisfying these conditions is called *symplectic Poisson algebra*. We show that the symplectic double extension process introduced in [10] permits the construction of all symplectic Poisson algebras. Lie groups whose Lie algebras are symplectic Poisson possess an important geometric property (see Theorem 5.2 and the following remarks). In Section 6, we study the problem of metrizability of special connections. Indeed, we consider a real Lie algebra $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ endowed with a nondegenerate symmetric bilinear metric. We denote by ℓ the Levi-Civita product associated to $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$. We show that if $\langle\ ,\ \rangle$ is positive definite $(\mathfrak{g}, \ell)$ is Poisson admissible iff $\langle\ ,\ \rangle$ is bi-invariant and in this case the associated Poisson product $\circ$ is trivial. We give a description of $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ for which $(\mathfrak{g}, \ell)$ is Poisson admissible in the case where $[\mathfrak{g}, \mathfrak{g}]$ is nondegenerate and $\langle\ ,\ \rangle$ has any signature.

All vector spaces, algebras, etc. in this paper are finite dimensional and will be over a ground field $\mathbb{K}$ of characteristic 0.

2. Geometric interpretation of finite dimensional Poisson structures

We give in this section an interesting geometric interpretation of Poisson structures involving the theory of connections and holonomy algebras. This theory is a fundamental topic in differential geometry and has its origin in the work of Elie Cartan [5,6]. For a detailed account of this theory, see Ehresmann [11], Chern [9], Lichnerowicz [16], Nomizu [20], and Kobayashi [14]. Let us recall some classical facts about linear connections and state some formulas which will lead naturally to the desired interpretation.

Given a linear connection on a smooth manifold M, we consider the covariant differentiation ∇ associated to it. Let T^∇ and K^∇ be, respectively, the torsion and curvature tensor fields on M with respect to ∇:

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad \text{and} \quad K^\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

For any closed curve τ at $p \in M$, the parallel displacement along τ is a linear transformation of T_pM, and the totality of these linear transformations for all closed curves forms the *holonomy group* $H(p)$. The *restricted holonomy group* $H(p)^0$ is the subgroup consisting of parallel displacements along all closed curves which are homotopic to zero. Its Lie algebra is called *holonomy Lie algebra*. On the other hand, consider linear endomorphisms of T_pM of the form $K^\nabla(X,Y)$, $(\nabla_Z K^\nabla)(X,Y)$, $(\nabla_W \nabla_Z K^\nabla)(X,Y)$, ... (all covariant derivatives), where X, Y, Z, W, ... are arbitrary tangent vectors at p. All these linear endomorphisms span a subalgebra $\mathfrak{h}_p^\nabla$ of the Lie algebra consisting of all linear endomorphisms of T_pM. We call it the *infinitesimal holonomy Lie algebra*. The Lie subgroup of $\mathrm{GL}(T_pM, \mathbb{R})$ generated by $\mathfrak{h}_p^\nabla$ is the *infinitesimal holonomy group* at p. The main result is that if the infinitesimal holonomy group has the same dimension at every point p of M (which is the case when M and ∇ are analytic), then the restricted holonomy group is equal to the infinitesimal holonomy group at every point (see [20]). The linear connection ∇ will be called invariant under parallelism in case T^∇ and K^∇ are both parallel with respect to ∇. The existence of a linear connection ∇ invariant under parallelism characterize (at least locally) reductive homogeneous spaces (see [15]). If ∇ is invariant under parallelism then

$$\mathfrak{h}_p^\nabla = \left\{ \sum K^\nabla(u_i, v_i),\ u_i, v_i \in T_pM \right\}. \tag{3}$$

A vector field A is an infinitesimal ∇-transformation if and only if for any couple of vector fields X, Y,

$$[A, \nabla_X Y] = \nabla_{[A,X]} Y + \nabla_X [A,Y]. \tag{4}$$

On can see easily that this relation is equivalent to

$$\nabla^2_{X,Y} A + \left[\nabla_X, T_A^\nabla\right] Y = K^\nabla(X,A)Y, \tag{5}$$

where $\nabla^2_{X,Y} A = \nabla_X \nabla_Y A - \nabla_{\nabla_X Y} A$.

Let $\overline{\nabla}$ be another linear connections on M. One knows that $S = \overline{\nabla} - \nabla$ is a tensor field of type $(1, 2)$. By using a terminology due to Kostant, we will say that $\overline{\nabla}$ is rigid with respect to ∇ whenever S is parallel with respect to ∇. In this case, we have the following formula (see [15], Lemma 2):

$$K^{\overline{\nabla}}(X,Y) = K^{\nabla}(X,Y) + [S_X, S_Y] + S_{T^{\nabla}(X,Y)}. \tag{6}$$

Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra. For any $u \in \mathfrak{g}$ we denote by u^+ (resp. u^-) the left invariant (resp. the right invariant) vector field associated to u.

It is obvious that G is a reductive homogeneous space and hence, according to a result of Kostant (see [15], Theorem 2), G admits a linear connection invariant under parallelism. In fact G admits many such connections and we will use in this paper a special one, namely, the linear connection ∇^0 given by

$$\nabla^0_{u^+} v^+ = \frac{1}{2}\left[u^+, v^+\right],$$

for any $u, v \in \mathfrak{g}$. This connection is torsion free, invariant under parallelism, bi-invariant, complete and its curvature and holonomy Lie algebra are given by

$$K^{\nabla^0}\left(u^+, v^+\right)w^+ = -\frac{1}{4}\left[\left[u^+, v^+\right], w^+\right], \quad u, v, w \in \mathfrak{g}, \tag{7}$$

$$\mathfrak{h}_e^{\nabla^0} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}. \tag{8}$$

A linear connection ∇ on G is called bi-invariant if ∇ is invariant by left and right multiplication. The following lemma gives different characterizations of bi-invariant linear connections on G.

Lemma 2.1. *Let ∇ be a linear connection on G. Then the following assertions are equivalent:*

1. *∇ is a bi-invariant linear connection.*
2. *For any couple of left invariant (resp. right invariant) vector fields X, Y, $\nabla_X Y$ is left invariant (resp. right invariant).*
3. *For any couple of left invariant vector fields X, Y, $\nabla_X Y$ is left invariant and the product $\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by $\alpha(u, v) = (\nabla_{u^+} v^+)(e)$ satisfies*

$$\left[u, \alpha(v, w)\right] = \alpha\left([u, v], w\right) + \alpha\left(v, [u, w]\right). \tag{9}$$

4. *For any couple of right invariant vector fields X, Y, $\nabla_X Y$ is right invariant and the product $\beta : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ given by $\beta(u, v) = (\nabla_{u^-} v^-)(e)$ satisfies*

$$\left[u, \beta(v, w)\right] = \beta\left([u, v], w\right) + \beta\left(v, [u, w]\right).$$

5. *∇ is left invariant and rigid with respect to ∇^0.*
6. *∇ is right invariant and rigid with respect to ∇^0.*

Proof. Since G is connected, ∇ is bi-invariant if and only if, for any $u \in \mathfrak{g}$, u^+ and u^- are infinitesimal ∇-transformations, i.e., according to (4), for any couple of vector fields X, Y,

$$\left[u^+, \nabla_X Y\right] = \nabla_{[u^+,X]} Y + \nabla_X\left[u^+, Y\right] \quad \text{and} \quad \left[u^-, \nabla_X Y\right] = \nabla_{[u^-,X]} Y + \nabla_X\left[u^-, Y\right].$$

Since G is a parallelizable by left invariant vector field and these vector fields commute with right invariant vector fields, these equations are equivalent to

$$\left[u^+, \nabla_{v^+} w^+\right] = \nabla_{[u^+,v^+]} w^+ + \nabla_{v^+}\left[u^+, w^+\right] \quad \text{and} \quad \left[u^-, \nabla_{v^+} w^+\right] = 0, \quad v, w \in \mathfrak{g}.$$

The group G is also parallelizable by right invariant vector field and hence these equations are also equivalent to

$$\left[u^-, \nabla_{v^-} w^-\right] = \nabla_{[u^-,v^-]} w^- + \nabla_{v^-}\left[u^-, w^-\right] \quad \text{and} \quad \left[u^+, \nabla_{v^-} w^-\right] = 0, \quad v, w \in \mathfrak{g}.$$

On the other hand, ∇ is left invariant and rigid with respect to ∇^0 is equivalent to

$$\left[u^-, \nabla_{v^+} w^+\right] = 0 \quad \text{and} \quad \nabla^0_{u^+}\left(\nabla - \nabla^0\right)\left(v^+, w^+\right) = 0.$$

Or

$$\begin{aligned}
\nabla^0_{u^+}\left(\nabla - \nabla^0\right)\left(v^+, w^+\right) &= \nabla^0_{u^+}\left(\nabla_{v^+} w^+ - \nabla^0_{v^+} w^+\right) - \left(\nabla_{\nabla^0_{u^+} v^+} w^+ - \nabla^0_{\nabla^0_{u^+} v^+} w^+\right) \\
&\quad - \left(\nabla_{v^+} \nabla^0_{u^+} w^+ - \nabla^0_{v^+} \nabla^0_{u^+} w^+\right) \\
&= \frac{1}{2}\left[u^+, \nabla_{v^+} w^+\right] - \frac{1}{4}\left[u^+, \left[v^+, w^+\right]\right] - \frac{1}{2}\nabla_{[u^+,v^+]} w^+ \\
&\quad - \frac{1}{4}\left[w^+, \left[u^+, v^+\right]\right] - \frac{1}{2}\nabla_{v^+}\left[u^+, v^+\right] - \frac{1}{4}\left[v^+, \left[w^+, u^+\right]\right] \\
&= \frac{1}{2}\left[u^+, \nabla_{v^+} w^+\right] - \frac{1}{2}\nabla_{[u^+,v^+]} w^+ - \frac{1}{2}\nabla_{v^+}\left[u^+, v^+\right].
\end{aligned}$$

A similar computation holds when one replaces left invariant vector field by right ones. Now we can get the desired equivalences easily. □

Remark 1. For any $u, v \in \mathfrak{g}$, $u^+(e) = u^-(e)$ and $[v^+, u^-] = 0$, so we get

$$\left(\nabla_{u^+} v^+\right)(e) = \left(\nabla_{u^-} v^+\right)(e) = \left(\nabla_{v^+} u^-\right)(e) = \left(\nabla_{v^-} u^-\right)(e).$$

Thus

$$\alpha(u, v) = \beta(v, u).$$

Let ∇ be torsion free bi-invariant linear connection on G. As above, we define $S = \nabla - \nabla^0$. It is clear that S is bi-invariant and define a product $\circ$ on $\mathfrak{g}$. We have

$$u \circ v = \alpha(u, v) - \frac{1}{2}[u, v] = \frac{1}{2}\alpha(u, v) + \frac{1}{2}\alpha(v, u) = \frac{1}{2}\alpha(u, v) + \frac{1}{2}\beta(u, v).$$

This product is obviously commutative and, according to Lemma 2.1 3 and 4, satisfies (1). Since ∇ is rigid with respect to ∇^0, (6) holds and can be written for any $u, v \in \mathfrak{g}$,

$$K^{\nabla}(u, v) = K^{\nabla^0}(u, v) + [S_u, S_v].$$

Thus $K^{\nabla} = K^{\nabla^0}$ if and only if $[S_u, S_v] = 0$ for any $u, v \in \mathfrak{g}$, which is equivalent to $\circ$ is associative. Hence $\circ$ defines a Poisson structure on $\mathfrak{g}$ if and only if ∇ and ∇^0 have the same curvature. So we get the desired interpretation.

Theorem 2.1. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then the following assertions hold:*

1. *Let ∇ be a left invariant linear connection on G and let $\circ$ the product on $\mathfrak{g}$ given by*

$$u \circ v = \left(\nabla_{u^+} v^+\right)(e) - \frac{1}{2}[u, v].$$

Then $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra if and only if ∇ is torsion free, bi-invariant and has the same curvature as ∇^0.

2. *Let $\circ$ be a product on $\mathfrak{g}$ such that $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra. Then the linear connection on G given by*

$$\nabla_{u^+} v^+ = \frac{1}{2}\left[u^+, v^+\right] + (u \circ v)^+$$

is torsion free, bi-invariant and has the same curvature as ∇^0.

We call *special* a torsion free bi-invariant linear connection which has the same curvature as ∇^0.

In Riemannian geometry there is a notion of semi-symmetric spaces which is a direct generalization of locally symmetric spaces, namely, Riemannian manifolds for which the curvature tensor K satisfies $K.K = 0$, i.e.,

$$\nabla_X \nabla_Y K - \nabla_Y \nabla_X K - \nabla_{[X,Y]} K = 0, \tag{10}$$

for any vector fields X, Y. Semi-Riemannian symmetric spaces were investigated first by E. Cartan [7] and studied by many authors ([17,8,22] etc.). More generally, we call a torsion free linear connection on a manifold semi-symmetric if its curvature tensor satisfies (10).

Proposition 2.1. *Any special connection is semi-symmetric.*

Proof. Let ∇ be a special connection on a Lie group G. According to Theorem 2.1, its curvature K is given by

$$K(X, Y)Z = -\frac{1}{4}\left[[X, Y], Z\right],$$

for any left invariant vector fields X, Y, Z. Now, it was shown in [22], p. 532 that Eq. (10) is equivalent to

$$\left[K(U, V), K(X, Y)\right] = K\left(K(U, V)X, Y\right) + K\left(X, K(U, V)Y\right),$$

for any left invariant vector fields X, Y, U, V. By replacing K in this relation by its expression above we get the desired result. $\square$

Let ∇ be a left invariant linear connection on G. The holonomy Lie algebra is the smallest subalgebra $\mathfrak{h}^\nabla$ of $\mathrm{End}(\mathfrak{g})$ which contains all $K^\nabla(u, v)$ and satisfying $[\nabla_u, \mathfrak{h}^\nabla] \subset \mathfrak{h}^\nabla$ for any $u \in \mathfrak{g}$ (see [20]). It is clear that it is difficult to compute $\mathfrak{h}^\nabla$ explicitly. However, the holonomy algebra of a special linear connection can be computed easily.

Lemma 2.2. *Let ∇ be a special connection on G. Then the holonomy Lie algebra of ∇ is given by*

$$\mathfrak{h}_e^\nabla = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{L}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{R}_{[[\mathfrak{g},\mathfrak{g}],\mathfrak{g}]},$$

where $\mathrm{L}, \mathrm{R} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ are given by $\mathrm{L}_u = \alpha(u, .)$ and $\mathrm{R}_u = \alpha(., u)$ and $\alpha(u, v) = (\nabla_{u^+} v^+)(e)$.

Proof. This is a consequence of the following formulas which hold for any special connection. We have, for any $u, v \in \mathfrak{g}$,

$$[\mathrm{ad}_u, \mathrm{L}_u] = \mathrm{L}_{[u,v]}, \qquad [\mathrm{ad}_u, \mathrm{R}_u] = \mathrm{R}_{[u,v]}, \qquad [\mathrm{L}_u, \mathrm{L}_v] = \mathrm{L}_{[u,v]} - \frac{1}{4}\mathrm{ad}_{[u,v]},$$
$$[\mathrm{R}_u, \mathrm{R}_v] = -\mathrm{R}_{[u,v]} - \frac{1}{4}\mathrm{ad}_{[u,v]}.$$

These formulas will be stated rigorously in the next section. □

A special connection which has also the same holonomy Lie algebra as ∇^0 is called *strongly special.*

3. Poisson algebras and Poisson admissible algebras

In this section, we study Poisson algebras and Poisson admissible algebras in algebraic point of view, having in mind the results of the previous section.

Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional Lie algebra and $\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ a product on $\mathfrak{g}$. For any $u \in \mathfrak{g}$, we define $\mathrm{L}_u, \mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\mathrm{L}_u v = \alpha(u, v) \quad \text{and} \quad \mathrm{R}_u v = \alpha(v, u).$$

Suppose that α is Lie-admissible, i.e., for any $u, v \in \mathfrak{g}$,

$$\alpha(u, v) - \alpha(v, u) = [u, v].$$

Suppose also that α is bi-invariant, i.e., it satisfies (9). It is obvious that the product $\circ$ on $\mathfrak{g}$ given by

$$u \circ v = \alpha(u, v) - \frac{1}{2}[u, v] = \frac{1}{2}\alpha(u, v) + \frac{1}{2}\alpha(v, u) \tag{11}$$

is commutative. It is also bi-invariant which is equivalent to

$$[S_u, \mathrm{ad}_v] = S_{[u,v]}, \tag{$*$}$$

for any $u, v \in \mathfrak{g}$ where $S_u v = u \circ v$. If we denote by K^α the curvature of α, we get that

$$\begin{aligned} K^\alpha(u, v) &:= [\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]} \\ &= \left[S_u + \frac{1}{2}\mathrm{ad}_u, S_v + \frac{1}{2}\mathrm{ad}_v\right] - S_{[u,v]} - \frac{1}{2}\mathrm{ad}_{[u,v]} \\ &\overset{(*)}{=} [S_u, S_v] - \frac{1}{4}\mathrm{ad}_{[u,v]}. \end{aligned}$$

This formula is the infinitesimal analogous of (6). The product $\circ$ being commutative, it is associative if, for any $u, v \in \mathfrak{g}$, $[S_u, S_v] = 0$. Thus we have the following result.

Proposition 3.1. *Let $\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ a Lie-admissible product on $\mathfrak{g}$ and $\circ$ given by (11). Then $(\mathfrak{g}, [\ ,\ ], \circ)$ is a Poisson algebra if and only if α is bi-invariant and, for any $u, v \in \mathfrak{g}$,*

$$K^\alpha(u, v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}.$$

According to this proposition and Lemma 2.2, we can introduce the following definition.

Definition 3.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra.

1. A product α on $\mathfrak{g}$ is called **quasi-canonical** if it is Lie-admissible and, for any $u, v \in \mathfrak{g}$,
$$[\mathrm{L}_u, \mathrm{ad}_v] = \mathrm{L}_{[u,v]} \quad \text{and} \quad K^\alpha(u,v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}.$$
2. The holonomy Lie algebra of a quasi-canonical product α on $\mathfrak{g}$ is the subalgebra of the Lie algebra $\mathrm{End}(\mathfrak{g})$ given by
$$\mathfrak{h}^\alpha = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{L}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} + \mathrm{R}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]}.$$
3. A quasi-canonical product α is called **parallel** if its curvature is parallel, i.e.,
$$\mathrm{L}_{[u,[v,w]]} = \mathrm{ad}_{[\mathrm{ad}_u v, w]} + \mathrm{ad}_{[v, \mathrm{ad}_u w]}.$$
4. A quasi-canonical product α is called **strongly quasi-canonical** if $\mathfrak{h}^\alpha = \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$, i.e., $\mathrm{L}_{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]} \subset \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$.

It is obvious that any parallel quasi-canonical product is strongly quasi-canonical.

According to Proposition 3.1 there is a correspondence between the set of Poisson products on a Lie algebra $\mathfrak{g}$ and the set of quasi-canonical products on $\mathfrak{g}$. We call *parallel* (resp. *strong*) a Poisson product whose corresponding quasi-canonical product is parallel (resp. strongly quasi-canonical). The corresponding quasi-canonical product to the trivial Poisson product is $\alpha_0(u,v) = \frac{1}{2}[u,v]$.

Let α be a quasi-canonical product on a Lie algebra $\mathfrak{g}$. Then (9) is equivalent to
$$[\mathrm{ad}_u, \mathrm{L}_v] = \mathrm{L}_{[u,v]}, \tag{12}$$
or
$$[\mathrm{ad}_u, \mathrm{R}_v] = \mathrm{R}_{[u,v]}, \tag{13}$$
for any $u, v \in \mathfrak{g}$. Since $\mathrm{ad}_u = \mathrm{L}_u - \mathrm{R}_u$, we get when we replace ad_u in (12) that
$$K^\alpha(u,v) = [\mathrm{R}_u, \mathrm{L}_v]. \tag{14}$$
Note that the curvature of α vanishes if and only if α is associative. In this case, the Lie algebra $\mathfrak{g}$ is 2-nilpotent because $K^\alpha(u,v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}$.

Let us give now some properties of Poisson admissible algebras. Recall that an algebra $(\mathrm{A}, .)$ is called *Poisson admissible* if $(\mathrm{A}, [\ ,\ ], \circ)$ is a Poisson algebra, where
$$[u,v] = u.v - v.u \quad \text{and} \quad u \circ v = \frac{1}{2}(u.v + v.u).$$
Note that
$$u.v = \frac{1}{2}[u,v] + u \circ v.$$

Remark 2.

1. For any Poisson admissible algebra $(\mathrm{A}, .)$ we denote by $\mathfrak{g}^A$ the associated Lie algebra.
2. Any Lie algebra is (trivially) Poisson admissible.
3. Any associative commutative algebra is Poisson admissible. In this case $\mathfrak{g}^A$ is abelian.

Proposition 3.2. *Let* $(\mathrm{A},.)$ *be an algebra. For any* $u \in \mathrm{A}$*, we denote by* $\mathrm{L}_u, \mathrm{R}_u : \mathrm{A} \longrightarrow \mathrm{A}$ *the endomorphisms given by* $\mathrm{L}_u v = u.v$ *and* $\mathrm{R}_u v = v.u$*. Then the following conditions are equivalent:*

1. $(\mathrm{A},.)$ *is a Poisson admissible algebra.*
2. *For any* $u, v \in \mathrm{A}$*,*
$$[\mathrm{L}_u, \mathrm{L}_v] + [\mathrm{R}_u, \mathrm{R}_v] + 2[\mathrm{L}_u, \mathrm{R}_v] = 0 \quad \textit{and} \quad K(u,v) = [\mathrm{R}_u, \mathrm{L}_v],$$
where $K(u,v) := [\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]}$.
3. *For any* $u, v \in \mathrm{A}$*,*
$$[\mathrm{R}_u, \mathrm{R}_v] + \mathrm{L}_{[u,v]} + 3[\mathrm{L}_u, \mathrm{R}_v] = 0.$$
4. *For any* $u, v \in \mathrm{A}$*,*
$$[\mathrm{L}_u, \mathrm{L}_v] - \mathrm{R}_{[u,v]} + 3[\mathrm{R}_u, \mathrm{L}_v] = 0.$$

Proof. For any $u, v \in \mathrm{A}$, put $\mathrm{L}_u v = \mathrm{R}_v u = u.v$, $[u,v] = u.v - v.u$, $u \circ v = \frac{1}{2}(u.v + v.u)$ and $K(u,v) = [\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]}$.

The algebra $(\mathrm{A},.)$ is Poisson admissible if and only if $(\mathrm{A},[\ ,\ ])$ is a Lie algebra and "." is quasi-canonical with respect to $(\mathrm{A},[\ ,\ ])$. This is equivalent to

- $K(u,v)w + K(v,w)u + K(w,u)v = 0$ (Bianchi identity),
- $K(u,v) \overset{(14)}{=} [\mathrm{R}_u, \mathrm{L}_v]$,
- $[\mathrm{L}_u + \mathrm{R}_u, \mathrm{L}_v + \mathrm{R}_v] = 0$ (the associativity of $\circ$),

for any $u, v \in \mathrm{A}$. Since from the second condition we deduce that $[\mathrm{R}_u, \mathrm{L}_v] = -[\mathrm{R}_v, \mathrm{L}_u]$ (the flexibility), the conditions above are equivalent to

- $K(u,v)w + K(v,w)u + K(w,u)v = 0$,
- $[\mathrm{L}_u, \mathrm{L}_v] - \mathrm{L}_{[u,v]} = [\mathrm{R}_u, \mathrm{L}_v]$,
- $[\mathrm{L}_u, \mathrm{L}_v] + [\mathrm{R}_u, \mathrm{R}_v] + 2[\mathrm{L}_u, \mathrm{R}_v] = 0$.

So statement 1 implies statement 2. Now it is obvious that statement 2 implies statement 3.

Let us show now that the third condition implies the first one. Note first that

$$K(u,v)w = \mathrm{ass}(v,u,w) - \mathrm{ass}(u,v,w) \tag{$*$}$$

where

$$\mathrm{ass}(u,v,w) = (u.v).w - u.(v.w) = [\mathrm{R}_w, \mathrm{L}_u](v). \tag{$**$}$$

Moreover, the third condition implies that $\mathrm{ass}(u,v,w) = -\mathrm{ass}(w,v,u)$ and hence the product "." is Lie-admissible if and only if

$$\mathrm{ass}(u,v,w) + \mathrm{ass}(v,w,u) + \mathrm{ass}(w,u,v) = 0. \tag{$*\!*\!*$}$$

Now from 3 we deduce that

$$3\mathrm{ass}(u, v, w) = \mathrm{R}_u \circ \mathrm{R}_w(v) - \mathrm{R}_w \circ \mathrm{R}_u(v) + \mathrm{R}_v \circ \mathrm{R}_w(u) - \mathrm{R}_v \circ \mathrm{R}_u(w)$$

and we get easily that "." is Poisson-admissible and consequently $K(u, v) = [\mathrm{R}_u, \mathrm{L}_v]$. Now this relation and 3 implies that

$$[\mathrm{L}_u, \mathrm{L}_v] + [\mathrm{R}_u, \mathrm{R}_v] + 2[\mathrm{L}_u, \mathrm{R}_v] = 0.$$

The condition 4 is equivalent to 3 is a consequence of the following remark. If $(\mathrm{A}, .)$ is an algebra and $\star$ is the product given by $u \star v = -v.u$ then $(\mathrm{A}, .)$ is Poisson admissible if and only if $(\mathrm{A}, \star)$ is Poisson admissible. In this case the structures of Lie algebras of $(\mathrm{A}, .)$ and $(\mathrm{A}, \star)$ coincident. □

Poisson admissible algebras are a subclass of Lie-admissible flexible algebras studied in [1]. Recall that an algebra is called *flexible* if its associator satisfies

$$\mathrm{ass}(u, v, w) + \mathrm{ass}(w, v, u) = 0,$$

for any u, v, w. The third characterization of Poisson admissible algebras in Proposition 3.2 appears first in [19].

Corollary 3.1. *An associative algebra* $(\mathrm{A}, .)$ *is Poisson admissible if and only if* $\mathfrak{g}^A$ *is a 2-nilpotent Lie algebra.*

An algebra $(\mathrm{A}, .)$ is called LR-algebra if, for any $u, v \in \mathrm{A}$,

$$[\mathrm{L}_u, \mathrm{L}_v] = [\mathrm{R}_u, \mathrm{R}_v] = 0.$$

It follows from Proposition 3.2 that an LR-algebra is Poisson admissible if and only if it is associative.

Let us introduce now an important class of strong Poisson admissible algebras. A *left Leibniz algebra* is an algebra $(\mathrm{A}, .)$ such that for any $u \in \mathrm{A}$, the left multiplication L_u is a derivation, i.e., for any $v, w \in \mathrm{A}$,

$$u.(v.w) = (u.v).w + v.(u.w).$$

This is equivalent to one of the two following relations

$$[\mathrm{L}_u, \mathrm{L}_v] = \mathrm{L}_{uv} \quad \text{or} \quad [\mathrm{L}_u, \mathrm{R}_v] = \mathrm{R}_{uv}. \tag{15}$$

A *right Leibniz algebra* is an algebra $(\mathrm{A}, .)$ such that, for any $u \in \mathrm{A}$, the right multiplication R_u is a derivation, i.e., for any $v, w \in \mathrm{A}$,

$$(v.w).u = (v.u).w + v.(w.u).$$

This is equivalent to one of the two following relations

$$[\mathrm{R}_u, \mathrm{R}_v] = \mathrm{R}_{vu} \quad \text{or} \quad [\mathrm{R}_u, \mathrm{L}_v] = \mathrm{L}_{vu}. \tag{16}$$

An algebra which is left and right Leibniz is called *symmetric Leibniz algebra*. Leibniz algebras were introduced by Loday in [18]. Many examples of symmetric Leibniz algebras can be found in [2]. By using (15) and (16), we get the following proposition.

Proposition 3.3. *The following assertions are equivalent:*

1. $(\mathrm{A}, .)$ *is a symmetric Leibniz algebra.*
2. *For any* $u, v \in \mathrm{A}$, $[\mathrm{L}_u, \mathrm{L}_v] = \mathrm{L}_{uv} = -\mathrm{R}_{uv}$.
3. *For any* $u, v \in \mathrm{A}$, $[\mathrm{R}_u, \mathrm{R}_v] = \mathrm{R}_{vu} = -\mathrm{L}_{vu}$.

Any Lie algebra is a symmetric Leibniz algebra and any Leibniz algebra is Lie-admissible. However, the class of symmetric Leibniz algebras contains strictly the class of Lie algebras. We can state now one of our main result.

Theorem 3.1. *Let* $(\mathrm{A}, .)$ *be a symmetric Leibniz algebra. Then* $(\mathrm{A}, .)$ *is Poisson admissible, the multiplication "." is parallel on* $(\mathrm{A}, [\ ,\ ])$ *and hence strongly quasi-canonical.*

Proof. Put

$$Q = [\mathrm{R}_u, \mathrm{R}_v] + \mathrm{L}_{[u,v]} + 3[\mathrm{L}_u, \mathrm{R}_v].$$

By using Proposition 3.3, we get

$$Q = -\mathrm{L}_{vu} + \mathrm{L}_{uv} - \mathrm{L}_{vu} - 3\mathrm{L}_{uv} = 0.$$

Now, according to Proposition 3.2 we get that $(\mathrm{A}, .)$ is Poisson admissible. On the other hand, by using Proposition 3.3, we get that, for any $u, v \in \mathrm{A}$,

$$\mathrm{ad}_{[u,v]} = 2\mathrm{L}_{[u,v]} = 4\mathrm{L}_{uv} \quad \text{and} \quad K(u, v) = \mathrm{L}_{vu}$$

and hence $\mathrm{ad}_{[\mathrm{A},\mathrm{A}]} = \mathrm{L}_{[\mathrm{A},\mathrm{A}]}$. So the holonomy Lie algebra of "." is $\mathrm{ad}_{[\mathrm{A},\mathrm{A}]}$ which prove that the multiplication "." is strongly quasi-canonical. Finally,

$$\begin{aligned}\nabla K(u, v, w) &= [\mathrm{L}_u, \mathrm{L}_{wv}] - \mathrm{L}_{(uw)v} - \mathrm{L}_{w(uv)} \\ &= \mathrm{L}_{u(wv)} - \mathrm{L}_{(uw)v} - \mathrm{L}_{w(uv)} = 0,\end{aligned}$$

which completes the proof. □

By using the geometric interpretation of Poisson structures introduced in Section 2, we get the following interesting corollary.

Corollary 3.2. *Let* $(\mathrm{A}, .)$ *be a real symmetric Leibniz algebra which is not a Lie algebra and* G *any connected Lie group associated to* $(\mathrm{A}, [\ ,\ ])$*. Then the left invariant connection on* G *given by*

$$\nabla_{u^+} v^+ = (u.v)^+$$

is different from ∇^0*, strongly special and its curvature is parallel.*

Example 1. We give here an example of a 4-dimensional real symmetric Leibniz algebra for which we give the connected and simply connected Lie group associated to the underlying Lie algebra and we give explicitly the two connections ∇^0 and ∇ appearing in the corollary above.

We consider the symmetric Leibniz algebra product on $\mathbb{R}^4$ given in the canonical basis (e_1, e_2, e_3, e_4) by

$$e_1.e_1 = e_4, \qquad e_2.e_1 = e_3, \qquad e_3.e_1 = e_4, \qquad e_1.e_2 = -e_3, \qquad e_1.e_3 = -e_4.$$

All the others products vanish. One can check easily by using Proposition 3.3 that this product defines actually a symmetric Leibniz algebra. The underlying Lie algebra say $\mathfrak{g} = \mathbb{R}^4$ has its non-vanishing Lie brackets given by

$$[e_1, e_2] = -2e_3 \quad \text{and} \quad [e_1, e_3] = -2e_4.$$

It is a 3-nilpotent Lie algebra. The associated connected and simply connected Lie group is $G = \mathbb{R}^4$ with the multiplication given by Campbell–Baker–Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \frac{1}{12}\big[x, [x, y]\big] + \frac{1}{12}\big[y, [y, x]\big].$$

This formula can be written

$$\begin{aligned} xy = \Big(&x_1 + y_1, x_2 + y_2, x_3 + y_3 - (x_1y_2 - x_2y_1), \\ &x_4 + y_4 - (x_1y_3 - x_3y_1) + \frac{1}{3}x_1(x_1y_2 - x_2y_1) + \frac{1}{3}y_1(y_1x_2 - y_2x_1)\Big). \end{aligned} \tag{17}$$

Recall that for any vector $u \in \mathfrak{g}$, u^+ denotes the left invariant vector on G associated to u. A straightforward computation using (17) gives

$$\begin{aligned} e_1^+ &= \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + \left(x_3 - \frac{1}{3}x_1x_2\right)\frac{\partial}{\partial x_4}, \\ e_2^+ &= \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3} + \frac{1}{3}x_1^2\frac{\partial}{\partial x_4}, \\ e_3^+ &= \frac{\partial}{\partial x_3} - x_1\frac{\partial}{\partial x_4}, \qquad e_4^+ = \frac{\partial}{\partial x_4}, \end{aligned}$$

where (x_1, x_2, x_3, x_4) are the canonical coordinates of $\mathbb{R}^4$. We consider the two torsion free linear connections on G defined by the formulas

$$\nabla^0_{x^+}y^+ = \frac{1}{2}\big[x^+, y^+\big] \quad \text{and} \quad \nabla_{x^+}y^+ = (x.y)^+. \tag{18}$$

The dot here is the symmetric Leibniz product. According to what above these two connections are bi-invariant, have the same curvature and the same holonomy Lie algebra. Moreover, they both have parallel curvature. Let us compute the Christoffel symbols of ∇^0 and ∇ in the canonical coordinates (x_1, x_2, x_3, x_4). The computation is straightforward consisting of replacing x and y in (18) by e_i, e_j, $i, j = 4, \ldots, 1$. We get that the only non-vanishing Christoffel symbols are given by

$$\nabla^0_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} = -\frac{2}{3}x_2\frac{\partial}{\partial x_4} \quad \text{and} \quad \nabla^0_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} = \frac{1}{3}x_1\frac{\partial}{\partial x_4},$$

and

$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} = \left(1 - \frac{2}{3}x_2\right)\frac{\partial}{\partial x_4} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} = \frac{1}{3}x_1\frac{\partial}{\partial x_4}.$$

We can also compute the exponential maps associated to ∇^0 and ∇ and we get that $\exp_0 : \mathfrak{g} \longrightarrow G$ is the identity, however $\exp : \mathfrak{g} \longrightarrow G$ is given by

$$\exp(a,b,c,d) = \left(a,b,c,d - \frac{1}{2}a^2\right).$$

Proposition 3.4. *A left (right) Leibniz algebra is Poisson admissible if and only if it is a symmetric Leibniz algebra.*

Proof. Suppose that $(\mathrm{A},.)$ is a left Leibniz algebra. According to Proposition 3.2, $(\mathrm{A},.)$ is Poisson admissible if and only if, for any $u,v \in \mathrm{A}$,

$$[\mathrm{L}_u, \mathrm{L}_v] - \mathrm{R}_{[u,v]} + 3[\mathrm{R}_u, \mathrm{L}_v] = 0.$$

From this relation and (15), we get that $\mathrm{L}_{uv} = 2\mathrm{R}_{vu} + \mathrm{R}_{uv}$. On the other hand, (15) implies that $\mathrm{L}_{uv} = -\mathrm{L}_{vu}$ so we deduce that $\mathrm{L}_{uv} = -\mathrm{R}_{uv}$ and we can achieve the proof by using Proposition 3.3 and Theorem 3.1. □

The following proposition gives a tool to build many symmetric Leibniz algebras from old ones.

Proposition 3.5. *Let* A *be a symmetric Leibniz and* U *an associative* LR*-algebra then* $\mathrm{A} \otimes \mathrm{U}$ *endowed with the product*

$$(u \otimes a)(v \otimes b) = (uv) \otimes (ab)$$

is a symmetric Leibniz algebra.

Proof. It is a straightforward computation. □

We can state now our second main result.

Theorem 3.2. *Let* $(\mathrm{A},.)$ *be a Poisson admissible algebra and* U *an associative* LR*-algebra. Then the product on* $\mathrm{A} \otimes \mathrm{U}$ *given by*

$$(u \otimes a) \star (v \otimes b) = \frac{1}{2}[u,v] \otimes (ab + ba) + \frac{1}{2}u.v \otimes (3ab + ba)$$

induces on $\mathrm{A} \otimes \mathrm{U}$ *a Poisson admissible algebra structure. Moreover, if "." is strongly quasi-canonical on* $(\mathrm{A}, [\ ,\ ])$ *then* $\star$ *is also strongly quasi-canonical on* $(\mathrm{A} \otimes \mathrm{U}, [\ ,\ ])$.

Proof. Note first that since U is an associative LR-algebra, for any $a_1, a_2, a_3 \in \mathrm{U}$ and for any permutation σ of $\{1,2,3\}$, $a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} = a_1a_2a_3$.

We will use Proposition 3.2 and show that, for any $u,v \in \mathrm{A}$ and $a,b \in \mathrm{U}$,

$$Q = [\mathrm{L}_{u\otimes a}, \mathrm{L}_{v\otimes b}] - \mathrm{R}_{[u\otimes a, v\otimes b]} + 3[\mathrm{R}_{u\otimes a}, \mathrm{L}_{v\otimes b}] = 0.$$

For any $w \in \mathrm{A}$ and $c \in \mathrm{U}$, we have

$$\begin{aligned}
[\mathrm{L}_{u\otimes a}, \mathrm{L}_{v\otimes b}](w \otimes c) &= (u \otimes a) \star \left(\frac{1}{2}[v,w] \otimes (bc + cb) + \frac{1}{2}v.w \otimes (3bc + cb)\right) \\
&\quad - (v \otimes b) \star \left(\frac{1}{2}[u,w] \otimes (ac + ca) + \frac{1}{2}u.w \otimes (3ac + ca)\right) \\
&= \big([u,[v,w]] + 2u.[v,w] + 2[u,vw] + 4u(vw)\big) \otimes (abc) \\
&\quad - \big([v,[u,w]] + 2v.[u,w] + 2[v,uw] + 4v(uw)\big) \otimes (abc) \\
&= \big([u,[v,w]] + [v,[w,u]] + 4[u,v].w + 4[\mathrm{L}_u, \mathrm{L}_v](w)\big) \otimes (abc).
\end{aligned}$$

Thus

$$[\mathrm{L}_{u\otimes a}, \mathrm{L}_{v\otimes b}](w\otimes c) = \big(\big[u,[v,w]\big] + \big[v,[w,u]\big] + 4[u,v].w + 4[\mathrm{L}_u, \mathrm{L}_v](w)\big) \otimes (abc).$$

A similar computation gives

$$\mathrm{R}_{[u\otimes a, v\otimes b]}(w\otimes c) = \big(4\big[w,[u,v]\big] + 8w.[u,v]\big) \otimes (abc),$$

and

$$[\mathrm{R}_{u\otimes a}, \mathrm{L}_{v\otimes b}](w\otimes c) = \big(\big[[v,w],u\big] - \big[v,[w,u]\big] + 2\big[[v,u],w\big] + 4[\mathrm{R}_u, \mathrm{L}_v](w)\big) \otimes (abc).$$

By using Jacobi identity and the relation

$$[\mathrm{L}_u, \mathrm{L}_v] - \mathrm{R}_{[u,v]} + 3[\mathrm{R}_u, \mathrm{L}_v] = 0,$$

we get that $Q = 0$ and hence $(\mathrm{A}\otimes\mathrm{U}, .)$ is a Poisson admissible algebra.

On the other hand, a direct computation gives, for any $u, v, w \in \mathrm{A}$ and any $a, b, c \in \mathrm{U}$,

$$\big[[u\otimes a, v\otimes b], w\otimes c\big] = 16\big[[u,v],w\big] \otimes (abc).$$

This shows that

$$\big[\mathrm{A}\otimes\mathrm{U}, [\mathrm{A}\otimes\mathrm{U}, \mathrm{A}\otimes\mathrm{U}]\big] = \big[\mathrm{A},[\mathrm{A},\mathrm{A}]\big] \otimes \mathrm{U}^3,$$

and

$$\mathrm{ad}_{[u\otimes a, v\otimes b]} = 16\mathrm{ad}_{[u,v]} \otimes \mathrm{L}_{ab}.$$

Moreover, one can check easily that for $u \in [\mathrm{A},[\mathrm{A},\mathrm{A}]]$ and $a \in \mathrm{U}^3$,

$$\mathrm{L}_{u\otimes a} = (\mathrm{ad}_u + 2\mathrm{L}_u) \otimes \mathrm{L}_a.$$

With all these formulas, one can show easily that if the multiplication "." is strongly quasi-canonical on $(\mathrm{A}, [\ ,\ ])$ then $\star$ is also strongly quasi-canonical on $(\mathrm{A}\otimes\mathrm{U}, [\ ,\ ])$. $\square$

Proposition 3.6. *Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra and "." is a strongly quasi-canonical product on $\mathfrak{g}$. Then $\mathfrak{g}^3 = [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$ is two sided ideal of $(\mathfrak{g}, .)$, $(\mathfrak{g}^3, .)$ is a symmetric Leibniz algebra and the sequence*

$$0 \longrightarrow \big(\mathfrak{g}^3, .\big) \longrightarrow (\mathfrak{g}, .) \longrightarrow \big(\mathfrak{g}/\mathfrak{g}^3, .\big) \longrightarrow 0$$

is an exact sequence of Poisson admissible algebras, $(\mathfrak{g}/\mathfrak{g}^3, .)$ is associative and $(\mathfrak{g}/\mathfrak{g}^3, [\ ,\ ])$ is 2-nilpotent.

Proof. Since "." is strongly quasi-canonical then its holonomy Lie algebra is equal to $\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$ and hence $\mathrm{L}_{\mathfrak{g}^3} \subset \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$ and $\mathrm{R}_{\mathfrak{g}^3} \subset \mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}$. Then for any $u \in \mathfrak{g}$ and $v \in \mathfrak{g}^3$ there exists $w, t \in [\mathfrak{g},\mathfrak{g}]$ such that $\mathrm{L}_u = \mathrm{ad}_w$ and $\mathrm{R}_u = \mathrm{ad}_t$. So $u.v \in \mathfrak{g}^3$, $v.u \in \mathfrak{g}^3$ and $\mathrm{L}_u, \mathrm{R}_u$ are derivations of the restriction of "." to $\mathfrak{g}^3$. We get that $\mathfrak{g}^3$ is a two side ideal and $(\mathfrak{g}^3, .)$ is a symmetric Leibniz algebra. On the other hand, $(\mathfrak{g}/\mathfrak{g}^3, .)$ is a Poisson algebra and $(\mathfrak{g}/\mathfrak{g}^3, [\ ,\ ])$ is 2-nilpotent so $(\mathfrak{g}/\mathfrak{g}^3, .)$ is associative. $\square$

In Proposition 23 of [13], it was proved that there is no non-trivial Poisson structure on a simple complex Lie algebra. We finish this section by generalizing this result to any semi-simple Lie algebra over any field.

We will show also that in a perfect Lie algebra the canonical product is the only strongly quasi-canonical product.

Theorem 3.3.

1. *Let $\mathfrak{g}$ be a perfect Lie algebra, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then the product $\alpha_0(u, v) = \frac{1}{2}[u, v]$ is the only strongly quasi-canonical product on $\mathfrak{g}$.*
2. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the product $\alpha_0(u, v) = \frac{1}{2}[u, v]$ is the only quasi-canonical product on $\mathfrak{g}$. In particular, there is no non-trivial Poisson structure on $\mathfrak{g}$.*

Proof.

1. Suppose that "." is a strongly quasi-canonical product on $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We have shown in Proposition 3.6 that in this case the restriction of "." to $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ is a Leibniz product. Or $\mathfrak{g} = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ and hence $(\mathfrak{g}, .)$ is a Leibniz algebra. Now from the relation $\mathrm{L}_{u.v} = -\mathrm{R}_{u.v}$ and the fact that $\mathfrak{g}.\mathfrak{g} = \mathfrak{g}$ we deduce that $u.v = \frac{1}{2}[u, v]$ for any $u, v \in \mathfrak{g}$ and hence "." is the canonical product on $\mathfrak{g}$.
2. Suppose that "." is a quasi-canonical product on a semi-simple Lie algebra $\mathfrak{g}$, denote by L_u and R_u, respectively, the left and the right multiplication by u associated to "." and put $S_u := \mathrm{L}_u - \frac{1}{2}\mathrm{ad}_u$. Note first that since $\mathfrak{g}$ is semi-simple, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and hence, by using (12), we get that for any $u \in \mathfrak{g}$,
$$\mathrm{tr}(S_u) = 0.$$
Consider
$$\mathcal{I} = \{u \in \mathfrak{g}, S_u = 0\}.$$
For any $u \in \mathfrak{g}$ and any $v \in \mathcal{I}$, we have from (12) that
$$\mathrm{L}_{[u,v]} = [\mathrm{ad}_u, \mathrm{L}_v] = \frac{1}{2}[\mathrm{ad}_u, \mathrm{ad}_v] = \frac{1}{2}\mathrm{ad}_{[u,v]},$$
and hence $\mathcal{I}$ is an ideal of $(\mathfrak{g}, [\ ,\ ])$. Let us show that $\mathcal{I} = \mathfrak{g}$.
Since $\mathfrak{g}$ is semi-simple we have
$$\mathfrak{g} = \bigoplus_{i=1}^{p} \mathfrak{g}_i,$$
where $(\mathfrak{g}_i)_{i=1}^p$ is a family of simple ideals of $\mathfrak{g}$,
$$[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] = \{0\} \quad \text{if } i \neq j.$$
We have for any $i, j = 1, \dots, p$,
$$\mathfrak{g}_i.\mathfrak{g}_i \subset \mathfrak{g}_i \quad \text{and} \quad \mathfrak{g}_i.\mathfrak{g}_j = \{0\} \quad \text{if } i \neq j.$$
Indeed, for any $u, v \in \mathfrak{g}_i$ and for $j \neq i$ and $w \in \mathfrak{g}_j$, we have
$$[w, u.v] = [w, u].v + u.[w, v] = 0.$$
By using a similar argument, we get that if $i \neq j$, $u \in \mathfrak{g}_i$ and $v \in \mathfrak{g}_j$, $u.v \in \mathfrak{g}_i \oplus \mathfrak{g}_j$. If $u = [a, b]$ with $a, b \in \mathfrak{g}_i$, we get

$$u.v = [a, b].v = [a, b.v] \in \mathfrak{g}_i.$$

Since $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ we get that $u.v \in \mathfrak{g}_i$ and in a similar way $u.v \in \mathfrak{g}_j$ and hence $u.v = 0$.
Suppose by contradiction that $\mathcal{I} \neq \mathfrak{g}$. Since $\mathcal{I}$ is an ideal, eventually by changing the indexation of the family $(\mathfrak{g}_i)_{i=1}^p$, we can suppose that there exists $1 \le r \le p$ such that

$$\mathfrak{g} = \mathcal{I} \oplus \mathcal{J} \quad \text{and} \quad \mathcal{J} = \bigoplus_{i=r}^{p} \mathfrak{g}_i.$$

For any $u \in \mathcal{J}$, we denote by $\overline{S}_u$ the restriction of S_u to $\mathcal{J}$. The product on $\mathcal{J}$ given by $u \circ v = \overline{S}_u v$ is a Poisson product and hence it is commutative and associative. So, for any $u \in \mathcal{J}$, and any $n \in \mathbb{N}^*$,

$$\operatorname{tr}\big((\overline{S}_u)^n\big) = \operatorname{tr}(\overline{S}_{u^n}) = \operatorname{tr}(S_{u^n}) = 0,$$

and hence $\overline{S}_u$ is nilpotent. Since for any $u, v \in \mathcal{J}$, $[\overline{S}_u, \overline{S}_v] = 0$, we deduce by using Engel's Theorem that there exists $u_0 \in \mathcal{J} \setminus \{0\}$ such that $\overline{S}_u(u_0) = \overline{S}_{u_0} u = 0$, and hence $\overline{S}_{u_0} = 0$. Since the restriction of S_{u_0} to $\mathcal{I}$ vanishes, we deduce that $S_{u_0} = 0$ and hence $u_0 \in \mathcal{I}$ which constitutes a contradiction and achieves the proof. □

4. Associative Poisson admissible algebras

We have shown in Corollary 3.1 that an associative algebra $(\mathrm{A}, .)$ is Poisson admissible if and only if $(\mathrm{A}, [\ ,\ ])$ is 2-nilpotent, i.e., for any $u, v \in \mathrm{A}$,

$$\mathrm{L}_{[u,v]} = \mathrm{R}_{[u,v]}. \tag{19}$$

An associative algebra satisfying this condition will be called *associative Poisson admissible algebra*. This section is devoted to a description of such algebras.

Let $(\mathrm{A}, .)$ be an associative Poisson admissible algebra. We consider

$$Z(\mathrm{A}) = \{u \in \mathrm{A}, \mathrm{L}_u = \mathrm{R}_u\}.$$

Since A is associative, $Z(\mathrm{A})$ is a commutative associative subalgebra of A. Put

$$\mathrm{A} = V \oplus Z(\mathrm{A}),$$

where V is a vectorial subspace of A. According to this splitting, we get that, for any $z \in Z(\mathrm{A})$ and $u, v \in V$,

$$z.u = u.z = \mathrm{P}_z(u) + \mathrm{Q}_u(z) \quad \text{and} \quad u.v = \mathfrak{a}(u, v) + \mathfrak{b}(u, v). \tag{20}$$

These relations define two bilinear maps $\mathfrak{a} : V \times V \longrightarrow V$, $\mathfrak{b} : V \times V \longrightarrow Z(\mathrm{A})$, and two linear maps $\mathrm{Q} : V \longrightarrow \mathrm{End}(Z(\mathrm{A}))$, $\mathrm{P} : Z(\mathrm{A}) \longrightarrow \mathrm{End}(V)$. Condition (19) is equivalent to $\mathfrak{a}$ symmetric and $\mathfrak{b}(u, v) - \mathfrak{b}(v, u) = [u, v]$. The associativity is equivalent to the following relations:

1. $\mathrm{P}_{zz'} = \mathrm{P}_z \circ \mathrm{P}_{z'}$, $[\mathrm{Q}_u, \mathrm{L}_z](z') = \mathrm{Q}_{\mathrm{P}_{z'}(u)}(z)$,
2. $\mathfrak{b}(\mathrm{P}_z(u), v) + \mathrm{Q}_v \circ \mathrm{Q}_u(z) = \mathfrak{b}(u, \mathrm{P}_z(v)) + \mathrm{Q}_u \circ \mathrm{Q}_v(z) = \mathrm{Q}_{\mathfrak{a}(u,v)}(z) + z\mathfrak{b}(u, v)$,
3. $\mathrm{P}_z(\mathfrak{a}(u, v)) = \mathfrak{a}(u, \mathrm{P}_z(v)) + \mathrm{P}_{\mathrm{Q}_v(z)}(u)$,
4. $\mathfrak{b}(\mathfrak{a}(u, v), w) - \mathfrak{b}(u, \mathfrak{a}(v, w)) = \mathrm{Q}_u(\mathfrak{b}(v, w)) - \mathrm{Q}_w(\mathfrak{b}(u, v))$,
5. $\mathfrak{a}(\mathfrak{a}(u, v), w) - \mathfrak{a}(u, \mathfrak{a}(v, w)) = \mathrm{P}_{\mathfrak{b}(v,w)}(u) - \mathrm{P}_{\mathfrak{b}(u,v)}(w)$.

So, we have shown that the associative and commutative algebra $Z(\mathrm{A})$, the vector space V, and P, Q, $\mathfrak{a}$, $\mathfrak{b}$ as above satisfying the conditions 1–5 describe entirely associative Poisson algebras.

Proposition 4.1. *Let $(\mathfrak{g}, [\ ,\ ])$ be a 2-nilpotent Lie algebra. Then there is on $\mathfrak{g}$ a quasi-canonical product different from the canonical one.*

Proof. Put $\mathfrak{g} = Z(\mathfrak{g}) \oplus V$ and consider the product on $\mathfrak{g}$ given, for any $z, z' \in Z(\mathfrak{g})$, $u, v \in V$, by

$$z.z' = u.z = z.u = 0 \quad \text{and} \quad u.v = \mathfrak{s}(u, v) + \frac{1}{2}[u, v],$$

where $\mathfrak{s}$ is any non-trivial symmetric bilinear map from $V \times V$ to $Z(\mathfrak{g})$. It is easy to check that this product is quasi-canonical and different from the canonical one. $\square$

5. Symplectic Poisson algebras

In this section, we study an important class of Poisson algebras. To introduce theses algebras we recall some classical results on symplectic Lie groups and introduce a new symplectic linear connection.

Let (G, Ω) be a symplectic Lie group, i.e., a Lie group G endowed with a left invariant symplectic form Ω. It is well-known that the linear connection given by the formula

$$\Omega(\nabla^{\mathrm{a}}_{u^+} v^+, w^+) = -\Omega(v^+, [u^+, w^+]), \tag{21}$$

where $u, v, w \in \mathfrak{g}$, defines a left invariant flat and torsion free connection ∇^{a}. Moreover, $\nabla^{\mathrm{a}}\Omega$ never vanishes unless G is abelian. So we can define a tensor field N by the relation

$$\nabla^{\mathrm{a}}_{u^+}\Omega(v^+, w^+) = \Omega(\mathrm{N}(u^+, v^+), w^+).$$

The linear connection given by

$$\nabla^{\mathrm{s}}_{u^+} v^+ = \nabla^{\mathrm{a}}_{u^+} v^+ + \frac{1}{3}\mathrm{N}(u^+, v^+) + \frac{1}{3}\mathrm{N}(v^+, u^+)$$

is left invariant torsion free and symplectic, i.e., $\nabla^{\mathrm{s}}\Omega = 0$. This construction follows a general scheme which permit to build symplectic connection from any connection (see [4]). A straightforward computation gives that ∇^{s} can be defined by the following formula

$$\Omega(\nabla^{\mathrm{s}}_{u^+} v^+, w^+) = \frac{1}{3}\Omega([u^+, v^+], w^+) + \frac{1}{3}\Omega([u^+, w^+], v^+). \tag{22}$$

This formula shows that on any symplectic Lie group there exists a canonical torsion free symplectic connection.

Let $(\mathfrak{g}, \omega)$ be the Lie algebra of G endowed with the value of Ω at e. We denote by α^{a} and α^{s} the product on $\mathfrak{g}$ induced, respectively, by ∇^{a} and ∇^{s}. We have, for any $u, v \in \mathfrak{g}$,

$$\alpha^{\mathrm{a}}(u, v) = -\mathrm{ad}^*_u v \quad \text{and} \quad \alpha^{\mathrm{s}}(u, v) = \frac{1}{3}(\mathrm{ad}_u v - \mathrm{ad}^*_u v), \tag{23}$$

where ad^*_u is the adjoint of ad_u with respect to ω.

Conversely, given a symplectic Lie algebra, formulas (23) define on $\mathfrak{g}$ two Lie-admissible products whose one is left symmetric and the other one is symplectic, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}^{\mathrm{a}}(u,v,w) = \mathrm{ass}^{\mathrm{a}}(v,u,w) \quad \text{and} \quad \omega\big(\alpha^{\mathrm{s}}(u,v),w\big) + \omega\big(v,\alpha^{\mathrm{s}}(u,w)\big) = 0,$$

where

$$\mathrm{ass}^{\mathrm{a}}(v,u,w) = \alpha^{\mathrm{a}}\big(\alpha^{\mathrm{a}}(u,v),w\big) - \alpha^{\mathrm{a}}\big(u,\alpha^{\mathrm{a}}(v,w)\big)$$

is the associator of α^{a}. Let us see under which conditions these products are quasi-canonical.

Proposition 5.1. *Let $(\mathfrak{g},\omega)$ be a symplectic Lie algebra and α^{a}, α^{s} the product given by (23). Then the following assertions are equivalent:*

1. *α^{a} is quasi-canonical.*
2. *α^{s} is quasi-canonical.*
3. *$\mathfrak{g}$ is 2-nilpotent and, for any $u,v\in\mathfrak{g}$, $[\mathrm{ad}_u,\mathrm{ad}_v^*]=0$.*

Moreover, if one of the conditions above holds then $(\mathfrak{g},\alpha^{\mathrm{a}})$ and $(\mathfrak{g},\alpha^{\mathrm{s}})$ are both associative LR*-algebras.*

Proof. Note first that $K^{\alpha^{\mathrm{a}}}=0$ and the left and right multiplications associated to α^a are given by

$$\mathrm{L}_u^{\mathrm{a}} = -\mathrm{ad}_u^* \quad \text{and} \quad \mathrm{R}_u^{\mathrm{a}} = -\mathrm{ad}_u^* - \mathrm{ad}_u.$$

The product α^{a} is quasi-canonical if and only if, for any $u,v\in\mathfrak{g}$,

$$K^{\alpha^{\mathrm{a}}}(u,v) = \big[\mathrm{R}_u^{\mathrm{a}},\mathrm{L}_v^{\mathrm{a}}\big] \quad \text{and} \quad \big[\mathrm{L}_u^{\mathrm{a}}+\mathrm{R}_u^{\mathrm{a}},\mathrm{L}_v^{\mathrm{a}}+\mathrm{R}_v^{\mathrm{a}}\big] = 0$$

which is obviously equivalent to

$$\mathrm{ad}_{[u,v]} = \big[\mathrm{ad}_u,\mathrm{ad}_v^*\big] = 0.$$

On the other hand, we have

$$\mathrm{L}_u^{\mathrm{s}} = \frac{1}{3}\big(\mathrm{ad}_u - \mathrm{ad}_u^*\big) \quad \text{and} \quad \mathrm{R}_u^{\mathrm{s}} = -\frac{1}{3}\big(2\mathrm{ad}_u + \mathrm{ad}_u^*\big),$$

and hence

$$\begin{aligned}
K^{\alpha^{\mathrm{s}}}(u,v) &= \big[\mathrm{L}_u^{\mathrm{s}},\mathrm{L}_v^{\mathrm{s}}\big] - \mathrm{L}_{[u,v]}^{\mathrm{s}} \\
&= \frac{1}{9}\big([\mathrm{ad}_u,\mathrm{ad}_v] - \big[\mathrm{ad}_u,\mathrm{ad}_v^*\big] - \big[\mathrm{ad}_u^*,\mathrm{ad}_v\big] + \big[\mathrm{ad}_u^*,\mathrm{ad}_v^*\big]\big) \\
&\quad - \frac{1}{3}\mathrm{ad}_{[u,v]} + \frac{1}{3}\mathrm{ad}_{[u,v]}^*, \\
\big[\mathrm{R}_u^{\mathrm{s}},\mathrm{L}_v^{\mathrm{s}}\big] &= -\frac{1}{9}\big(2[\mathrm{ad}_u,\mathrm{ad}_v] - 2\big[\mathrm{ad}_u,\mathrm{ad}_v^*\big] + \big[\mathrm{ad}_u^*,\mathrm{ad}_v\big] - \big[\mathrm{ad}_u^*,\mathrm{ad}_v^*\big]\big).
\end{aligned}$$

Thus $K^{\alpha^{\mathrm{s}}}(u,v) = [\mathrm{R}_u^{\mathrm{s}},\mathrm{L}_v^{\mathrm{s}}]$ if and only if

$$\big[\mathrm{ad}_u,\mathrm{ad}_v^*\big] = \mathrm{ad}_{[u,v]}^*. \tag{24}$$

Let us compute

$$Q = \big[\mathrm{L}_u^{\mathrm{s}} + \mathrm{R}_u^{\mathrm{s}},\mathrm{L}_v^{\mathrm{s}} + \mathrm{R}_v^{\mathrm{s}}\big].$$

We have

$$Q = \frac{1}{9}\left([\mathrm{ad}_u, \mathrm{ad}_v] + 2\left[\mathrm{ad}_u, \mathrm{ad}_v^*\right] + 2\left[\mathrm{ad}_u^*, \mathrm{ad}_v\right] + 4\left[\mathrm{ad}_u^*, \mathrm{ad}_v^*\right]\right).$$

So we get that $K^{\alpha^{\mathrm{s}}}(u,v) = [\mathrm{R}_u^{\mathrm{s}}, \mathrm{L}_v^{\mathrm{s}}]$ and $Q = 0$ if and only if

$$\left[\mathrm{ad}_u, \mathrm{ad}_v^*\right] = \mathrm{ad}_{[u,v]} = 0. \qquad \square$$

A *symplectic Poisson algebra* is a 2-nilpotent symplectic Lie algebra $(\mathfrak{g}, \omega)$ satisfying, for any $u, v \in \mathfrak{g}$,

$$\left[\mathrm{ad}_u, \mathrm{ad}_v^*\right] = 0. \tag{25}$$

Proposition 5.2. *Let $(\mathfrak{g}, \omega)$ be 2-nilpotent symplectic Lie algebra which carries a bi-invariant pseudo-Euclidean product B. Then $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra.*

Proof. We consider the isomorphism of $\mathfrak{g}$ given by

$$\omega(u,v) = B(Du, v).$$

It is easy to check by using the fact that B is bi-invariant and ω is symplectic that D is derivation of $\mathfrak{g}$ and that, for any $u \in \mathfrak{g}$, $\mathrm{ad}_u^* = -D^{-1} \circ \mathrm{ad}_u \circ D$. Now, for any $u, v \in \mathfrak{g}$,

$$\left[\mathrm{ad}_u^*, \mathrm{ad}_v\right] = \mathrm{ad}_v \circ D^{-1} \circ \mathrm{ad}_u \circ D - D^{-1} \circ \mathrm{ad}_u \circ D \circ \mathrm{ad}_v.$$

Since $D \circ \mathrm{ad}_v = \mathrm{ad}_v \circ D + \mathrm{ad}_{Dv}$ and $\mathfrak{g}$ is 2-nilpotent we get that

$$D^{-1} \circ \mathrm{ad}_u \circ D \circ \mathrm{ad}_v = 0.$$

On the other hand, $D^{-1}[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]$ so we get since $\mathfrak{g}$ is 2-nilpotent that $\mathrm{ad}_v \circ D^{-1} \circ \mathrm{ad}_u \circ D = 0$ and finally, $[\mathrm{ad}_u^*, \mathrm{ad}_v] = 0$ which show that $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra. $\square$

Example 2. Let $(\mathfrak{g}, [\ ,\ ])$ be a 2-nilpotent Lie algebra. Then $\mathfrak{g} = \mathcal{V} \oplus Z(\mathfrak{g})$, where $\mathcal{V}$ is a vector subspace of $\mathfrak{g}$ such that $[\mathcal{V}, \mathcal{V}] \subseteq Z(\mathfrak{g})$. The endomorphism D of $\mathfrak{g}$ defined by:

$$D(v) = v \quad \text{and} \quad D(z) = 2z, \quad \text{for all } v \in \mathcal{V},\ z \in Z(\mathfrak{g}),$$

is an invertible derivation of $\mathfrak{g}$.

Now, the vector space $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{g}^*$ endowed with the following product:

$$[u + \alpha, v + \beta] = [u, v] + \alpha \circ \mathrm{ad}_v - \beta \circ \mathrm{ad}_u, \quad \text{for all } u, v \in \mathfrak{g},\ \alpha, \beta \in \mathfrak{g}^*,$$

is a 2-nilpotent Lie algebra. Moreover, the bilinear form $B : \mathcal{G} \times \mathcal{G} \to \mathbb{K}$ defined by:

$$B(u + \alpha, v + \beta) = \alpha(v) + \beta(u), \quad \text{for all } u, v \in \mathfrak{g},\ \alpha, \beta \in \mathfrak{g}^*,$$

is non-degenerate, bi-invariant and symmetric. Then $(\mathcal{G}, B)$ is a 2-nilpotent quadratic Lie algebra. An easy computation shows that the endomorphism δ of $\mathcal{G}$ defined by:

$$\delta(u) = D(u) \quad \text{and} \quad \delta(\alpha) = -\alpha \circ D, \quad \text{for all } u \in \mathfrak{g},\ \alpha \in \mathfrak{g}^*,$$

is an invertible derivation of $\mathcal{G}$ which is skew-symmetric with respect to B. Consequently, the bilinear form defined by:

$$\omega(X,Y) = B\bigl(\delta(X),Y\bigr), \quad \text{for all } X,Y \in \mathcal{G},$$

is a symplectic structure on $\mathcal{G}$. Finally, $(\mathcal{G},\omega)$ is symplectic Poisson algebra.

Let us give now the inductive description of symplectic Poisson algebras. Let $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\omega)$ be a symplectic Poisson algebra. Since $\mathfrak{g}$ is nilpotent Lie algebra, according to [10], $(\mathfrak{g},[\ ,\ ]_{\mathfrak{g}},\omega)$ is the symplectic double extension of a symplectic Lie algebra $(\mathfrak{h},[\ ,\ ]_{\mathfrak{h}},\varpi)$ of dimension $\dim\mathfrak{g}-2$ by the one-dimensional Lie algebra by means of an element (D,z) of $\mathrm{Der}(\mathfrak{h})\times\mathfrak{h}$. This means that $\mathfrak{g}=\mathbb{K}e\oplus\mathfrak{h}\oplus\mathbb{K}d$ and

1. for any $a,b\in\mathfrak{h}$,

$$[a,b]_{\mathfrak{g}} = [a,b]_{\mathfrak{h}} + \varpi\bigl(\bigl(D+D^*\bigr)(a),b\bigr)e, \qquad [d,d]_{\mathfrak{g}} = 0,$$
$$[a,d]_{\mathfrak{g}} = D(a) + \varpi(z,a)e, \qquad [e,\mathfrak{g}]_{\mathfrak{g}} = \{0\},$$

   where D^* the adjoint of D with respect to ϖ,
2. $\omega_{|\mathfrak{h}\times\mathfrak{h}} = \varpi$, $\omega(e,d)=1$, $\omega(e,\mathfrak{h})=\omega(d,\mathfrak{h})=\{0\}$.

The fact that $(\mathfrak{g},\omega)$ is a symplectic Poisson algebra is equivalent to

$$\mathrm{ad}_{[u,v]} = \bigl[\mathrm{ad}_u,\mathrm{ad}_v^*\bigr] = 0,$$

for any $u,v\in\mathfrak{g}$.

The first condition which means that $\mathfrak{g}$ is 2-nilpotent Lie algebra is equivalent to:

- $\mathfrak{h}$ is a 2-nilpotent Lie algebra,
- $D(\mathfrak{h})\subset Z(\mathfrak{h})$,
- $D_{|[\mathfrak{h},\mathfrak{h}]_{\mathfrak{h}}} = D^*_{|[\mathfrak{h},\mathfrak{h}]_{\mathfrak{h}}} = 0$, $\varpi([\mathfrak{h},\mathfrak{h}]_{\mathfrak{h}},z)=0$,
- $D^2 = D^*\circ D = 0$ and $D^*(z)=0$.

Let us compute ad_u^* for any $u\in\mathfrak{g}$. A straightforward computation gives, for any $a,b\in\mathfrak{h}$,

$$\mathrm{ad}_a^*b = \mathrm{ad}_a^{\mathfrak{h}*}b + \varpi\bigl(b,D(a)\bigr)e, \qquad \mathrm{ad}_a^*d = -\bigl(D+D^*\bigr)(a) + \varpi(a,z)e,$$
$$\mathrm{ad}_a^*e = \mathrm{ad}_d^*e = 0, \qquad \mathrm{ad}_d^*d = -z, \qquad \mathrm{ad}_d^*a = -D^*(a).$$

So

$$\begin{aligned}
\bigl[\mathrm{ad}_a,\mathrm{ad}_b^*\bigr](c) &= \bigl[a,\mathrm{ad}_b^{\mathfrak{h}*}c\bigr] - \mathrm{ad}_b^*[a,c] = \bigl[a,\mathrm{ad}_b^{\mathfrak{h}*}c\bigr] - \mathrm{ad}_b^{\mathfrak{h}*}[a,c]_{\mathfrak{h}} + \varpi\bigl([a,c]_{\mathfrak{h}},D(b)\bigr)e\\
&= \bigl[\mathrm{ad}_a^{\mathfrak{h}},\mathrm{ad}_b^{\mathfrak{h}*}\bigr](c) + \varpi\bigl(\bigl(D+D^*\bigr)(a),\mathrm{ad}_b^{\mathfrak{h}*}c\bigr)e + \varpi\bigl([a,c]_{\mathfrak{h}},D(b)\bigr)e,\\
\bigl[\mathrm{ad}_a,\mathrm{ad}_b^*\bigr](d) &= -\bigl[a,\bigl(D+D^*\bigr)(b)\bigr] - \mathrm{ad}_b^*D(a)\\
&= -\bigl[a,\bigl(D+D^*\bigr)(b)\bigr]_{\mathfrak{h}} - \mathrm{ad}_b^{\mathfrak{h}*}D(a)\\
&\quad - \varpi\bigl(\bigl(D+D^*\bigr)(a),\bigl(D+D^*\bigr)(b)\bigr)e - \varpi\bigl(D(a),D(b)\bigr)e,\\
\bigl[\mathrm{ad}_a,\mathrm{ad}_d^*\bigr](b) &= -\bigl[a,D^*(b)\bigr] + D^*\bigl([a,b]_{\mathfrak{h}}\bigr)\\
&= -\bigl[a,D^*(b)\bigr]_{\mathfrak{h}} - \varpi\bigl(\bigl(D+D^*\bigr)(a),D^*(b)\bigr)e + D^*\bigl([a,b]_{\mathfrak{h}}\bigr),
\end{aligned}$$

$$\begin{aligned}
\left[\mathrm{ad}_a, \mathrm{ad}_d^*\right](d) &= -[a,z] + D^* \circ D(a) \\
&= -[a,z]_{\mathfrak{h}} - \bar{\omega}\big((D+D^*)(a), z\big)e + D^* \circ D(a), \\
\left[\mathrm{ad}_d, \mathrm{ad}_d^*\right](a) &= -\left[d, D^*(a)\right] - \mathrm{ad}_d^*[a,d] = D \circ D^*(a) - \bar{\omega}\big(z, D^*(a)\big)e + D^* \circ D(a), \\
\left[\mathrm{ad}_d, \mathrm{ad}_d^*\right](d) &= -D(z).
\end{aligned}$$

From these relations, we get that $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra if and only if

- $(\mathfrak{h}, \bar{\omega})$ is a symplectic Poisson algebra,
- $D(\mathfrak{h}) \subset Z(\mathfrak{h})$, $D^*(\mathfrak{h}) \subset Z(\mathfrak{h})$,
- $D^2 = D^* \circ D = D \circ D^* = 0$, $D^*(z) = D(z) = 0$ and $z \in Z(\mathfrak{h})$.

An element (D, z) of $\mathrm{Der}(\mathfrak{h}) \times \mathfrak{h}$ which verifies the conditions above will be called *admissible*.

To summarize, we have proved the following theorem.

Theorem 5.1. *Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra. Then $(\mathfrak{g}, \omega)$ is a symplectic Poisson algebra if and only if it is a symplectic double extension of a symplectic Poisson algebra $(\mathfrak{h}, \bar{\omega})$ of dimension $\dim \mathfrak{g} - 2$ by the one dimensional Lie algebra by means of an admissible element $(D, z) \in \mathrm{Der}(\mathfrak{h}) \times \mathfrak{h}$.*

There is only one 2-dimensional symplectic Poisson algebra, namely the two 2-dimensional abelian Lie algebra $\mathfrak{h}_0$ endowed with a symplectic form ω_0. There exists a basis $\mathbb{B} = \{e_1, e_2\}$ of $\mathfrak{h}_0$ such that $\omega_0(e_1, e_2) = 1$. An element $(D, z) \in \mathrm{Der}(\mathfrak{h}_0) \times \mathfrak{h}_0$ is admissible if and only if z is any element of $\mathfrak{h}_0$ and the matrix of D in the basis $\mathbb{B}$ has one of the following forms

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}, \qquad b \neq 0.$$

So we get all four-dimensional symplectic Poisson algebras.

Proposition 5.3. *Let $\mathfrak{g}$ be a 4-dimensional Lie algebra. Then $\mathfrak{g}$ is a symplectic Poisson algebra iff it is isomorphic to one of the following symplectic Lie algebras:*

1. $\mathrm{span}\{e, e_1, e_2, d\}$ *with the non-vanishing brackets*

$$[e_1, d] = -z_2 e, \qquad [e_2, d] = -a e_1 + z_1 e,$$

and the symplectic form satisfying

$$\omega(e, d) = \omega(e_1, e_2) = 1, \qquad \omega(e, e_1) = \omega(e, e_2) = \omega(d, e_1) = \omega(d, e_2) = 0.$$

2. $\mathrm{span}\{e, e_1, e_2, d\}$ *with the non-vanishing brackets*

$$[e_1, d] = a e_1 - \frac{a^2}{b} e_2 - z_2 e, \qquad [e_2, d] = b e_1 - a e_2 + z_1 e,$$

and the symplectic form satisfying

$$\omega(e, d) = \omega(e_1, e_2) = 1, \qquad \omega(e, e_1) = \omega(e, e_2) = \omega(d, e_1) = \omega(d, e_2) = 0.$$

We finish this section by giving an important geometric property of real symplectic Poisson algebras.

Let $(\mathfrak{g},\omega)$ be a non-abelian real symplectic Poisson algebra and G a connected Lie group having $\mathfrak{g}$ as its Lie algebra. The symplectic form ω defines on G a symplectic left invariant form Ω. Consider the two linear connections ∇^{a} and ∇^{s} defined on G by (21)–(22). These two connections are bi-invariant, flat, complete and $\nabla^{\mathrm{s}}\Omega=0$. It was shown in [3] that Ω is polynomial of degree at most $\dim G-1$ in any affine coordinates chart associated to ∇^{a}. The following result gives a more accurate statement on the polynomial nature of Ω.

Theorem 5.2. *With the hypothesis and the notations above we have*

$$\left(\nabla^{\mathrm{a}}\right)^3\Omega=0.$$

In particular, Ω is polynomial of degree at least one and at most 2 in any affine coordinates chart associated to ∇^{a}. Moreover, if the restriction of ω to $[\mathfrak{g},\mathfrak{g}]$ does not vanish then the degree is 2.

Proof. For any $u,v,x,y\in\mathfrak{g}$, an easy computation gives

$$\nabla^{\mathrm{a}}_{u^+}\Omega\left(x^+,y^+\right)=\Omega\left(u^+,\left[x^+,y^+\right]\right),$$

and hence

$$\nabla^{\mathrm{a}}_{u^+}\nabla^{\mathrm{a}}_{v^+}\Omega\left(x^+,y^+\right)=\Omega\left(\left[\nabla^{\mathrm{a}}_{u^+}x^+,y^+\right]+\left[x^+,\nabla^{\mathrm{a}}_{u^+}y^+\right],v^+\right).$$

Now since ∇^{a} is bi-invariant then

$$\left[\nabla^{\mathrm{a}}_{u^+}x^+,y^+\right]+\left[x^+,\nabla^{\mathrm{a}}_{u^+}y^+\right]=\nabla^{\mathrm{a}}_{[u^+,y^+]}x^++\nabla^{\mathrm{a}}_{[x^+,u^+]}y^++2\nabla^{\mathrm{a}}_{u^+}\left[x^+,y^+\right].$$

By using (21) and the fact that $\mathfrak{g}$ is 2-nilpotent, we get

$$\nabla^{\mathrm{a}}_{u^+}\nabla^{\mathrm{a}}_{v^+}\Omega\left(x^+,y^+\right)=2\Omega\left(\left[x^+,y^+\right],\left[u^+,v^+\right]\right).$$

By using the same arguments as above one can get easily that $(\nabla^{\mathrm{a}})^3\Omega=0$. The properties of the degree of Ω are an immediate consequence of formulas above. $\square$

It was proved in [12] that a compact affine manifold M has a polynomial Riemannian metric iff M is finitely covered by a complete affine nilmanifold. An affine nilmanifold is of the form Γ/N where N is a simply-connected nilpotent Lie group with a left invariant affine structure and Γ is a discrete subgroup of N. According to the results of this section, if G is the simply-connected Lie group associated to a non-abelian symplectic Poisson Lie algebra and Γ is a co-compact discrete subgroup of G then Γ/G is a compact nilmanifold which carries two affine structures and a symplectic form which is parallel for one affine structure and polynomial of degree at least 1 and at most 2 for the other one. It is natural to ask if there is a symplectic analog of Goldman's Theorem in [12].

6. Metrizability of special connections

In this section we study the problem of metrizability of special connections on Lie groups. Given a connected Lie group G with ∇ a special connection, does exist on G a left invariant pseudo-Riemannian metric whose associated Levi-Civita connection is ∇? Remark that if such a metric exists and it is bi-invariant then ∇ coincides with ∇^0. The following proposition gives an answer to this question when the metric is Riemannian.

Proposition 6.1. *Let $\mathfrak{g}$ be a real Lie algebra and $\langle\ ,\ \rangle$ an Euclidean product on $\mathfrak{g}$ such that the associated Levi-Civita product is quasi-canonical. Then $\langle\ ,\ \rangle$ is bi-invariant and hence the Levi-Civita product coincides with the canonical product.*

Proof. We have

$$\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus [\mathfrak{g},\mathfrak{g}]^{\perp}.$$

Since for any $u,v \in \mathfrak{g}$, $K(u,v) = -\frac{1}{4}\mathrm{ad}_{[u,v]}$ and $K(u,v)$ is skew-symmetric, we deduce that, for any $w \in [\mathfrak{g},\mathfrak{g}]$, ad_w is skew-symmetric. From this remark and the relation

$$2\langle u.v,w\rangle = \big\langle [u,v],w\big\rangle + \big\langle [w,v],u\big\rangle + \big\langle [w,u],v\big\rangle,$$

one can deduce easily that, for any $u,v \in [\mathfrak{g},\mathfrak{g}]$ and any $x,y \in [\mathfrak{g},\mathfrak{g}]^{\perp}$,

$$u.v = \frac{1}{2}[u,v] \quad \text{and} \quad x.y = \frac{1}{2}[x,y].$$

Let $u \in [\mathfrak{g},\mathfrak{g}]$ and $v \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, since ad_u is skew-symmetric, we get for any $w \in \mathfrak{g}$,

$$\big\langle [u,v],w\big\rangle = -\big\langle v,[u,w]\big\rangle = 0,$$

and hence $u.v = v.u$. Moreover, for any $x \in [\mathfrak{g},\mathfrak{g}]$,

$$\langle u.v,x\rangle = -\langle v,u.x\rangle = \frac{1}{2}\big\langle v,[x,u]\big\rangle = 0.$$

Thus $u.v = v.u \in [\mathfrak{g},\mathfrak{g}]^{\perp}$. Now

$$\begin{aligned} 2\langle u.v,u.v\rangle &= \big\langle [u,v],u.v\big\rangle + \big\langle [u.v,u],v\big\rangle + \big\langle [u.v,v],u\big\rangle \\ &= 0, \end{aligned}$$

since $[u,v] = 0$ and $[v,u.v] = [v,u].v + u.[v,v] = 0$. Thus $u.v = 0$ which completes the proof. □

The proposition above is not true in general when the $\langle\ ,\ \rangle$ is not positive definite. We give now a description of all real Lie algebras endowed with a pseudo-Euclidean product such that the associated Levi-Civita product is quasi-canonical and the derived ideal is non-degenerate.

Consider $(\mathfrak{h},\langle\ ,\ \rangle_0)$ a Lie algebra endowed with a bi-invariant pseudo-Euclidean product. Let (V,B) be a vector space with a nondegenerate symmetric bilinear form. We can split $V = V_0 \oplus U \oplus \overline{V}_0$ such that the restriction of B to U is positive definite and the map $V_0 \times \overline{V}_0 \longrightarrow \mathbb{R}$, $(u,v) \longrightarrow B(u,v)$ is non-degenerate. Finally, consider any bilinear skew-symmetric map $\gamma : \overline{V}_0 \times \overline{V}_0 \longrightarrow Z(\mathfrak{h})$. We consider now $\mathfrak{g} = \mathfrak{h} \oplus V$ endowed with $\langle\ ,\ \rangle = \langle\ ,\ \rangle_0 + B$ and the bracket for which $V_0 \oplus U \subset Z(\mathfrak{g})$, $[\mathfrak{h},V] = 0$, the restriction to $\mathfrak{h}$ coincides with the initial bracket and for any $u,v \in \overline{V}_0$, $[u,v] = \gamma(u,v)$. Then one can check easily that the Levi-Civita product of $\langle\ ,\ \rangle$ is quasi-canonical and $\langle\ ,\ \rangle$ is not bi-invariant. By a direct computation we can see easily that the curvature tensor is parallel which gives examples of locally symmetric pseudo-Riemannian spaces.

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