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Four-dimensional homogeneous semi-symmetric Lorentzian manifolds

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ABSTRACT

We determine all four-dimensional homogeneous semi-symmetric Lorentzian manifolds.

1. Introduction

A pseudo-Riemannian manifold (M, g) is said to be semi-symmetric if its curvature tensor K satisfies $\mathrm{K.K} = 0$. This is equivalent to

$$[\mathrm{K}(X,Y), \mathrm{K}(Z,T)] = \mathrm{K}(\mathrm{K}(X,Y)Z,T) + \mathrm{K}(Z,\mathrm{K}(X,Y)T), \tag{1}$$

for any vector fields X, Y, Z, T. Semi-symmetric pseudo-Riemannian manifolds generalize obviously locally symmetric manifolds ($\nabla \mathrm{K} = 0$). They also generalize second-order locally symmetric manifolds ($\nabla^2 \mathrm{K} = 0$ and $\nabla \mathrm{K} \neq 0$). Semi-symmetric Riemannian manifolds have been first investigated by E. Cartan [7] and the first example of a semi-symmetric not locally symmetric Riemannian manifold was given by Takagi [13]. More recently, Szabo [11,12] gave a complete description of these manifolds. In this study, Szabo used strong

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results proper to the Riemannian sitting which suggests that a similar study of semi-symmetric Lorentzian manifolds is far more difficult. To our knowledge, there are only few results on three dimensional locally homogeneous semi-symmetric Lorentzian manifolds [3,4] and second-order locally symmetric Lorentzian manifolds have been classified by D. Alekseevsky and A. Galaev in [1]. While in the Riemannian case every homogeneous semi-symmetric manifold is actually locally symmetric, in the Lorentzian case they are homogeneous semi-symmetric Lorentzian manifolds which are not locally symmetric.

This paper is devoted to the study of semi-symmetric curvature algebraic tensors on a Lorentzian vector space and to the classification of 4-dimensional simply-connected semi-symmetric homogeneous Lorentzian manifolds. There are our main results:

1. Let $(V, \langle\ ,\ \rangle)$ be a Lorentzian vector space and $\mathrm{K} : V \wedge V \longrightarrow V \wedge V$ a semi-symmetric algebraic curvature tensor, i.e., K satisfies the algebraic Bianchi identity and (1). Let $\mathrm{Ric}_K : V \longrightarrow V$ be its Ricci operator. The main result here (see Propositions 2.1 and 2.2) is that Ric_K has only real eigenvalues and, if $\lambda_1, \ldots, \lambda_r$ are the non null ones then V splits orthogonally
$$V = V_0 \oplus V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}, \tag{2}$$
where $V_{\lambda_i} = \ker(\mathrm{Ric}_K - \lambda_i \mathrm{Id}_V)$ and $V_0 = \ker(\mathrm{Ric}_K)^2$. Moreover, $\dim V_{\lambda_i} \geq 2$, $\mathrm{K}(V_{\lambda_i}, V_{\lambda_j}) = \mathrm{K}(V_0, V_{\lambda_i}) = 0$ for $i \neq j$, $\mathrm{K}(u,v)(V_{\lambda_i}) \subset V_{\lambda_i}$ and $\mathrm{K}(u,v)(V_0) \subset V_0$. This reduces the study of semi-symmetric algebraic curvature tensors to the ones who are Einstein ($\mathrm{Ric}_K = \lambda \mathrm{Id}_V$) or the ones who are Ricci isotropic ($\mathrm{Ric}_K \neq 0$ and $(\mathrm{Ric}_K)^2 = 0$).
2. In [8], Derdzinsky gave a classification of four dimensional Lorentzian Einstein manifolds whose curvature treated as a complex linear operator is diagonalizable and has constant eigenvalues. In [5], Calvaruso and Zaeim described locally homogeneous Lorentzian four-manifolds with diagonalizable Ricci operator. In [2], Astrakhantsev gave all semi-symmetric curvature tensors on a four dimensional Lorentzian vector space. Based on these three results, we prove the following two results.

   Theorem 1.1. *Let M be a four-dimensional Einstein Lorentzian manifold with non null scalar curvature. Then M is semi-symmetric if and only if it is locally symmetric.*

   Theorem 1.2. *Let M be a simply connected homogeneous semi-symmetric 4-dimensional Lorentzian manifold. If the Ricci tensor of M has a non zero eigenvalue then M is symmetric and in this case it is a product of a space of constant curvature and a Cahen–Wallace space.*

   We start in Section 3 by proving Theorem 1.2 when M is Lie group endowed with a left invariant Lorentzian metric. In Section 4, we prove Theorems 1.1 and 1.2.
3. Having Theorem 1.2 in mind, to complete the classification of simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds, we determine all simply connected four-dimensional semi-symmetric homogeneous Lorentzian manifolds with isotropic Ricci curvature. We will show that in this case $(\mathrm{Ric}_K)^2 = 0$ and $K^2 = 0$. To determine these spaces we distinguish two cases:
   (a) Simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds with non trivial isotropy and satisfying $(\mathrm{Ric}_K)^2 = 0$. In Section 5, by using Komrakov's classification of four-dimensional homogeneous pseudo-Riemannian manifolds [9], we give the list of such spaces. In Theorem 5.1, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are Ricci flat. In Theorem 5.2, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are not Ricci flat. We point out that there are four-dimensional homogeneous symmetric Lorentzian manifolds which are Ricci isotropic even Ricci flat non flat (see Remark 2).

(b) Four dimensional semi-symmetric Lorentzian Lie groups with $(\mathrm{Ric}_K)^2 = 0$. We study these Lie groups in Section 6. We point out that the Ricci flatness of these Lie groups implies flatness and when $\mathrm{Ric}_K \neq 0$ they are of two types: indecomposable with 2-dimensional holonomy Lie algebra, and decomposable with one dimensional holonomy Lie algebra.

The computations in Sections 5 and 6 have been performed using a computation software.

2. Semi-symmetric curvature tensors on Lorentzian vector spaces

In this section, we prove the first result listed in the introduction, we recall Astrakhantsev's list of semi-symmetric curvature tensors on a four dimensional Lorentzian vector space (see [2]) and we pull out from this list the results we will use later.

Let $(V, \langle\ ,\ \rangle)$ be a n-dimensional Lorentzian vector space. We identify V and its dual V^* by the means of $\langle\ ,\ \rangle$. This implies that the Lie algebra $V \otimes V^*$ of endomorphisms of V is identified with $V \otimes V$, the Lie algebra $\mathrm{so}(V, \langle\ ,\ \rangle)$ of skew-symmetric endomorphisms is identified with $V \wedge V$ and the space of symmetric endomorphisms is identified with $V \vee V$ (the symbol $\wedge$ is the outer product and $\vee$ is the symmetric product). For any $u, v \in V$,

$$(u \wedge v)w = \langle v, w\rangle u - \langle u, w\rangle v \quad \text{and} \quad (u \vee v)w = \frac{1}{2}\left(\langle v, w\rangle u + \langle u, w\rangle v\right).$$

Through this paper, we denote by $A_{u,v}$ the endomorphism $u \wedge v$. On the other hand, $V \wedge V$ carries also a nondegenerate symmetric product also denoted by $\langle\ ,\ \rangle$ and given by

$$\langle u \wedge v, w \wedge t\rangle := \langle u \wedge v(w), t\rangle = \langle v, w\rangle\langle u, t\rangle - \langle u, w\rangle\langle v, t\rangle.$$

We identify $V \wedge V$ with its dual by means of this metric.

A *curvature tensor* on $(V, \langle\ ,\ \rangle)$ is a $\mathrm{K} \in (V \wedge V) \vee (V \wedge V)$ satisfying the algebraic Bianchi's identity:

$$\mathrm{K}(u, v)w + \mathrm{K}(v, w)u + \mathrm{K}(w, u)v = 0, \quad u, v, w \in V.$$

The Ricci curvature associated to K is the symmetric bilinear form on V given by $\mathrm{ric}_K(u, v) = \mathrm{tr}(\tau(u, v))$, where $\tau(u, v) : V \longrightarrow V$ is given by $\tau(u, v)(a) = \mathrm{K}(u, a)v$. The Ricci operator is the symmetric endomorphism $\mathrm{Ric}_K : V \longrightarrow V$ given by $\langle \mathrm{Ric}_K(u), v\rangle = \mathrm{ric}_K(u, v)$. We call K Einstein (resp. Ricci isotropic) if $\mathrm{Ric}_K = \lambda \mathrm{Id}_V$ (resp. $\mathrm{Ric}_K \neq 0$ and $\mathrm{Ric}_K^2 = 0$). Note that if $K = (u \wedge v) \vee (w \wedge t)$ then

$$\mathrm{ric}_K = \langle u, w\rangle t \vee v + \langle v, t\rangle u \vee w - \langle v, w\rangle t \vee u - \langle u, t\rangle v \vee w.$$

We denote by $\mathfrak{h}(\mathrm{K})$ the vector subspace of $V \wedge V$ image of K, i.e., $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{\mathrm{K}(u, v)/\ u,\ v \in V\}$. A curvature tensor K is called *semi-symmetric* if it is invariant by $\mathfrak{h}(\mathrm{K})$, i.e.,

$$[\mathrm{K}(u, v), \mathrm{K}(a, b)] = \mathrm{K}(\mathrm{K}(u, v)a, b) + \mathrm{K}(a, \mathrm{K}(u, v)b), \quad u, v, a, b \in V. \tag{3}$$

In this case, $\mathfrak{h}(\mathrm{K})$ is a Lie subalgebra of $\mathrm{so}(V, \langle\ ,\ \rangle)$ called *primitive holonomy algebra* of K. If K is semi-symmetric then its Ricci operator is also invariant by $\mathfrak{h}(\mathrm{K})$, i.e.,

$$\mathrm{K}(u, v) \circ \mathrm{Ric}_K = \mathrm{Ric}_K \circ \mathrm{K}(u, v), \quad u, v \in V. \tag{4}$$

We recall now the different types of symmetric endomorphisms in a Lorentzian vector space in order to determine the types of Ricci operator of a semi-symmetric curvature tensor.

Theorem 2.1 *(see [10]). Let $(V,\langle\ ,\ \rangle)$ be a Lorentzian vector space of dimension $n\geq 3$ and $f:V\longrightarrow V$ a symmetric endomorphism. Then there exists a basis $\mathbb{B}$ of V such that the matrices of f and $\langle\ ,\ \rangle$ in $\mathbb{B}$ are given by one of the following types:*

1. *type* $\{\mathrm{diag}\}$: $\mathrm{M}(f,\mathbb{B})=\mathrm{diag}(\alpha_1,...,\alpha_n)$, $\mathrm{M}(\langle\ ,\ \rangle,\mathbb{B})=\mathrm{diag}(+1,...,+1,-1)$,
2. *type* $\{n-2,z\bar{z}\}$: $\mathrm{M}(f,\mathbb{B})=\mathrm{diag}(\alpha_1,...,\alpha_{n-2})\oplus\begin{pmatrix} a & b\\ -b & a\end{pmatrix}, b\neq 0$, $\mathrm{M}(\langle\ ,\ \rangle,\mathbb{B})=\mathrm{diag}(+1,...,+1,-1)$,
3. *type* $\{n,\alpha 2\}$: $\mathrm{M}(f,\mathbb{B})=\mathrm{diag}(\alpha_1,...,\alpha_{n-2})\oplus\begin{pmatrix} \alpha & 1\\ 0 & \alpha\end{pmatrix}$, $\mathrm{M}(\langle\ ,\ \rangle,\mathbb{B})=I_{n-2}\oplus\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$,
4. *type* $\{n,\alpha 3\}$: $\mathrm{M}(f,\mathbb{B})=\mathrm{diag}(\alpha_1,...,\alpha_{n-3})\oplus\begin{pmatrix} \alpha & 1 & 0\\ 0 & \alpha & 1\\ 0 & 0 & \alpha\end{pmatrix}$, $\mathrm{M}(\langle\ ,\ \rangle,\mathbb{B})=I_{n-3}\oplus\begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$.

The following proposition gives the type of the Ricci operator associated to a curvature tensor satisfying (4).

Proposition 2.1. *Let K be a curvature tensor on a Lorentzian vector space $(V,\langle\ ,\ \rangle)$ satisfying (4). Then its Ricci operator is either of type $\{\mathrm{diag}\}$ or $\{n,02\}$. In particular, all its eigenvalues are real.*

Proof. Since Ric_K is a symmetric endomorphism of $(V,\langle\ ,\ \rangle)$ then there exists a basis $\mathbb{B}$ of V such that the matrices of Ric_K and $\langle\ ,\ \rangle$ in $\mathbb{B}$ have one of the forms listed in the statement of Theorem 2.1.

1. Suppose that the matrices of Ric_K and $\langle\ ,\ \rangle$ are of type $\{n-2,z\bar{z}\}$. Put $\mathbb{B}=(e_1,\ldots,e_{n-1},e,\bar{e})$. Then, for $i=1,\ldots,n-2$,
$$\mathrm{Ric}_K(e_i)=\alpha_ie_i,\ \mathrm{Ric}_K(e)=ae-b\bar{e}\quad\text{and}\quad\mathrm{Ric}_K(\bar{e})=be+a\bar{e},\qquad b\neq 0.$$
This shows that the sum of the eigenspaces associated to the real eigenvalues of Ric_K is $E=\mathrm{span}\{e_1,\ldots,e_{n-2}\}$. From (4), we can deduce that $\mathfrak{h}(\mathrm{K})$ leaves invariant E and hence its orthogonal $E^\perp=\mathrm{span}\{e,\bar{e}\}$. So
$$b=\langle\mathrm{Ric}_K(\bar{e}),e\rangle=\langle\mathrm{K}(\bar{e},e)e,e\rangle-\langle\mathrm{K}(\bar{e},\bar{e})e,\bar{e}\rangle+\sum_{i=1}^{n-2}\langle\mathrm{K}(\bar{e},e_i)e,e_i\rangle=0,$$
which contradicts the fact that $b\neq 0$.
2. Suppose that the matrices of Ric_K and $\langle\ ,\ \rangle$ are of type $\{n,\alpha 2\}$. Put $\mathbb{B}=(e_1,\ldots,e_{n-2},e,\bar{e})$ and remark that, for $i=1,\ldots,n-2$,
$$\mathrm{Ric}_K(e_i)=\alpha_ie_i,\ \mathrm{Ric}_K(e)=\alpha e,\ \mathrm{Ric}_K(\bar{e})=e+\alpha\bar{e}\quad\text{and}\quad\langle e,e\rangle=\langle\bar{e},\bar{e}\rangle=0,\langle\bar{e},e\rangle=1.$$
This shows that Ric_K has only real eigenvalues and the sum of the associated eigenspaces is $E=\mathrm{span}\{e,e_1,\ldots,e_{n-2}\}$. From (4), we can deduce that $\mathfrak{h}(\mathrm{K})$ leaves invariant E. We have then
$$\alpha=\langle\mathrm{Ric}_K(\bar{e}),e\rangle=\langle\mathrm{K}(\bar{e},e)e,\bar{e}\rangle+\langle\mathrm{K}(\bar{e},\bar{e})e,e\rangle+\sum_{i=1}^{n-2}\langle\mathrm{K}(\bar{e},e_i)e,e_i\rangle=\langle\mathrm{K}(\bar{e},e)e,\bar{e}\rangle.$$
On the other hand,

$$\begin{aligned}\langle \mathrm{K}(\overline{e},e)e,\overline{e}\rangle &= \langle \mathrm{K}(\overline{e},e)(\mathrm{Ric}_K(\overline{e})-\alpha\overline{e}),\overline{e}\rangle = \langle \mathrm{K}(\overline{e},e)\circ \mathrm{Ric}_K(\overline{e}),\overline{e}\rangle \\ &\overset{(4)}{=} \langle \mathrm{Ric}_K \circ \mathrm{K}(\overline{e},e)\overline{e},\overline{e}\rangle = \langle \mathrm{K}(\overline{e},e)\overline{e},\mathrm{Ric}_K(\overline{e})\rangle \\ &= \langle \mathrm{K}(\overline{e},e)\overline{e},e+\alpha\overline{e}\rangle = \langle \mathrm{K}(\overline{e},e)\overline{e},e\rangle = -\langle \mathrm{K}(\overline{e},e)e,\overline{e}\rangle.\end{aligned}$$

So $\alpha = 0$.

3. Suppose that the matrices of Ric_K and $\langle\ ,\ \rangle$ are of type $\{n, \alpha 3\}$. Put $\mathbb{B} = (e_1, \dots, e_{n-3}, e, f, \overline{e})$ and remark that, for $i = 1, \dots, n-3$,

$$\mathrm{Ric}_K(e_i) = \alpha_i e_i,\ \mathrm{Ric}_K(e) = \alpha e,\ \mathrm{Ric}_K(f) = e + \alpha f \quad \text{and} \quad \mathrm{Ric}_K(\overline{e}) = f + \alpha\overline{e}.$$

This shows that Ric_K has only real eigenvalues and the sum of the associated eigenspaces is $E = \mathrm{span}\{e, e_1, \dots, e_{n-3}\}$. From (4), we can deduce that $\mathfrak{h}(\mathrm{K})$ leaves invariant E. We have then

$$\alpha = \langle \mathrm{Ric}_K(\overline{e}), e\rangle = \langle \mathrm{K}(\overline{e},e)e,\overline{e}\rangle + \langle \mathrm{K}(\overline{e},f)e,f\rangle + \langle \mathrm{K}(\overline{e},\overline{e})e,e\rangle + \sum_{i=1}^{n-3}\langle \mathrm{K}(\overline{e},e_i)e,e_i\rangle = \langle \mathrm{K}(\overline{e},e)e,\overline{e}\rangle.$$

Furthermore,

$$\begin{aligned}\langle \mathrm{K}(\overline{e},e)e,\overline{e}\rangle &= \langle \mathrm{K}(\overline{e},e)(\mathrm{Ric}_K(f)-\alpha f),\overline{e}\rangle \\ &= \langle \mathrm{K}(\overline{e},e)\circ \mathrm{Ric}_K(f),\overline{e}\rangle - \alpha\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle \\ &\overset{(4)}{=} \langle \mathrm{K}(\overline{e},e)f,\mathrm{Ric}_K(\overline{e})\rangle - \alpha\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle \\ &= \langle \mathrm{K}(\overline{e},e)f, f+\alpha\overline{e}\rangle - \alpha\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle \\ &= 0.\end{aligned}$$

So $\alpha = 0$. Thus

$$1 = \langle \mathrm{Ric}_K(\overline{e}), f\rangle = \langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle + \langle \mathrm{K}(\overline{e},f)f,f\rangle + \langle \mathrm{K}(\overline{e},\overline{e})f,e\rangle + \sum_{i=1}^{n-3}\langle \mathrm{K}(\overline{e},e_i)f,e_i\rangle = \langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle.$$

On the other hand,

$$\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle = \langle \mathrm{K}(\overline{e},e)\mathrm{Ric}_K(\overline{e}),\overline{e}\rangle \overset{(4)}{=} \langle \mathrm{K}(\overline{e},e)\overline{e},\mathrm{Ric}_K(\overline{e})\rangle = \langle \mathrm{K}(\overline{e},e)\overline{e},f\rangle = -\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle.$$

This shows that $\langle \mathrm{K}(\overline{e},e)f,\overline{e}\rangle = 0$ which contradicts what above and completes the proof. □

We give now the main result of this section which gives a useful decomposition of semi-symmetric curvature tensors in a Lorentzian spaces.

Proposition 2.2. *Let* K *be a semi-symmetric curvature tensor on a Lorentzian vector space* $(V, \langle\ ,\ \rangle)$. *Then all eigenvalues of* Ric_K *are real. Denote by* $\alpha_1, \dots, \alpha_r$ *the non null eigenvalues and* $V_1, \dots, V_r$ *the corresponding eigenspaces. Then:*

1. V *splits orthogonally as* $V = V_0 \oplus V_1 \oplus \ldots \oplus V_r$, *where* $V_0 = \ker(\mathrm{Ric})^2$,
2. *for any* $u, v \in V$ *and* $i = 0, \dots, r$, V_i *is* $\mathfrak{h}(\mathrm{K})$*-invariant,*
3. *for any* $i, j = 0, \dots, r$ *with* $i \neq j$, $\mathrm{K}_{|V_i \wedge V_j} = 0$,
4. *for any* $i = 1, \dots, r$, $\dim V_i \geq 2$.

Proof.

1. This is a consequence of Proposition 2.1.
2. This statement follows from (4).
3. Let $u \in V_i$, $v \in V_j$ and $a, b \in V$. Since $\mathrm{K}(a,b)(V_i) \subset V_i$ and $\langle V_i, V_j \rangle = 0$, we get
$$0 = \langle \mathrm{K}(a,b)u, v \rangle = \langle \mathrm{K}(u,v)a, b \rangle$$
and hence $\mathrm{K}(u,v) = 0$.
4. Suppose that $\dim V_i = 1$ for $i = 1, \ldots, r$ and choose a generator e of V_i such that $\langle e, e \rangle = \epsilon$ with $\epsilon^2 = 1$ and complete to get an orthonormal basis $(e, e_1, \ldots, e_{n-1})$ with $\langle e_i, e_i \rangle = \epsilon_i$, $\epsilon_i^2 = 1$. For any $a, b \in V$, $\mathrm{K}(a,b)$ is skew-symmetric and leaves V_i invariant so $\mathrm{K}(a,b)e = 0$. Now
$$\epsilon \alpha_i = \langle \mathrm{Ric}_K(e), e \rangle = \epsilon \langle \mathrm{K}(e,e)e, e \rangle + \sum_{i=1}^{n-1} \epsilon_i \langle \mathrm{K}(e, e_i)e, e_i \rangle = 0,$$
which is a contradiction and achieves the proof. □

This proposition reduces the determination of semi-symmetric curvature tensors on Lorentzian vector spaces to the determination of three classes of semi-symmetric curvature tensors: Einstein semi-symmetric curvature tensors on an Euclidean vector space, Einstein semi-symmetric curvature tensors on a Lorentzian vector space and Ricci isotropic semi-symmetric curvature tensors on a Lorentzian vector space.

We end this section by recalling the classification of semi-symmetric curvature tensors on four dimensional vector spaces given by Astrakhantsev in [2] and pulling out from it some results we will use later.

The idea behind Astrakhantsev's classification is the following. Let K be a semi-symmetric curvature tensor on a Lorentzian vector space $(V, \langle\ ,\ \rangle)$. The space $\mathfrak{h}(K)$ is actually a subalgebra of $\mathrm{so}(V, \langle\ ,\ \rangle)$ and the semi-symmetry is equivalent to $\mathfrak{h}(K).K = 0$. So, one way to determine all semi-symmetric curvature tensors is to classify, up to equivalence, all proper subalgebras of $\mathrm{so}(V, \langle\ ,\ \rangle)$ and for each one of them, say $\mathfrak{g}$, determine all the curvature tensors K satisfying $\mathfrak{h}(K) = \mathfrak{g}$ and $\mathfrak{g}.K = 0$. In dimension four, this was done successfully in [2] and led to the following result.

Theorem 2.2 *([2]). Let $(V, \langle\ ,\ \rangle)$ be a four dimensional Lorentzian vector space and K a semi-symmetric curvature tensor on V. Then there exists an orthonormal basis (x, y, z, t) of V with $\langle t, t \rangle = -1$ such that one of the following situations occurs:*

1. $\dim \mathfrak{h}(K) = 1$*:*
   (a) $K = aA_{p,x} \vee A_{p,x}$, $2[\mathrm{Ric}_K] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & -a \\ 0 & 0 & a & -a \end{pmatrix}$, $\mathrm{Ric}_K^2 = 0$,
   (b) $K = aA_{t,z} \vee A_{t,z}$, $[\mathrm{Ric}_K] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$,
   (c) $K = aA_{y,x} \vee A_{y,x}$, $[\mathrm{Ric}_K] = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

2. $\dim \mathfrak{h}(K) = 2$:

   (a) $K = aA_{p,x} \vee A_{p,x} + bA_{p,y} \vee A_{p,y} + cA_{p,x} \vee A_{p,y}$, $2[\mathrm{Ric}_K] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a+b & -a-b \\ 0 & 0 & a+b & -a-b \end{pmatrix}$, $\mathrm{Ric}_K^2 = 0$,

   (b) $K = aA_{t,z} \vee A_{t,z} + bA_{y,x} \vee A_{y,x}$, $[\mathrm{Ric}_K] = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}$,

3. $\dim \mathfrak{h}(K) = 3$:

   (a) $K = a(A_{t,z} \vee A_{t,z} + 2A_{p,y} \vee A_{y,q})$, $[\mathrm{Ric}_K] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2a & 0 & 0 \\ 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & -2a \end{pmatrix}$,

   (b) $K = a(A_{x,y} \vee A_{x,y} + A_{x,z} \vee A_{x,z} + A_{y,z} \vee A_{y,z})$, $[\mathrm{Ric}_K] = \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

4. $\dim \mathfrak{h}(K) = 6$, $K = a\mathrm{Id}_{V\wedge V}$ *and* $\mathrm{Ric}_K = -3a\mathrm{Id}_V$.

On what above $p = \frac{1}{\sqrt{2}}(z+t)$, $q = \frac{1}{\sqrt{2}}(z-t)$, a, b, c *are real parameters and* $[\mathrm{Ric}_K]$ *is the matrix of* Ric_K *in the basis* (x, y, z, t).

As a consequence of this theorem we get the following result.

Corollary 2.1. *Let* $(V, \langle\ ,\ \rangle)$ *be a four dimensional Lorentzian vector space and* K *a semi-symmetric curvature tensor on* V. *Then:*

1. *If* K *is Einstein with* $\dim \mathfrak{h}(K) \neq 6$ *and* $\mathrm{Ric}_K \neq 0$ *then there exists an orthonormal basis* (x, y, z, t) *of* V *with* $\langle t, t\rangle = -1$ *such that*

$$K = a(A_{t,z} \vee A_{t,z} - A_{y,x} \vee A_{y,x}), \quad a \in \mathbb{R}^*.$$

   In particular, K *is diagonalizable with* a *as an eigenvalue of multiplicity 2 and 0 as an eigenvalue of multiplicity 4.*
2. *If* K *is Ricci flat then there exists an orthonormal basis* (x, y, z, t) *of* V *with* $\langle t, t\rangle = -1$ *such that*

$$K = aA_{p,x} \vee A_{p,y}, \quad a \in \mathbb{R}, p = \frac{1}{\sqrt{2}}(z+t).$$

   In particular, $K^2 = 0$.
3. *If* K *is Ricci isotropic then there exists an orthonormal basis* (x, y, z, t) *of* V *with* $\langle t, t\rangle = -1$ *such that*

$$K = aA_{p,x} \vee A_{p,x} + bA_{p,y} \vee A_{p,y} + cA_{p,x} \vee A_{p,y}, \quad a, b, c \in \mathbb{R},\ a+b \neq 0, p = \frac{1}{\sqrt{2}}(z+t).$$

   In particular, $K^2 = 0$.

Actually, in Section 6 we need a more simple form of Ricci isotropic semi-symmetric curvature tensors.

Proposition 2.3. *Let $(V, \langle\ ,\ \rangle)$ be a four dimensional Lorentzian vector space and K a semi-symmetric Ricci isotropic curvature tensor on V. Then there exists a basis (e, f, g, h) such that the non vanishing products are $\langle e, e\rangle = \langle f, f\rangle = \langle g, h\rangle = 1$ and*

$$K = \omega_1 A_{e,g} \vee A_{e,g} + \omega_2 A_{f,g} \vee A_{f,g}, \quad \omega_1 + \omega_2 = \pm 1.$$

Proof. As in Corollary 2.1, put $K = aA_{p,x} \vee A_{p,x} + bA_{p,y} \vee A_{p,y} + cA_{p,x} \vee A_{p,y}$ with $a + b \neq 0$. If $c = 0$ we take $e = x$, $f = y$, $g = \sqrt{|a + b|}p$ and $h = \frac{q}{\sqrt{|a+b|}}$, $\omega_1 = \frac{a}{|a+b|}$ and $\omega_2 = \frac{b}{|a+b|}$. If $c \neq 0$, we look for $e = \cos(\alpha)x + \sin(\alpha)y$, $f = -\sin(\alpha)x + \cos(\alpha)y$ and ω_1' and ω_2' such that $K = \omega_1' A_{e,p} \vee A_{e,p} + \omega_2' A_{f,p} \vee A_{f,p}$. This is equivalent to

$$\omega_1' \cos^2(\alpha) + \omega_2' \sin^2(\alpha) = a,\ \omega_1' \cos^2(\alpha) + \omega_2' \sin^2(\alpha) = b \quad \text{and} \quad (\omega_1' - \omega_2') \sin(2\alpha) = c.$$

This equivalent to

$$\omega_1' + \omega_2' = a + b,\ \omega_1' - \omega_2' = \frac{c}{\sin(2\alpha)} \quad \text{and} \quad \omega_1' \cos^2(\alpha) + \omega_2' \sin^2(\alpha) = a.$$

Which is also equivalent to

$$\omega_1' = \frac{1}{2}\left(a + b + \frac{c}{\sin(2\alpha)}\right),\ \omega_2' = \frac{1}{2}\left(a + b - \frac{c}{\sin(2\alpha)}\right) \quad \text{and} \quad \tan^2(\alpha) + \frac{2(a-b)}{c}\tan(\alpha) - 1 = 0.$$

The last equation has a solution which completes the proof. □

We end this section by the following interesting remark.

Remark 1. By using Theorem 2.2, one can see easily that if Ric_K has a non zero eigenvalue then it is diagonalizable. Otherwise, $\mathrm{Ric}_K^2 = 0$.

3. Four dimensional semi-symmetric Lorentzian Lie groups with Ricci curvature having a non zero eigenvalue are locally symmetric

In this section, we give some general properties of semi-symmetric Lorentzian Lie groups and we prove Theorem 1.2 when M is a Lorentzian Lie group.

A Lie group G together with a left-invariant pseudo-Riemannian metric g is called a *pseudo-Riemannian Lie group*. The metric g defines a pseudo-Euclidean product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g} = T_eG$ of G, and conversely, any pseudo-Euclidean product on $\mathfrak{g}$ gives rise to an unique left-invariant pseudo-Riemannian metric on G.

We will refer to a Lie algebra endowed with a pseudo-Euclidean product as a *pseudo-Euclidean Lie algebra*. The Levi-Civita connection of (G, g) defines a product $\mathrm{L} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the Levi-Civita product and given by Koszul's formula

$$2\langle \mathrm{L}_u v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle. \tag{5}$$

For any $u, v \in \mathfrak{g}$, $\mathrm{L}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is skew-symmetric and $[u, v] = \mathrm{L}_u v - \mathrm{L}_v u$. We will also write $u.v = \mathrm{L}_v u$. The curvature on $\mathfrak{g}$ is given by $\mathrm{K}(u, v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v]$. It is well-known that K is a curvature tensor on $(\mathfrak{g}, \langle\ ,\ \rangle)$ and, moreover, it satisfies the differential Bianchi identity

$$\mathrm{L}_u(\mathrm{K})(v, w) + \mathrm{L}_v(\mathrm{K})(w, u) + \mathrm{L}_w(\mathrm{K})(u, v) = 0, \quad u, v, w \in \mathfrak{g} \tag{6}$$

where $\mathrm{L}_u(\mathrm{K})(v,w) = [\mathrm{L}_u, \mathrm{K}(v,w)] - \mathrm{K}(\mathrm{L}_u v, w) - \mathrm{K}(v, \mathrm{L}_u w)$. Denote by $\mathfrak{h}(\mathfrak{g})$ the holonomy Lie algebra of (G,g). It is the smallest Lie algebra containing $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{\mathrm{K}(u,v) : u,v \in \mathfrak{g}\}$ and satisfying $[\mathrm{L}_u, \mathfrak{h}(\mathfrak{g})] \subset \mathfrak{h}(\mathfrak{g})$, for any $u \in \mathfrak{g}$.

If we denote by $\mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ the right multiplication given by $\mathrm{R}_u v = \mathrm{L}_v u$, it is easy to check the following useful relation

$$\mathrm{K}(u,.)v = -\mathrm{R}_v \circ \mathrm{R}_u + \mathrm{R}_{u.v} + [\mathrm{R}_v, \mathrm{L}_u]. \tag{7}$$

We can also see easily that

$$[\mathfrak{g},\mathfrak{g}]^\perp = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\} \quad \text{and} \quad (\mathfrak{g}.\mathfrak{g})^\perp = \{u \in \mathfrak{g}, \mathrm{R}_u = 0\}. \tag{8}$$

(G,g) is semi-symmetric iff K is a semi-symmetric curvature tensor of $(\mathfrak{g}, \langle\ ,\ \rangle)$. Without reference to any Lie group, we call a pseudo-Euclidean Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ semi-symmetric if its curvature is semi-symmetric.

We introduce, for any nonunimodular pseudo-Euclidean Lie algebras $\mathfrak{g}$, the vector $\mathbf{h}$ defined by $\langle u, \mathbf{h}\rangle = \mathrm{tr}(\mathrm{ad}_u)$. We have obviously, $\mathbf{h}.\mathbf{h} = 0$ and since $\mathbf{h} \in [\mathfrak{g},\mathfrak{g}]^\perp$, $\mathrm{R}_{\mathbf{h}}$ is a symmetric endomorphism.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a semi-symmetric Lorentzian Lie algebra. According to Proposition 2.2, $\mathfrak{g}$ splits orthogonally as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r, \tag{9}$$

where $\mathfrak{g}_0 = \ker(\mathrm{Ric}^2)$ and $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ are the eigenspaces associated to the non zero eigenvalues of Ric. Moreover, $\mathrm{K}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ for any $i \neq j$ and $\dim \mathfrak{g}_i \geq 2$ if $i \neq 0$. The following proposition gives more properties of the $\mathfrak{g}_i$'s involving the Levi-Civita product.

Proposition 3.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a semi-symmetric Lorentzian Lie algebra. Then, for any $i,j \in \{1,\ldots,r\}$ and $i \neq j$,*

$$\mathfrak{g}_j.\mathfrak{g}_i \subset \mathfrak{g}_i,\ \mathfrak{g}_i.\mathfrak{g}_i \subset \mathfrak{g}_0 + \mathfrak{g}_i,\ \mathfrak{g}_0.\mathfrak{g}_i \subset \mathfrak{g}_i,\ \mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0,\ \mathfrak{g}_i.\mathfrak{g}_0 \subset \mathfrak{g}_0 + \mathfrak{g}_i.$$

Moreover, if $\dim \mathfrak{g}_0 = 1$ then for any $u \in \mathfrak{g}_0$, $u.u = 0$ and, for any $k \in \mathbb{N}^*$, $[\mathrm{R}_u^k, \mathrm{L}_u] = k\mathrm{R}_u^{k+1}$. In particular, R_u is a nilpotent endomorphism.

Proof. We start by proving that, for any $i \in \{1,\ldots,r\}$ and any $x \in \mathfrak{g}_i^\perp$, $\mathrm{L}_x \mathfrak{g}_i \subset \mathfrak{g}_i$. Fix $i \in \{1,\ldots,r\}$ and $x \in \mathfrak{g}_i^\perp$. For any $u,v,w \in \mathfrak{g}_i$, by using the differential Bianchi identity, we get

$$\begin{aligned}
\mathrm{L}_x(\mathrm{K})(u,v,w) &= -\mathrm{L}_u(\mathrm{K})(v,x,w) - \mathrm{L}_v(\mathrm{K})(x,u,w)\\
&= -\mathrm{L}_u(\mathrm{K}(v,x)w) + \mathrm{K}(\mathrm{L}_u v, x)w + \mathrm{K}(v, \mathrm{L}_u x)w + \mathrm{K}(v,x)\mathrm{L}_u w\\
&\quad - \mathrm{L}_v(\mathrm{K}(u,x)w) + \mathrm{K}(\mathrm{L}_v u, x)w + \mathrm{K}(u, \mathrm{L}_v x)w + \mathrm{K}(u,x)\mathrm{L}_v w\\
&= \mathrm{K}(\mathrm{L}_u v, x)w + \mathrm{K}(v, \mathrm{L}_u x)w + \mathrm{K}(\mathrm{L}_v u, x)w + \mathrm{K}(u, \mathrm{L}_v x)w,
\end{aligned}$$

since, by virtue of Proposition 2.1, $\mathrm{K}(u,x) = \mathrm{K}(v,x) = 0$. This shows, also according to Proposition 2.1, that $\mathrm{L}_x(\mathrm{K})(u,v,w) \in \mathfrak{g}_i$. Now

$$\begin{aligned}
\mathrm{L}_x(\mathrm{K})(u,v,w) &= \mathrm{L}_x(\mathrm{K}(u,v)w) - \mathrm{K}(\mathrm{L}_x u, v)w - \mathrm{K}(u, \mathrm{L}_x v)w - \mathrm{K}(u,v)\mathrm{L}_x w\\
&= \mathrm{L}_x(\mathrm{K}(u,v)w) - \mathrm{K}(\mathrm{L}_x u, v)w - \mathrm{K}(u, \mathrm{L}_x v)w + \mathrm{K}(v, \mathrm{L}_x w)u + \mathrm{K}(\mathrm{L}_x w, u)v.
\end{aligned}$$

Since $\mathrm{L}_x(\mathrm{K})(u, v, w) \in \mathfrak{g}_i$ and $\mathrm{K}(\mathfrak{g}, \mathfrak{g})\mathfrak{g}_i \subset \mathfrak{g}_i$, we get $\mathrm{L}_x(\mathrm{K}(u, v)w) \in \mathfrak{g}_i$. Having this property in mind, we will prove now that $\mathrm{L}_x\mathrm{Ric}(u) \in \mathfrak{g}_i$. Choose an orthonormal basis $(e_1, \dots, e_n)$ which is adapted to the splitting (9) and put $\epsilon_i = \langle e_i, e_i\rangle$. For any $z \in \mathfrak{g}_i^\perp$, we have

$$\langle \mathrm{L}_x\mathrm{Ric}(u), z\rangle = -\langle \mathrm{Ric}(u), \mathrm{L}_x z\rangle = \sum_{k=1}^{n} \epsilon_i \langle \mathrm{K}(u, e_k)e_k, \mathrm{L}_x z\rangle = -\sum_{k=1}^{n} \epsilon_i \langle \mathrm{L}_x(\langle \mathrm{K}(u, e_k)e_k), z\rangle = 0.$$

We have used the fact that if $e_k \in \mathfrak{g}_i$ then $\mathrm{L}_x(\langle \mathrm{K}(u, e_k)e_k) \in \mathfrak{g}_i$ and if $e_k \in \mathfrak{g}_i^\perp$ then $\mathrm{K}(u, e_k) = 0$. Thus $\mathrm{L}_x\mathrm{Ric}(u) = \lambda_i \mathrm{L}_x u \in \mathfrak{g}_i$ where λ_i is the eigenvalues of Ric associated to the eigenspace $\mathfrak{g}_i$. We conclude that $\mathrm{L}_x\mathfrak{g}_i \subset \mathfrak{g}_i$ which shows that, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$, $\mathrm{L}_{\mathfrak{g}_j}\mathfrak{g}_i \subset \mathfrak{g}_i$ and $\mathrm{L}_{\mathfrak{g}_0}\mathfrak{g}_i \subset \mathfrak{g}_i$. Since L takes its values in $\mathrm{so}(\mathfrak{g})$, the other inclusions follow immediately.

Suppose that $\dim \mathfrak{g}_0 = 1$ an choose a non null vector $u \in \mathfrak{g}_0$. Since $\dim \mathfrak{g}_0 = 1$, $\mathfrak{g}_0$ is nondegenerate and $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ we get $u.u = 0$. Moreover, $\mathrm{K}(u, .) = 0$ and hence from (7) $[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2$. By induction, we deduce that, for any $k \in \mathbb{N}^*$, $[\mathrm{R}_u^k, \mathrm{L}_u] = k\mathrm{R}_u^{k+1}$. This implies that $\mathrm{tr}(\mathrm{R}_u^k) = 0$ for any $k \geq 2$ and hence R_u is a nilpotent endomorphism. □

We will use the following lemma later.

Lemma 3.1. *Let V be a pseudo-Euclidean vector space of dimension ≤ 3 and A, B are, respectively, an endomorphism and a skew-symmetric endomorphism such that $[A, B] = A^2$. Then $A = 0$ or $B = 0$.*

Proof. The relation $[A, B] = A^2$ implies that, for any $k \in \mathbb{N}^*$, $[A^k, B] = kA^{k+1}$ and $\mathrm{tr}(A^k) = 0$ for $k \geq 2$ which implies that A is nilpotent. If $\dim V = 2$ we have $[A, B] = 0$ and if $\dim V = 3$ we have $[A^2, B] = 0$. To conclude it suffices to show that in a pseudo-Euclidean vector space of dimension ≤ 3 if N and B are, respectively, nilpotent and skew-symmetric satisfying $[N, B] = 0$ then $B = 0$ or $N = 0$. Suppose $N \neq 0$ and denote by N^c and B^c the associated complex endomorphisms of $V \otimes \mathbb{C}$.

If $\dim V = 2$ and since $[N, B] = 0$ then there exists a basis of $V \otimes \mathbb{C}$ such that

$$[N^c] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ [B^c] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \ \{\alpha, \beta\} = \{\imath a, -\imath a\} \text{ or } \{\alpha, \beta\} = \{a, -a\}.$$

The condition $[N, B] = 0$ implies $a = 0$ and hence $B = 0$.

If $\dim V = 3$ and since $[N, B] = 0$ then there exists a basis of $V \otimes \mathbb{C}$ such that

$$[N^c] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, [B^c] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \{\alpha, \beta\} = \{\imath a, -\imath a\} \text{ or } \{\alpha, \beta\} = \{a, -a\}.$$

The condition $[N, B] = 0$ implies $a = 0$ and hence $B = 0$. □

Let (G, h) be a four dimensional semi-symmetric Lorentzian Lie group with Ricci curvature having a non zero eigenvalue. By virtue of Remark 1, the Ricci tensor is diagonalizable and, according to (9) and Proposition 3.1, the Lie algebra $\mathfrak{g}$ of G has one of the following types:

($S4\lambda$) $\dim \mathfrak{g} = 4$ and $\mathfrak{g} = \mathfrak{g}_\lambda$ with $\lambda \neq 0$.

($S4\mu\lambda$) $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_\mu = \dim \mathfrak{g}_\lambda = 2$, $\lambda \neq \mu$, $\lambda \neq 0$, $\mu \neq 0$, $\mathfrak{g}_\mu.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$.

($S40^1\lambda$) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_0 = 1$, $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and $\lambda \neq 0$.

$(S40^2\lambda)$ $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_\lambda = 2$, $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and $\lambda \neq 0$.

Here $\mathfrak{g}_\lambda = \ker(\mathrm{Ric} - \lambda \mathrm{Id}_\mathfrak{g})$ and $\mathfrak{g}_0 = \ker(\mathrm{Ric}^2)$.

In [6], there is a classification of four-dimensional Lorentzian Einstein Lie algebras which are all locally symmetric. To complete showing that G is locally symmetric we need the following three propositions.

Proposition 3.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Lorentzian Lie algebra of type $(S4\mu\lambda)$. Then $\mathfrak{g}_\lambda.\mathfrak{g}_\mu = \mathfrak{g}_\mu.\mathfrak{g}_\lambda = 0$ and hence $\mathfrak{g}$ is the product of a two dimensional Euclidean Lie algebra with a two dimensional Lorentzian Lie algebra.*

Proof. We have $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\lambda$ with $\mu \neq 0$, $\lambda \neq 0$, $\mu \neq \lambda$ $\mathfrak{g}_\mu.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda.\mathfrak{g}_\mu \subset \mathfrak{g}_\mu$ and $\mathfrak{g}_\mu.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$. We can suppose that $\mathfrak{g}_\mu$ is Euclidean and $\mathfrak{g}_\lambda$ is Lorentzian. According to Proposition 3.1, there exists an orthonormal basis (e, f) of $\mathfrak{g}_\mu$ and an orthonormal basis (g, h) of $\mathfrak{g}_\lambda$ such that, in restriction to $\mathfrak{g}_\mu$, L_f vanishes and, in restriction to $\mathfrak{g}_\lambda$, L_h vanishes. So

$$\mathrm{L}_e = ae\wedge f + bg\wedge h,\ \mathrm{L}_f = dg\wedge h,\ \mathrm{L}_g = ue\wedge f + vg\wedge h,\ \mathrm{L}_h = pe\wedge f,\ \mathrm{K}(e, f) = -\lambda e\wedge f \quad \text{and} \quad \mathrm{K}(g, h) = -\mu g\wedge h.$$

We have

$$[e, f] = ae,\ [e, g] = bh + uf,\ [e, h] = bg + pf,\ [f, g] = dh - ue,\ [f, h] = dg - pe,\ [g, h] = vg.$$

The relations

$$-\mu e \wedge f = \mathrm{L}_{[e,f]} - [\mathrm{L}_e, \mathrm{L}_f] \quad \text{and} \quad -\lambda g \wedge h = \mathrm{L}_{[g,h]} - [\mathrm{L}_g, \mathrm{L}_h]$$

are equivalent to $a^2 = -\lambda$, $v^2 = \mu$, $ab = vu = 0$ and hence $u = b = 0$. Now the relation $0 = \mathrm{L}_{[f,h]} - [\mathrm{L}_f, \mathrm{L}_h]$ is equivalent to $ap = dv - bp = 0$ and hence $p = d = 0$ and we get the result. □

Proposition 3.3. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Lorentzian Lie algebra of type $(S40^1\lambda)$. Then $\mathfrak{g}.\mathfrak{g}_0 = 0$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and hence $\mathfrak{g}$ the semi-direct product of $\mathfrak{g}_0$ with the three dimensional pseudo-Euclidean Lie algebra $\mathfrak{g}_\lambda$ of constant curvature and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_\lambda$ is by a skew-symmetric derivation.*

Proof. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_0 = 1$, $\lambda \neq 0$ and $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and $\mathfrak{g}_0.\mathfrak{g}_0 = \{0\}$. This implies that $\mathfrak{g}_\lambda.\mathfrak{g}_0 \subset \mathfrak{g}_\lambda$. Choose a generator u of $\mathfrak{g}_0$. According to Proposition 3.1, R_u is nilpotent and $[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2$. But $\mathrm{R}_u(u) = 0$ and $\mathrm{R}_u(\mathfrak{g}_\lambda) \subset \mathfrak{g}_\lambda$ and hence, according to Lemma 3.1, $\mathrm{R}_u = 0$ or $\mathrm{L}_u = 0$. Moreover, by virtue of Propositions 2.2 and 3.1, for any $v, w \in \mathfrak{g}_\lambda$, $\mathrm{K}(v, w) = -\frac{\lambda}{2} v \wedge w$ and $\mathrm{K}(u, .). = \mathrm{K}(., .)u = 0$. Let show that $\mathrm{R}_u = 0$.

Suppose that $\mathrm{R}_u \neq 0$, hence $\mathrm{L}_u = 0$ and $\mathrm{R}_u^2 = 0$. Then $\mathrm{Im}\mathrm{R}_u$ is a one dimensional subspace of $\mathfrak{g}_\lambda$. Choose a generator $v = x.u \in \mathrm{Im}\mathrm{R}_u$. We have,

$$0 = \mathrm{L}_{[u,x]} - [\mathrm{L}_u, \mathrm{L}_x] = \mathrm{L}_{x.u}.$$

So $\mathrm{L}_v = 0$. Then, for any $w \in \mathfrak{g}_\lambda$,

$$-\frac{\lambda}{2} v \wedge w = \mathrm{L}_{[v,w]} - [\mathrm{L}_v, \mathrm{L}_w] = \mathrm{L}_{w.v}.$$

Consider $\mathrm{R}_v : \mathfrak{g}_\lambda \longrightarrow \mathfrak{g}$. From the relation above, we have $\ker \mathrm{R}_v = \mathbb{R}v$. So there exists two linearly independent vectors $v_1, v_2 \in \mathfrak{g}_\lambda$ such that $\{v, v_1, v_2\}$ is a basis of $\mathfrak{g}_\lambda$, $\{v_1.v, v_2.v\}$ are linearly independent with

$$\mathrm{L}_{v_1.v} = -\lambda v \wedge v_1 \quad \text{and} \quad \mathrm{L}_{v_2.v} = -\lambda v \wedge v_2.$$

This implies that $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and hence $\mathrm{R}_u = 0$. Finally, $\mathrm{R}_u = 0$.

Now $D = \mathrm{L}_u = \mathrm{ad}_u$ is a skew-symmetric derivation of $\mathfrak{g}_\lambda$. If $\mathfrak{g}_\lambda$ is unimodular then $D = \mathrm{ad}_v$ with $v \in \mathfrak{g}_\lambda$ and since the metric on $\mathfrak{g}_\lambda$ is bi-invariant, for any $w \in \mathfrak{g}_\lambda$, $\mathrm{L}_w = \frac{1}{2}\mathrm{ad}_w$. So

$$\mathrm{K}(u, w) = \mathrm{L}_{[u,w]} - [\mathrm{L}_u, \mathrm{L}_w] = \frac{1}{2}\mathrm{ad}_{[v,w]} - \frac{1}{2}[\mathrm{ad}_v, \mathrm{ad}_w] = 0.$$

If $\mathfrak{g}_\lambda$ is nonunimodular then, for any $v \in \mathfrak{g}_\lambda$ $\mathrm{ad}_{Dv} = [D, \mathrm{ad}_v]$ and hence $0 = \mathrm{tr}(\mathrm{ad}_{Dv}) = \langle Dv, \mathbf{h}\rangle$ which implies that $D\mathbf{h} = 0$. One can check easily that this condition suffices to insure that $\mathrm{K}(u, v) = 0$ for any $v \in \mathfrak{g}_\lambda$. □

Proposition 3.4. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Lorentzian Lie algebra of type $(S40^2\lambda)$. Then $\mathfrak{g}_0.\mathfrak{g} = 0$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$, $\mathfrak{g}_\lambda.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and hence $\mathfrak{g}$ is the semi-direct product of the pseudo-Euclidean Lie algebra $\mathfrak{g}_\lambda$ with the abelian Lie algebra $\mathfrak{g}_0$ and the action of $\mathfrak{g}_\lambda$ on $\mathfrak{g}_0$ is given by skew-symmetric endomorphisms.*

Proof. We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda$ with $\dim \mathfrak{g}_0 = 2$, $\mathfrak{g}_0.\mathfrak{g}_0 \subset \mathfrak{g}_0$ and $\mathfrak{g}_0.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$. Moreover, for any $u \in \mathfrak{g}_0$ and $v, w \in \mathfrak{g}_\lambda$, $\mathrm{K}(u, .). = \mathrm{K}(., .)u = 0$ and $\mathrm{K}(v, w) = -\frac{\lambda}{2} u \wedge v$. Let first show that $\mathfrak{g}_0.\mathfrak{g}_0 = \{0\}$. Since $\mathfrak{g}_0$ is a pseudo-Euclidean Lie algebra with vanishing curvature then $\mathfrak{g}_0.\mathfrak{g}_0 = \{0\}$ when $\mathfrak{g}_0$ is Euclidean. If $\mathfrak{g}_0$ is Lorentzian then there exists a basis (e, f) of $\mathfrak{g}_0$ with $\langle e, f\rangle = 1$ such that

$$\mathrm{L}_e = ag \wedge h,\ \mathrm{L}_f = ce \wedge f + bg \wedge h \quad \text{and} \quad [e, f] = cf.$$

But the Lie algebra of skew-symmetric endomorphisms of a 2-dimensional pseudo-Euclidean vector space is abelian then, for any $u, v \in \mathfrak{g}_0$, we have $[\mathrm{L}_u, \mathrm{L}_v] = 0$ and hence $\mathrm{L}_{[u,v]} = 0$. Thus $c\mathrm{L}_f = 0$ which implies $\mathfrak{g}_0.\mathfrak{g}_0 = \{0\}$.

Consider $N = \{u \in \mathfrak{g}_0, \mathrm{L}_u = 0\}$. Since $\mathfrak{g}_0.\mathfrak{g}_0 = \{0\}$ and $\dim \mathrm{L}(\mathfrak{g}_0) \leq 1$ we have $\dim N \geq 1$. Suppose that $\dim N = 1$. Therefore, we can choose an orthonormal basis (e, f) of $\mathfrak{g}_0$ such that $\mathrm{L}_e \neq 0$ and $\mathrm{L}_f \neq 0$. Since $e.e = 0$, L_e left invariant $e^\perp$. We have also $\langle \mathrm{R}_e v, e\rangle = 0$ and hence R_e leaves invariant $e^\perp$. Since $e.e = 0$, we get from (7) that $[\mathrm{R}_e, \mathrm{L}_e] = \mathrm{R}_e^2$. According to Lemma 3.1, the restriction of R_e to $e^\perp$ vanishes and hence its vanishes. A same argument shows that $\mathrm{R}_f = 0$ and hence for any $u \in \mathfrak{g}_0$, $\mathrm{R}_u = 0$. This implies that $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$. Now, for any $u \in \mathfrak{g}_0$, L_u is a skew-symmetric derivation of $\mathfrak{g}_\lambda$ and hence $\mathrm{L}_u = 0$. So we have shown that, for any $u \in \mathfrak{g}_0$, $\mathrm{L}_u = 0$. Let show now that $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$. Remark first that is equivalent to $\mathrm{Im}\mathrm{R}_u \subset \mathfrak{g}_0$ for any $u \in \mathfrak{g}_0$.

Suppose that there exists $u \in \mathfrak{g}_0$ such that $\mathrm{Im}\mathrm{R}_u \not\subset \mathfrak{g}_0$. This means that there exists $v \in \mathfrak{g}_\lambda$ such that $v.u = v_0 + v_1$ where $v_0 \in \mathfrak{g}_0$ and $v_1 \in \mathfrak{g}_\lambda$ with $v_1 \neq 0$. Then $\mathrm{L}_{v.u} = \mathrm{L}_{[\mathrm{v},\mathrm{u}]} = \mathrm{L}_{v_1} = 0$. Therefore, for any $w \in \mathfrak{g}_\lambda$, $\mathrm{L}_{w.v} = -\frac{\lambda}{2} w \wedge v$. This implies that $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ which is a contradiction. So we have proved so far that, for any $u \in \mathfrak{g}_0$, $\mathrm{L}_u = 0$, $\mathfrak{g}_\lambda.\mathfrak{g}_\lambda \subset \mathfrak{g}_\lambda$ and $\mathfrak{g}_\lambda.\mathfrak{g}_0 \subset \mathfrak{g}_0$. So $\mathfrak{g}$ is the semi-direct product of $\mathfrak{g}_\lambda$ with $\mathfrak{g}_0$ and the action of $\mathfrak{g}_\lambda$ on $\mathfrak{g}_0$ is given by skew-symmetric endomorphisms. □

4. Proof of Theorems 1.1 and 1.2

The proof is based on Corollary 2.1 and the following two theorems proved, respectively, in [8] and [5].

Theorem 4.1 *([8]). Let (M, g) be an oriented four-dimensional Lorentzian Einstein manifold whose curvature operator, treated as a complex-linear vector bundle morphism $\widetilde{K} : \wedge^2 TM \longrightarrow \wedge^2 TM$, is diagonalizable at every point and has complex eigenvalues that form constant functions $M \longrightarrow \mathbb{C}$. Then (M, g) is locally homogeneous, and one of the following three cases occurs:*

(a) (M, g) *is a space of constant curvature.*
(b) (M, g) *is locally isometric to the Riemannian product of two pseudo-Riemannian surfaces having the same constant Gaussian curvature.*
(c) (M, g) *is locally isometric to a Petrov's Ricci-flat manifold.*

Furthermore, (M, g) *is locally symmetric in cases* $(a)-(b)$*, but not in* (c)*, and in case* (c) *it is locally isometric to a Lie group with a left-invariant metric.*

Theorem 4.2 *([5]). Let* (M, g) *be a locally homogeneous Lorentzian four-manifold. If its Ricci operator is diagonalizable then* (M, g) *is either Ricci-parallel or locally isometric to a Lie group equipped with a left invariant Lorentzian metric.*

Proof of Theorem 1.1. Suppose that M is semi-symmetric. For any $p \in M$, K_p is a semi-symmetric curvature tensor on T_pM. According to Corollary 2.1, its total curvature operator is diagonalizable as $\mathbb{C}$-linear endomorphism of $\wedge^2 T_pM$ with eigenvalues 0 and $-\frac{\lambda}{4}$ where λ is the scalar curvature. So the eigenvalues are constant and, according to Theorem 4.1, M is locally symmetric. $\square$

Proof of Theorem 1.2. Let (M, g) be a simply connected homogeneous semi-symmetric Lorentzian four-manifold with Ricci curvature having a non zero eigenvalue. According to Remark 1, Ric must be diagonalizable. So, according to Theorem 4.2, (M, g) is either Ricci-parallel or locally isometric to a Lie group equipped with a left invariant Lorentzian metric. If (M, g) is Ricci-parallel and has two distinct eigenvalues then, according to Theorem 7.3 in [5], M is locally symmetric. Suppose now that (M, g) is Einstein with non null scalar curvature. According to Corollary 2.1, the total curvature is diagonalizable and we can apply Theorem 4.1 to get that M is locally symmetric. If M is a Lorentzian Lie group, we have shown in section 3 that M is locally symmetric. This completes the proof. $\square$

5. Four-dimensional Ricci flat and Ricci isotropic homogeneous semi-symmetric Lorentzian manifolds

In this section, we deal with non flat semi-symmetric four-dimensional Lorentzian manifolds with isotropic Ricci curvature. According to Remark 1, these manifolds satisfy $\mathrm{Ric}^2 = 0$.

We use Komrakov's classification [9] of four-dimensional homogeneous pseudo-Riemannian manifolds and we apply the following algorithm to find among Komrakov's list the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to four-dimensional Ricci flat or Ricci isotropic homogeneous semi-symmetric Lorentzian manifolds.

Let $M = \overline{G}/G$ be an homogeneous manifold with G connected and $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{m}$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{m}$ an arbitrary complementary of $\mathfrak{g}$ (not necessary $\mathfrak{g}$-invariant). The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ uniquely defines the isotropy representation $\rho : \mathfrak{g} \longrightarrow \mathrm{gl}(\mathfrak{m})$ by $\rho(x)(y) = [x, y]_{\mathfrak{m}}$, for all $x \in \mathfrak{g}$, $y \in \mathfrak{m}$. Let $\{e_1, \dots, e_r, u_1, \dots, u_n\}$ be a basis of $\bar{\mathfrak{g}}$ where $\{e_i\}$ and $\{u_j\}$ are bases of $\mathfrak{g}$ and $\mathfrak{m}$, respectively. The algorithm goes as follows.

1. Determination of invariant pseudo-Riemannian metrics on M. It is well-known that invariant pseudo-Riemannian metrics on M are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms on $\mathfrak{m}$. A symmetric bilinear form on $\mathfrak{m}$ is determined by its matrix B in $\{u_i\}$ and its invariant if $\rho(e_i)^t \circ B + B \circ \rho(e_i) = 0$ for $i = 1, \dots, r$.
2. Determination of the Levi-Civita connection. Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{m}$. It defines uniquely an invariant linear Levi-Civita connection $\nabla : \bar{\mathfrak{g}} \longrightarrow \mathrm{gl}(\mathfrak{m})$ given by

$$\nabla(x) = \rho(x),\ \nabla(y)(z) = \frac{1}{2}[y, z]_{\mathfrak{m}} + \nu(y, z),\ x \in \mathfrak{g}, y, z \in \mathfrak{m},$$

where $\nu : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ is given by the formula

$$2B(\nu(a,b),c) = B([c,a]_{\mathfrak{m}},b) + B([c,b]_{\mathfrak{m}},a),\ a,b,c \in \mathfrak{m}.$$

3. Determination of the curvature. The curvature of B is the bilinear map $\mathrm{K} : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathrm{gl}(\mathfrak{m})$ given by

$$\mathrm{K}(a,b) = [\nabla(a), \nabla(b)] - \nabla([a,b]_{\mathfrak{m}}) - \rho([a,b]_{\mathfrak{g}}),\ a,b \in \mathfrak{m}.$$

4. Determination of the Ricci curvature. It is given by its matrix in $\{u_i\}$, i.e., $\mathrm{ric} = (\mathrm{ric}_{ij})_{1\le i,j\le n}$ where

$$\mathrm{ric}_{ij} = \sum_{r=1}^{n} \mathrm{K}_{ri}(u_r, u_j).$$

5. Determination of the Ricci operator. We have $\mathrm{Ric} = B^{-1}\mathrm{ric}$.
6. Checking the semi-symmetry condition.

The following theorem gives the list of homogeneous with non trivial isotropy four dimensional semi-symmetric non symmetric Lorentzian manifolds which are Ricci flat non flat.

Theorem 5.1. *Let $M = \bar{G}/G$ be four-dimensional semi-symmetric non symmetric Ricci flat homogeneous Lorentzian manifold. Then M is isometric to one of the following models, where $\mathfrak{g} = \mathbb{R}e_1$ and that the only non trivial brackets $[e_1, u_i]$ are indicated:*

I) 1.4^1, $\bar{\mathfrak{g}} = span\{e_1, u_1, u_2, u_3, u_4\}$ *with* $[e_1, u_2] = u_1,\ [e_1, u_3] = u_2$ *and* $B_0 = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & b & d \\ 0 & 0 & d & c \end{pmatrix}$ $(ac < 0)$*;*

(a) $1.4^1 : 9$*;*

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2 + u_4,\ [u_3, u_4] = pu_4 \quad \textit{with } c = 2ap^2 + 2ap + 2ar,$$

(b) $1.4^1 : 10$*;*

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2,\ [u_3, u_4] = pu_4 \quad \textit{with } p^2 + p + r = 0,$$

(c) $1.4^1 : 11$*;*

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2 + u_4,\ [u_3, u_4] = u_1 - u_4 \quad \textit{with } c = 2ar,$$

(d) $1.4^1 : 13$*;*

$$[u_2, u_3] = re_1 + u_4,\ [u_3, u_4] = u_4 \quad \textit{with } c = 2a(1 + r),$$

(e) $1.4^1 : 14$*;*

$$[u_2, u_3] = re_1,\ [u_3, u_4] = u_4 \quad \textit{with } r = -1,$$

(f) $1.4^1 : 16$*;*

$$[u_2, u_3] = -e_1 + u_4,\ [u_3, u_4] = u_1 \quad \textit{with } c = -2a,$$

(g) $1.4^1 : 19$;

$$[u_2, u_3] = -e_1 + u_4, \quad \textit{with } c = -2a,$$

II) 2.5^2, $\bar{\mathfrak{g}} = span\{e_1, e_2, u_1, u_2, u_3, u_4\}$ *with* $[e_1, u_2] = -[e_2, u_4] = u_1$, $[e_1, u_3] = -u_2, [e_2, u_3] = u_4$ *and* $B_0 = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$;

(a) $2.5^2 : 2$;

$$[u_2, u_3] = (1+s)e_1, \ [u_3, u_4] = (1-s)e_2 \quad \textit{with } p = -r^2, \quad s > 0,$$

(b) $2.5^2 : 3$;

$$[u_2, u_3] = (r+s)e_1 - u_4, \ [u_2, u_4] = u_1, \ [u_3, u_4] = (s-r)e_2 - u_2 \ \textit{with } 4r = 1, \quad s > 0.$$

The following theorem gives the list of homogeneous with non trivial isotropy four dimensional semi-symmetric non symmetric Lorentzian manifolds which are not Ricci flat.

Theorem 5.2. *Let $M = \bar{G}/G$ be four-dimensional semi-symmetric non symmetric homogeneous Lorentzian manifold satisfying* $\mathrm{Ric}^2 = 0$ *and* $\mathrm{Ric} \neq 0$. *Then M is isometric to one of the following models, where $\mathfrak{g} = \mathbb{R}e_1$ and that the only non trivial brackets $[e_1, u_i]$ are indicated:*

I) $\mathbf{1.1}^2$, $\bar{\mathfrak{g}} = span\{e_1, u_1, u_2, u_3, u_4\}$ *with* $[e_1, u_2] = u_3$, $[e_1, u_3] = -u_1$, $B_0 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & a & 0 \\ 0 & c & 0 & d \end{pmatrix}$ *($ac < 0$);*

(a) $1.1^2 : 1$;

$$[u_1, u_3] = -u_2, \ [u_1, u_4] = u_1, \ [u_2, u_4] = 2u_2, \ [u_3, u_4] = u_3 \quad \textit{with } 2ap^2 + 2ap + 2ar - c \neq 0,$$

(b) $1.1^2 : 2$;

$$[u_1, u_4] = u_1, \ [u_2, u_4] = pu_2, \ [u_3, u_4] = u_3 \quad \textit{with } p \neq 0, \ 1,$$

II) $\mathbf{1.4}^1$, $\bar{\mathfrak{g}} = span\{e_1, u_1, u_2, u_3, u_4\}$ *with* $[e_1, u_2] = u_1$, $[e_1, u_3] = u_2$ *and* $B_0 = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & b & d \\ 0 & 0 & d & c \end{pmatrix}$ *($ac < 0$);*

(a) $1.4^1 : 2$;

$$[e_1, u_4] = e_1, \quad [u_1, u_4] = u_1, \quad [u_3, u_4] = -u_3 \quad \textit{with } b \neq 0$$

(b) $1.4^1 : 9$;

$$[u_1, u_3] = u_1, \ [u_2, u_3] = re_1 + u_2 + u_4, \quad [u_3, u_4] = pu_4 \quad \textit{with } 2ap^2 + 2ap + 2ar - c \neq 0,$$

(c) $1.4^1 : 10$;

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2,\ [u_3, u_4] = pu_4 \quad \textit{with } p^2 + p + r \neq 0,$$

(d) $1.4^1 : 11$;

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2 + u_4,\ [u_3, u_4] = u_1 - u_4 \quad \textit{with } c \neq 2ar,$$

(e) $1.4^1 : 12$;

$$[u_1, u_3] = u_1,\ [u_2, u_3] = re_1 + u_2,\ [u_3, u_4] = u_1 - u_4 \quad \textit{with } r \neq 0,$$

(f) $1.4^1 : 13$;

$$[u_2, u_3] = re_1 + u_4,\ [u_3, u_4] = u_4 \quad \textit{with } c \neq 2a(1 + r),$$

(g) $1.4^1 : 15$ *and* 17;

$$[u_2, u_3] = \epsilon e_1 + u_4,\ [u_3, u_4] = u_1 \quad \textit{with } c + 2\epsilon a \neq 0, \quad \epsilon = 0, 1,$$

(h) $1.4^1 : 16$;

$$[u_2, u_3] = -e_1 + u_4,\ [u_3, u_4] = u_1 \quad \textit{with } c \neq -2a, \quad a \neq -c,$$

(i) $1.4^1 : 18$ *and* 20;

$$[u_2, u_3] = \epsilon e_1 + u_4, \qquad \epsilon = 0, 1,$$

(j) $1.4^1 : 19$;

$$[u_2, u_3] = -e_1 + u_4, \quad \textit{with } c \neq -2a,$$

III) $\mathbf{2.5^2}$, $\bar{\mathfrak{g}} = span\{e_1, e_2, u_1, u_2, u_3, u_4\}$ *with* $[e_1, u_2] = -[e_2, u_4] = u_1$, $[e_1, u_3] = -u_2, [e_2, u_3] = u_4$ *and*

$$B_0 = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & a & 0 & 0 \\ a & 0 & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix};$$

(a) $2.5^2 : 2$;

$$[u_1, u_3] = u_1,\ [u_2, u_3] = A,\ [u_2, u_4] = 2ru_1, [u_2, u_3] = B,$$

$$\textit{with } A = (p+s)e_1 + re_2 + u_2 - 2ru_4, \quad B = -re_1 + (p-s)e_2 - 2ru_2 - u_4, \quad r \geq 0, \quad s \geq \quad \textit{and } p + r^2 \neq 0,$$

(b) $2.5^2 : 3$;

$$[u_2, u_3] = -(r + s)e_1 - u_4,\ [u_2, u_4] = u_1 \quad , \ [u_3, u_4] = (s - r)e_2 - u_2 \ \textit{with } 4r \neq 1, \quad s > 0.$$

Remark 2. They are four dimensional homogeneous Lorentzian manifolds which are symmetric and Ricci isotropic. For instance, $1.4^1 : 14 - 21 - 22 - 24 - 25$ and $2.5^2 : 4 - 5 - 6$ symmetric and satisfy $\mathrm{Ric}^2 = 0$. Moreover, $2.5^2 : 6$ is symmetric, Ricci flat non flat with $K^2 = 0$.

6. Semi-symmetric Ricci isotropic four dimensional Lorentzian Lie algebras

Note first that if G is a Lie group with a semi-symmetric Ricci flat Lorentzian metric then, according to Corollary 2.1, $\mathrm{K}^2 = 0$. In the list obtained by Calvaruso–Zaeim [6], the condition $\mathrm{K}^2 = 0$ is equivalent to $\mathrm{K} = 0$. To 'complete our study, we devote the reminder of this section to the determination of the list of the Lie algebras with their Lorentzian metrics associated to Lie groups with a left invariant Lorentzian metric which is semi-symmetric and Ricci isotropic.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a Lorentzian Lie algebra with semi-symmetric Ricci isotropic curvature. According to Proposition 2.3, there exists a basis (e, f, g, h) a basis of $\mathfrak{g}$ such that the non vanishing products are $\langle e, e\rangle = \langle f, f\rangle = \langle g, h\rangle = 1$ and

$$K = \omega_1 A_{e,g} \vee A_{e,g} + \omega_2 A_{f,g} \vee A_{f,g},\ |\omega_1 + \omega_2| = 1, \quad \text{and} \quad \mathfrak{h}(K) = \mathrm{span}\{\omega_1 A_{e,g}, \omega_2 A_{f,g}\}. \tag{10}$$

Let $\mathfrak{h}(\mathfrak{g})$ be the holonomy Lie algebra of $\mathfrak{g}$. It is the smallest Lie algebra containing $\mathfrak{h}(K)$ and satisfying $[\mathrm{L}_u, \mathfrak{h}(\mathfrak{g})] \subset \mathfrak{h}(\mathfrak{g})$, for any $u \in \mathfrak{g}$. Before starting the computation, remark that if $\dim \mathfrak{h}(K) = 2$ then $\mathfrak{g}$ is indecomposable, i.e., $\mathfrak{h}(\mathfrak{g})$ doesn't leave any proper nondegenerate vector subspace. Indeed, if E is a nondegenerate vector subspace invariant by $\mathfrak{h}(\mathfrak{g})$ then E is invariant by $A_{e,g}$ and $A_{f,g}$ and we can suppose that $\dim E = 1$ or 2. If $\dim E = 1$ then $A_{e,g}(E) = A_{f,g}(E) = 0$ and hence $E \subset \{e, g\}^\perp \cap \{f, g\}^\perp = \mathbb{R}g$ which is impossible. A same argument leads to a contradiction when $\dim E = 2$.

Let us compute now the Levi-Civita product from the curvature. We distinguish three cases:

1. $\mathfrak{g}$ is indecomposable with $\dim \mathfrak{h}(K) = 2$. In this case we will show that $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}(K)$.
2. $\mathfrak{g}$ is indecomposable with $\dim \mathfrak{h}(K) = 1$. In this case we will show that $\mathfrak{h}(\mathfrak{g}) = \mathrm{span}\{A_{e,g}, A_{f,g}\}$.
3. $\mathfrak{g}$ is decomposable. In this case we will show that $\dim \mathfrak{h}(\mathfrak{g}) = 1$.

Theorem 6.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Ricci isotropic Lorentzian Lie algebra with $\dim \mathfrak{h}(K) = 2$. Then, there exists a basis (e, f, g, h) with the non vanishing products $\langle e, e\rangle = \langle f, f\rangle = \langle g, h\rangle = 1$ and the non vanishing brackets have one of the following forms:*

1. $[e, f] = (a - b)g,\ [e, h] = \epsilon\sqrt{ab + \frac{1}{2}}e + (b + x)f + zg,\ [f, h] = (a - x)e + \epsilon\sqrt{ab + \frac{1}{2}}f + yg,\ [g, h] = 2\epsilon\sqrt{ab + \frac{1}{2}}g, a \neq b.$
2. $[e, f] = (a - \frac{2bc-1}{2a})g,\ [e, h] = ce + \frac{2bc-1}{2a}f + zg,\ [f, h] = ae + bf + yg,\ [g, h] = (c + b)g, a - \frac{2bc-1}{2a} \neq 0.$
3. $[e, h] = ae + xf + ag,\ [f, h] = -xe + af + yg,\ [g, h] = \frac{2a^2+1}{2a}g.$
4. $[e, h] = \epsilon\sqrt{\frac{2a^2+1}{2}}e + (a + x)f + zg,\ [f, h] = (a - x)e + \epsilon\sqrt{\frac{2a^2+1}{2}}f + yg,\ [g, h] = 2\epsilon\sqrt{\frac{2a^2+1}{2}}g.$
5. $[e, h] = ce + (a + \frac{2a^3+a-2abc}{b^2-c^2})f + zg,\ [f, h] = (a - \frac{2a^3+a-2abc}{b^2-c^2})e + bf + yg,\ [g, h] = \frac{2a^2+b^2+c^2+1}{b+c}g.$

In all what above $\epsilon^2 = 1$. Moreover, all the models above are not second-order locally symmetric and satisfy $\mathfrak{h}(\mathrm{K}) = \mathfrak{h}(\mathfrak{g})$.

Proof. In this case, the curvature is given by (10) with $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Put

$$[\mathrm{L}_e] = \begin{pmatrix} 0 & a & u_1 & c \\ -a & 0 & u_2 & l \\ -c & -l & -k & 0 \\ -u_1 & -u_2 & 0 & k \end{pmatrix}, \quad [\mathrm{L}_f] = \begin{pmatrix} 0 & m & v_1 & d \\ -m & 0 & v_2 & q \\ -d & -q & -r & 0 \\ -v_1 & -v_2 & 0 & r \end{pmatrix},$$

$$[\mathrm{L}_g] = \begin{pmatrix} 0 & s & w_1 & u \\ -s & 0 & w_2 & z \\ -u & -z & -w & 0 \\ -w_1 & -w_2 & 0 & w \end{pmatrix}, \quad [\mathrm{L}_h] = \begin{pmatrix} 0 & x & p_1 & n \\ -x & 0 & p_2 & y \\ -n & -y & -b & 0 \\ -p_1 & -p_2 & 0 & b \end{pmatrix}.$$

The notation $[A]$ designs the matrix of A in the basis (e, f, g, h). The differential Bianchi identity gives

$$\begin{aligned}
0 &= \mathrm{L}_e(\mathrm{K})(f,g) + \mathrm{L}_f(\mathrm{K})(g,e) + \mathrm{L}_g(\mathrm{K})(e,f) = -w_2\omega_1 A_{e,g} + w_1\omega_2 A_{e,f},\\
0 &= \mathrm{L}_e(\mathrm{K})(f,h) + \mathrm{L}_f(\mathrm{K})(h,e) + \mathrm{L}_h(\mathrm{K})(e,f)\\
&= (a(\omega_1-\omega_2) - (2r+p_2)\omega_1)A_{e,g} - (m(\omega_1-\omega_2) - 2\omega_2 k - p_1\omega_2)A_{f,g} + (u_1\omega_2 + v_2\omega_1)A_{e,f}\\
&\quad - (u_2\omega_2 - v_1\omega_1)A_{g,h},\\
0 &= \mathrm{L}_e(\mathrm{K})(g,h) + \mathrm{L}_g(\mathrm{K})(h,e) + \mathrm{L}_h(\mathrm{K})(e,g)\\
&= -(2w-u_1)\omega_1 A_{e,g} - (s(\omega_1-\omega_2) - u_2\omega_2)A_{f,g} + w_2\omega_1 A_{e,f} - w_1\omega_1 A_{g,h},\\
0 &= \mathrm{L}_f(\mathrm{K})(g,h) + \mathrm{L}_g(\mathrm{K})(h,f) + \mathrm{L}_h(\mathrm{K})(f,g)\\
&= (s(\omega_2-\omega_1) + v_1\omega_1)A_{e,g} - (2w-v_2)\omega_2 A_{f,g} + w_1\omega_2 A_{e,f} - w_2\omega_2 A_{g,h}.
\end{aligned}$$

So $w_1 = w_2 = 0$ and

$$\begin{aligned}
&0 = u_2\omega_2 - v_1\omega_1 = u_1\omega_2 + v_2\omega_1 = (2w-u_1)\omega_1 = (2w-v_2)\omega_2 \qquad (11)\\
&0 = s(\omega_1-\omega_2) - v_1\omega_1 = s(\omega_1-\omega_2) - u_2\omega_2 = a(\omega_2-\omega_1) + (2r+p_2)\omega_1 = m(\omega_2-\omega_1) + (2k+p_1)\omega_2.
\end{aligned}$$

Since $\omega_1 \neq 0$ and $\omega_2 \neq 0$ from (11) we get $u_1 = v_2 = w = 0$. On the other hand, since $g.g = 0$, we get from (7) $[\mathrm{R}_g, \mathrm{L}_g] = \mathrm{R}_g^2$. This implies that $[\mathrm{R}_g^k, \mathrm{L}_g] = k\mathrm{R}_g^{k+1}$ for any $k \in \mathbb{N}$. Thus $\mathrm{tr}(\mathrm{R}_g^k) = 0$, of any $k \geq 2$ and hence R_g is nilpotent, i.e, $\mathrm{R}_g^4 = 0$. Or,

$$[\mathrm{R}_g] = \begin{pmatrix} 0 & v_1 & 0 & p_1 \\ u_2 & 0 & 0 & p_2 \\ -k & -r & 0 & -b \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and a direct computation shows that $\mathrm{R}_g^4 = 0$ implies $v_1u_2 = 0$ and from (11) we get $v_1 = u_2 = 0$. The relation $[\mathrm{R}_g, \mathrm{L}_g] = \mathrm{R}_g^2$ is equivalent to

$$sp_1 = sp_2 = sr = sk = -uk - rz + kp_1 + rp_2 + up_1 + zp_2 = 0.$$

We have two cases.

1. $s \neq 0$. Then $\omega_1 = \omega_2$, $|\omega_1| = 1/2$ and $p_1 = p_2 = k = r = 0$. We consider the equations

$$\begin{cases}
\mathrm{K}(e,f) = \mathrm{L}_{[e,f]} - [\mathrm{L}_e, \mathrm{L}_f] = a\mathrm{L}_e + m\mathrm{L}_f + (d-l)\mathrm{L}_g - [\mathrm{L}_e, \mathrm{L}_f] = 0,\\
\mathrm{K}(e,g) = \mathrm{L}_{[e,g]} - [\mathrm{L}_e, \mathrm{L}_g] = s\mathrm{L}_f + (u-k)\mathrm{L}_g - [\mathrm{L}_e, \mathrm{L}_g] = 0,\\
\mathrm{K}(e,h) = \mathrm{L}_{[e,h]} - [\mathrm{L}_e, \mathrm{L}_h] = c\mathrm{L}_e + (l+x)\mathrm{L}_f + n\mathrm{L}_g + k\mathrm{L}_h - [\mathrm{L}_e, \mathrm{L}_h] = -\omega_1 A_{e,g},\\
\mathrm{K}(f,g) = \mathrm{L}_{[f,g]} - [\mathrm{L}_f, \mathrm{L}_g] = -s\mathrm{L}_e + (z-r)\mathrm{L}_g - [\mathrm{L}_f, \mathrm{L}_g] = 0,\\
\mathrm{K}(f,h) = \mathrm{L}_{[f,h]} - [\mathrm{L}_f, L_h] = (d-x)\mathrm{L}_e + q\mathrm{L}_f + y\mathrm{L}_g + r\mathrm{L}_h - [\mathrm{L}_f, L_h] = -\omega_2 A_{f,g},\\
\mathrm{K}(g,h) = \mathrm{L}_{[g,h]} - [\mathrm{L}_g, \mathrm{L}_h] = u\mathrm{L}_e + z\mathrm{L}_f + b\mathrm{L}_g - [\mathrm{L}_g, \mathrm{L}_h] = 0.
\end{cases} \qquad (12)$$

From the second equation $m = -u$. From the fourth equation we get $z = a$ and from the sixth equation we get $b = 0$. The equations become

$$\begin{cases} a^2 + u^2 + sd - sl &= 0, \\ ac - 2ul - aq &= 0, \\ -uq + 2ad + cu &= 0, \\ sl - a^2 + u^2 + sd &= 0, \\ -cs + 2au + sq &= 0, \\ ac - ul - ux + ns &= 0, \\ c^2 + dl + dx + nu + lx - ay + \omega_1 &= 0, \\ cl + lq + qx + 2an - cx &= 0, \\ ad - ax - uq + ys &= 0, \\ cd - cx + qd + 2yu + qx &= 0, \\ dl - lx + q^2 + ay - dx - nu + \omega_2 &= 0, \\ cu + ad + ax - ys &= 0, \\ ul + aq - ux + ns &= 0. \end{cases}$$

Then $u^2 = -sd$, $a^2 = sl$ and $c - q = 2s^{-1}au$. So

$$c^2 + q^2 + 2dl + 1 = (c - q)^2 + 2qc + 2dl + 1 = 4s^{-2}a^2u^2 - 2s^{-2}a^2u^2 + 2qc + 1 = 0.$$

Thus $cq = -\frac{1}{2} - s^{-2}a^2u^2$. Since $c - q = 2s^{-1}au$ we get that c and $-q$ are solutions of the equation $X^2 - 2s^{-1}auX + \frac{1}{2} + s^{-2}a^2u^2 = 0$ and this equation has no real solution. In conclusion the case $s \neq 0$ is impossible.

2. $s = 0$. From the first equation in (12) we get $a^2 + m^2 = 0$ and hence $a = m = 0$. From the second equation, we get $u = 0$, from the third equation we get $p_1 = 0$, from the fourth equation we get $z = 0$ and from the fifth equation we get $p_2 = 0$. Then (12) is now equivalent to

$$\begin{aligned} \omega_2 k = kx = \omega_1 r = rx = 0, \\ -cr + kd = -lr + kq = 0, \\ r(c + b + q) = k(c + b + q) = 0, \\ d(c + q - b) + 2rn + (q - c)x = 0, \\ l(c + q - b) + 2ky + (q - c)x = 0, \\ dl - lx + q^2 + 2ry - qb - dx + \omega_2 = 0, \\ c^2 + dl + dx + 2kn - cb + lx + \omega_1 = 0. \end{aligned}$$

Then $k = r = 0$ and

$$\begin{cases} d(c + q - b) + (q - c)x &= 0, \\ l(c + q - b) + (q - c)x &= 0, \\ dl - lx + q^2 - qb - dx + \omega_2 &= 0, \\ c^2 + dl + dx - cb + lx + \omega_1 &= 0. \end{cases}$$

This is equivalent to

$d(c+q-b)+(q-c)x=0,\ (d-l)(c+q-b)=0,\ c^2+q^2+2dl+1=b(q+c)$ and $B=dl-lx+q^2-qb-dx$.

If $d\neq l$ then $b=c+q$, $2(dl-qc)=-1$ and $(q-c)x=0$. Since d and l play symmetric roles we can suppose $d\neq 0$. We get two types of Lie algebras

$$[e,f]=(d-l)g,\ [e,h]=\epsilon\sqrt{dl+\frac{1}{2}}e+(l+x)f+ng,$$
$$[f,h]=(d-x)e+\epsilon\sqrt{dl+\frac{1}{2}}f+yg,\ [g,h]=2\epsilon\sqrt{dl+\frac{1}{2}}g, d\neq l$$

or

$$[e,f]=(d-\frac{2qc-1}{2d})g,\ [e,h]=ce+\frac{2qc-1}{2d}f+ng,\ [f,h]=de+qf+yg,\ [g,h]=(c+q)g, d-\frac{2qc-1}{2d}\neq 0.$$

If $d=l$ then $b(q+c)=c^2+q^2+2l^2+1$ and hence $b+c\neq 0$. If $q=c$ then we have two types of Lie algebras

$$[e,h]=ce+xf+ng,\ [f,h]=-xe+cf+yg,\ [g,h]=\frac{2c^2+1}{2c}g,$$
$$[e,h]=\epsilon\sqrt{\frac{2l^2+1}{2}}e+(l+x)f+ng,\ [f,h]=(l-x)e+\epsilon\sqrt{\frac{2l^2+1}{2}}f+yg,\ [g,h]=2\epsilon\sqrt{\frac{2l^2+1}{2}}g.$$

If $q\neq c$ then $b=\frac{2l^2+q^2+c^2+1}{q+c}$ and $x=\frac{2l^3+l-2lqc}{q^2-c^2}$. So

$$[e,h]=ce+(l+\frac{2l^3+l-2lqc}{q^2-c^2})f+ng,\ [f,h]=(d-\frac{2l^3+l-2lqc}{q^2-c^2})e+qf+yg,\ [g,h]=\frac{2l^2+q^2+c^2+1}{q+c}g.$$

For all these models we have $\mathrm{L}^2_{h,h}\mathrm{K}(f,h)\neq 0$ which shows that there are not second-order locally symmetric. Moreover, $\mathfrak{h}(\mathrm{K})$ is invariant by L which shows that $\mathfrak{h}(\mathrm{K})=\mathfrak{h}(\mathfrak{g})$. □

Theorem 6.2. *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Ricci isotropic indecomposable Lorentzian Lie algebra with $\dim\mathfrak{h}(\mathrm{K})=1$. Then, there exists a basis (e,f,g,h) with the non vanishing products $\langle e,e\rangle=\langle f,f\rangle=\langle g,h\rangle=1$ and the non vanishing brackets*

$$[e,f]=\frac{2a^2+1}{2a}g,\ [e,h]=\frac{1}{2a(2a^2-1)}f+xg,\ [f,h]=\frac{2a(a^2-1)}{2a^2-1}e+yg.$$

Moreover, $\mathfrak{h}(\mathfrak{g})=\mathrm{span}\{A_{e,g},A_{f,g}\}$ and $\mathfrak{g}$ is not second-order locally symmetric.

Proof. We proceed as in the proof of Theorem 6.1 and we suppose that $\omega_1=0$. Then (11) is equivalent to $u_1=u_2=s=a=0$, $v_2=2w$ and $m=-2k-p_1$. We have

$$[\mathrm{R}_e]=\begin{pmatrix}0&0&0&0\\0&-m&0&-x\\-c&-d&-u&-n\\0&-v_1&0&-p_1\end{pmatrix}\quad\text{and}\quad[\mathrm{R}_g]=\begin{pmatrix}0&v_1&0&p_1\\0&2w&0&p_2\\-k&-r&-w&-b\\0&0&0&0\end{pmatrix}.$$

From (7), we get $[\mathrm{R}_g,\mathrm{L}_g]-\mathrm{R}_g^2-w\mathrm{R}_g=0$ which is equivalent to

$$w=v_1(z-p_2)=v_1(u+k)=-uk-rz+kp_1+rp_2+up_1+zp_2=0. \qquad (13)$$

We have also $[\mathrm{R}_e, \mathrm{L}_e] - \mathrm{R}_e^2 - w\mathrm{R}_e = 0$ which is equivalent to

$$\begin{aligned} cv_1 = cp_1 = -lv_1 - m^2 + v_1x + cm = -lm - kx + lp_1 - xm - p_1x + cx = 0, \\ -ck + c^2 = lu - lm - kd - md - du + v_1n + cd \\ = -u^2 + uc = -c^2 - ld - 2kn - lx - xd - nu - p_1n + cn = 0, \\ (-k + m + p_1 - c)v_1 = lv_1 + v_1x - p_1^2 + cp_1 = 0. \end{aligned} \tag{14}$$

If $v_1 \neq 0$ we get from (13) and (14) $c = u = k = z = p_2 = 0$ and $m = -p_1$ and (14) becomes

$$-lv_1 - m^2 + v_1x = -2lm = lu - lm - md + v_1n = -ld - lx - xd - nu + mn = lv_1 + v_1x - m^2 = 0.$$

This implies that $l = 0$. We return to (12) and we find that the first equation implies $m = x = y = n = b = 0$ and from the fifth equation we deduce that $q = 0$ and $\omega_2 = 0$ which is impossible.

Thus $v_1 = 0$ and hence $p_1 = 0$. From the first equation in (12) we get $m = 0$, from the second equation we get $u = 0$, from the fourth equation we get $z = 0$, from the fifth equation we get $p_2 = 0$. From $m = -2k - p_1$ we get $k = 0$ and since $c^2 = ck$ we deduce that $c = 0$. Thus

$$\mathrm{L}_e = lA_{f,g},\ \mathrm{L}_f = dA_{e,g} + qA_{f,g} - rA_{g,h},\ \mathrm{L}_g = 0 \quad \text{and} \quad \mathrm{L}_h = xA_{e,f} + nA_{e,g} + yA_{f,g} - bA_{g,h},$$

and (12) is now equivalent to

$$xr = lr = ld + xd + lx = lq + xq - lb = qd + 2rn + xq - db = ld + q^2 + 2ry - lx - xd - qb - B = r(q + b) = 0.$$

If $x = 0$ then one can check easily that $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{A_{f,g}\}$ is invariant by L and hence $\mathfrak{h}(\mathrm{K}) = \mathfrak{h}(\mathfrak{g})$ which leaves invariant $\mathbb{R}e$ and hence it is decomposable. Thus $x \neq 0$. Then

$$r = 0, ld + xd + lx = lq + xq - lb = qd + xq - db = ld + q^2 - lx - xd - qb + 1 = 0.$$

Since $xq = lb - lq = db - qd$ we get $(l - d)(q - b) = 0$. If $q = b$ then $q = b = 0$ and hence

$$r = 0, ld + xd + lx = 2ld + 1 = 0.$$

So $x = \frac{d}{2d^2-1}$ and $l = -\frac{1}{2d}$. In this case the Lie brackets are

$$[e, f] = \frac{2d^2+1}{2d}g,\ [e, h] = \frac{1}{2d(2d^2-1)}f + ng,\ [f, h] = \frac{2d(d^2-1)}{2d^2-1}e + yg.$$

If $q \neq b$ then $l = d$ and

$$r = 0,\ d = l = \frac{xq}{b-q}, l^2 + 2lx = 2l^2 + q^2 - qb + 1 = 0.$$

So

$$r = 0, d = l = \frac{xq}{b-q}, x = -\frac{l}{2}, b = \frac{2l^2+q^2+1}{q}.$$

This implies that $l(1 + \frac{q^2}{2l^2+1}) = 0$ and hence $x = 0$. The semi-symmetric Lie algebras obtained satisfy $\mathrm{L}_{h,h}^2\mathrm{K}(f, h) \neq 0$ and hence are not second-order locally symmetric. $\square$

To determine four-dimensional semi-symmetric Ricci isotropic Lorentzian decomposable Lie algebras, we need the following proposition.

Proposition 6.1. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a three dimensional semi-symmetric Ricci isotropic Lorentzian Lie algebra. Then there exists a basis (e, f, g) of $\mathfrak{g}$ such that the non vanishing products are $\langle e, e\rangle = \langle f, g\rangle = 1$ and the non vanishing Lie brackets have one of the following types:*

(i) $[e, f] = af,\ [e, g] = be - ag + \frac{1+2b^2}{2a} f,\ [f, g] = -bf, a, b \in \mathbb{R}, a \neq 0.$
(ii) $[e, g] = ae + bf,\ [f, g] = \frac{1+a^2}{a} f,\ a, b \in \mathbb{R},\ a \neq 0.$

In both cases, $\mathfrak{g}$ is not second-order locally symmetric and $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}(\mathrm{K}) = \mathrm{span}\{A_{e,f}\}$.

Proof. It is easy to see that there exists a basis (e, f, g) of $\mathfrak{g}$ such that the non vanishing products are $\langle e, e\rangle = \langle g, f\rangle = 1$ and $\mathrm{K} = A_{e,f} \vee A_{e,f}$. Put

$$\mathrm{L}_e = aA_{e,f} + bA_{e,g} + cA_{f,g},\ \mathrm{L}_f = xA_{e,f} + yA_{e,g} + zA_{f,g} \quad \text{and} \quad \mathrm{L}_g = pA_{e,f} + qA_{e,g} + rA_{f,g}.$$

We have

$$\begin{aligned}
&\mathrm{L}_e(\mathrm{K})(f, g) = -\mathrm{K}(\mathrm{L}_e f, g) - \mathrm{K}(f, \mathrm{L}_e g) = bA_{e,f},\\
&\mathrm{L}_f(\mathrm{K})(g, e) = [\mathrm{L}_f, e \wedge f] - \mathrm{K}(\mathrm{L}_f g, e) - \mathrm{K}(g, \mathrm{L}_f e) = A_{\mathrm{L}_f e,f} + A_{e,\mathrm{L}_f f} + z\mathrm{K}(g, e) = -yA_{g,f} + 2zA_{e,f},\\
&\mathrm{L}_g(\mathrm{K})(e, f) = -\mathrm{K}(\mathrm{L}_g e, f) - \mathrm{K}(e, \mathrm{L}_g f) = 0.
\end{aligned}$$

So the differential Bianchi identity gives $y = 0$ and $b = -2z$. On the other hand, the relation $0 = \mathrm{L}_{[e,f]} - [\mathrm{L}_e, \mathrm{L}_f]$ is equivalent to $z^2 = x^2 - az = 3xz - cz = 0$ and hence $z = y = b = x = 0$. Now the relations $-A_{e,f} = \mathrm{L}_{[e,g]} - [\mathrm{L}_e, \mathrm{L}_g]$ and $\mathrm{L}_{[e,g]} - [\mathrm{L}_e, \mathrm{L}_g] = 0$ are equivalent to

$$q^2 = a^2 + 1 - 2pc + pq + ar = ac - rc + rq + aq = aq = qc = 0.$$

Thus $q = 0$ and $c(a - r) = a^2 - 2cp + 1 + ar = 0$. Therefore, the solutions are

$$(x = y = z = b = c = q = 0 \quad \text{and} \quad a^2 + ar + 1 = 0) \quad \text{or} \quad (x = y = z = b = q = 0, c \neq 0, a = r \quad \text{and} \quad p = \frac{2r^2 + 1}{2c}).$$

Hence

$$\mathrm{L}_e = aA_{e,f} + cA_{f,g},\ \mathrm{L}_f = 0 \quad \text{and} \quad \mathrm{L}_g = pA_{e,f} + rA_{f,g},$$

where $(c = 0, a^2 + ar + 1 = 0)$ or $(c \neq 0, p = \frac{2r^2+1}{2c})$. In both cases it is easy to check that $\mathfrak{h}(\mathrm{K}) = \mathrm{span}\{A_{e,f}\}$ is invariant by L which shows that it is the holonomy Lie algebra. Moreover, for the first case we have $\mathrm{L}^2_{g,g}(\mathrm{K})(e, g) = -\frac{6(1+a^2)^2}{a^2} A_{e,f}$ and in the second case $\mathrm{L}^2_{e,g}(\mathrm{K})(e, g) = -4rcA_{e,f}$ and $\mathrm{L}^2_{g,g}(\mathrm{K})(e, g) = (1 - 4r^2)A_{e,f}$. This shows that in both cases $\mathfrak{g}$ is not second-order locally symmetric. □

Proposition 6.2. *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a four-dimensional semi-symmetric Ricci isotropic Lorentzian decomposable Lie algebra. Then $\mathfrak{g}$ is a product of $\mathbb{R}$ with a Lie algebra as in Proposition 6.1.*

Proof. In this case $\omega_1 = 0$ or $\omega_2 = 0$. We suppose $\omega_1 = 0$ and we consider the basis (e, f, g, h) where $\mathrm{K} = \epsilon A_{f,g} \vee A_{f,g}$ with $\epsilon = \pm 1$. Let E be a nondegenerate vector subspace of $\mathfrak{g}$ invariant by the holonomy Lie algebra. We can suppose that $\dim E = 1$ or $\dim E = 2$. If $\dim E = 2$ and since E must be invariant by $A_{f,g}$ then $E \subset \mathrm{span}\{f, g\}$ or $E \subset \mathrm{span}\{f, g\}^\perp$ which is impossible so $\dim E = 1$. Let u be a generator of E. Since $A_{f,g}(u) = 0$ then $u \in \mathrm{span}\{e, g\}$. So $u = e + \alpha g$. By making the change of basis (e, f, g, h)

into $(e + \alpha g, f, g, h - \alpha e)$ we can suppose $u = e$. Then the left invariant vector field associated to e must be parallel and hence $\mathrm{R}_e = 0$. Hence span$\{f, g, h\}$ is a semi-symmetric Lie algebra of dimension 3 with isotropic Ricci curvature. According to Proposition 6.1 and its proof, $\mathrm{L}_f = aA_{f,g} + cA_{g,h}$, $\mathrm{L}_g = 0$ and $\mathrm{L}_h = pA_{f,g} + rA_{g,h}$ with $(c = 0, a^2 + ar + 1 = 0)$ or $(c \neq 0, p = \frac{2r^2+1}{2c})$. Put $\mathrm{L}_e = xA_{f,g} + yA_{f,h} + zA_{g,h}$. The relation $\mathrm{K}(e, f) = 0$ is equivalent to

$$y(pA_{f,g} + rA_{g,h}) + xcA_{g,f} + yaA_{g,h} + ycA_{f,h} + zaA_{f,g} = 0.$$

If $c = 0$ then $y(r + a) = yp + za = 0$. Since $a \neq -r$ we get $y = z = 0$. On the other hand, the relation $\mathrm{K}(e, h) = 0$ gives $xaA_{f,g} + xrA_{f,g} = 0$ and hence $x = 0$.

If $c \neq 0$ then $y = 0$ and $xc = za$. The relation $\mathrm{K}(e, h) = 0$ gives

$$x(aA_{f,g} + cA_{g,h}) - xrA_{g,f} - zpA_{f,g} = 0.$$

So $x = 0$ and $z = 0$. Thus $\mathrm{L}_e = 0$ which completes the proof. $\square$

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