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Contravariant pseudo-Hessian manifolds and their associated Poisson structures

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A B S T R A C T

A contravariant pseudo-Hessian manifold is a manifold M endowed with a pair (∇, h) where ∇ is a flat connection and h is a symmetric bivector field satisfying a contravariant Codazzi equation. When h is invertible we recover the known notion of pseudo-Hessian manifold. Contravariant pseudo-Hessian manifolds have properties similar to Poisson manifolds and, in fact, to any contravariant pseudo-Hessian manifold (M, ∇, h) we associate naturally a Poisson tensor on TM. We investigate these properties and we study in details many classes of such structures in order to highlight the richness of the geometry of these manifolds.

1. Introduction

A contravariant pseudo-Hessian manifold is an affine manifold (M, ∇) endowed with a symmetric bivector field h such that, for any $\alpha, \beta, \gamma \in \Omega^1(M)$,

$$(\nabla_{\alpha^\#} h)(\beta, \gamma) = (\nabla_{\beta^\#} h)(\alpha, \gamma), \tag{1.1}$$

where $\alpha^\# = h_\#(\alpha)$, $\beta^\# = h_\#(\beta)$ and $h_\# : T^*M \longrightarrow TM$ is the contraction. We will refer to (1.1) as contravariant Codazzi equation. These manifolds were introduced in [2] as a generalization of pseudo-Hessian manifolds. Recall that a pseudo-Hessian manifold is an affine manifold (M, ∇) with a pseudo-Riemannian metric g satisfying the Codazzi equation

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$$\nabla_X g(Y,Z) = \nabla_Y g(X,Z), \tag{1.2}$$

for any $X, Y, Z \in \Gamma(TM)$. The book [13] is devoted to the study of Hessian manifolds which are pseudo-Hessian manifolds with a Riemannian metric.

In this paper, we study contravariant pseudo-Hessian manifolds. The passage from pseudo-Hessian manifolds to contravariant pseudo-Hessian manifolds is similar to the passage from symplectic manifolds to Poisson manifolds and this similarity will guide our study. Let (M, ∇, h) be a contravariant pseudo-Hessian manifold. We will show that T^*M has a Lie algebroid structure, M has a singular foliation whose leaves are pseudo-Hessian manifolds and TM has a Poisson tensor whose symplectic leaves are pseudo-Kählerian manifolds. We investigate an analog of Darboux-Weinstein's theorem and we show that it is not true in general but holds in some cases. We will study in details the correspondence which maps a contravariant pseudo-Hessian bivector field on (M, ∇) to a Poisson bivector field on TM. We study affine, linear and quadratic contravariant pseudo-Hessian structures on vector spaces and we show that an affine contravariant pseudo-Hessian structure on a vector space V is equivalent to an associative commutative algebra product and a 2-cocycle on V^*. We study right invariant contravariant pseudo-Hessian structures on a Lie group G and we show that TG has a structure of Lie group (different from the one associated to the adjoint action) for which the associated Poisson tensor is right invariant. We show that a right invariant contravariant pseudo-Hessian structure on a Lie group is equivalent to a S-matrix on the associated left symmetric algebra (see [1,3]) and we associate to any S-matrix on a left symmetric algebra $\mathfrak{g}$ a solution of the classical Yang-Baxter equation on $\mathfrak{g} \times \mathfrak{g}$. Finally, we show that an action of a left symmetric algebra $\mathfrak{g}$ on an affine manifold (M, ∇) transforms a S-matrix on $\mathfrak{g}$ to a contravariant pseudo-Hessian bivector field on (M, ∇). Since the Lie algebra of affine vector fields of (M, ∇) has a natural structure of finite dimensional associative algebra, we have a mean to define contravariant pseudo-Hessian structures on any affine manifold. The paper contains many examples of contravariant pseudo-Hessian structures.

The paper is organized as follows. In Section 2, we give the definition of a contravariant pseudo-Hessian manifold and we investigate its properties. In Section 3, we study in details the Poisson structure of the tangent bundle of a contravariant pseudo-Hessian manifold. Section 4 is devoted to the study of linear and affine contravariant pseudo-Hessian structures. Quadratic contravariant pseudo-Hessian structures will be studied in Section 5. In Section 6, we study right invariant pseudo-Hessian structures on Lie groups.

2. Contravariant pseudo-Hessian manifolds: definition and principal properties

2.1. Definition of a contravariant pseudo-Hessian manifold

Recall that an affine manifold is a n-manifold M endowed with a maximal atlas such that all transition functions are restrictions of elements of the affine group $\mathrm{Aff}(\mathbb{R}^n)$. This is equivalent to the existence on M of a flat connection ∇, i.e., torsionless and with vanishing curvature (see [13] for more details). An affine coordinates system on an affine manifold (M, ∇) is a coordinates system $(x_1, \ldots, x_n)$ satisfying $\nabla \partial_{x_i} = 0$ for any $i = 1, \ldots, n$.

Let g be a pseudo-Riemannian metric on an affine manifold (M, ∇). The triple (M, ∇, g) is called a *pseudo-Hessian manifold* if g can be locally expressed in any affine coordinates system $(x_1, \ldots, x_n)$ as

$$g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

That is equivalent to g satisfying the *Codazzi equation* (1.2). When g is Riemannian, we call (M, ∇, g) a Hessian manifold. The geometry of Hessian manifolds was studied intensively in [13].

We consider now a more general situation.

Definition 2.1 *([2])*. Let h be a symmetric bivector field on an affine manifold (M, ∇) and $h_\# : T^*M \to TM$ the associated contraction given by $\beta(\alpha^\#) = h(\alpha, \beta)$ where $\alpha^\# := h_\#(\alpha)$. The triple (M, ∇, h) is called a *contravariant pseudo-Hessian manifold* if h satisfies the *contravariant Codazzi equation*

$$(\nabla_{\alpha^\#} h)(\beta, \gamma) = (\nabla_{\beta^\#} h)(\alpha, \gamma), \tag{2.1}$$

for any $\alpha, \beta, \gamma \in \Omega^1(M)$. We call such h a pseudo-Hessian bivector field.

One can see easily that if (M, ∇, g) is a pseudo-Hessian manifold then (M, ∇, g^{-1}) is a contravariant pseudo-Hessian manifold.

The following proposition is obvious and gives the local expression of the equation (2.1) in affine charts.

Proposition 2.2. *Let (M, ∇, h) be an affine manifold endowed with a symmetric bivector field. Then h satisfies (2.1) if and only if, for any $m \in M$, there exists an affine coordinates system $(x_1, \ldots, x_n)$ around m such that for any $1 \le i < j \le n$ and any $k = 1, \ldots, n$*

$$\sum_{l=1}^{n} [h_{il} \partial_{x_l}(h_{jk}) - h_{jl} \partial_{x_l}(h_{ik})] = 0, \tag{2.2}$$

where $h_{ij} = h(dx_i, dx_j)$.

Example 2.3.

1. Take $M = \mathbb{R}^n$ endowed with its canonical affine structure and consider

$$h = \sum_{i=1}^{n} f_i(x_i) \partial_{x_i} \otimes \partial_{x_i},$$

where $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ for $i = 1, \ldots, n$. Then one can see easily that h satisfies (2.2) and hence defines a contravariant pseudo-Hessian structure on $\mathbb{R}^n$.

2. Take $M = \mathbb{R}^n$ endowed with its canonical affine structure and consider

$$h = \sum_{i,j=1}^{n} x_i x_j \partial_{x_i} \otimes \partial_{x_j}.$$

Then one can see easily that h satisfies (2.2) and hence defines a contravariant pseudo-Hessian structure on $\mathbb{R}^n$.

3. Let (M, ∇) be an affine manifold, $(X_1, \ldots, X_r)$ a family of parallel vector fields and $(a_{i,j})_{1 \le i,j \le n}$ a symmetric n-matrix. Then

$$h = \sum_{i,j} a_{i,j} X_i \otimes X_j$$

defines a contravariant pseudo-Hessian structure on M.

2.2. The Lie algebroid of a contravariant pseudo-Hessian manifold

We show that associated to any contravariant pseudo-Hessian manifold there is a Lie algebroid structure on its cotangent bundle and a Lie algebroid flat connection. The reader can consult [9,12] for more details on Lie algebroids and their connections.

Let (M, ∇, h) be an affine manifold endowed with a symmetric bivector field. We associate to this triple a bracket on $\Omega^1(M)$ by putting

$$[\alpha, \beta]_h := \nabla_{\alpha^\#}\beta - \nabla_{\beta^\#}\alpha, \tag{2.3}$$

and a map $\mathcal{D} : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$ given by

$$\prec \mathcal{D}_\alpha\beta, X \succ := (\nabla_X h)(\alpha, \beta) + \prec \nabla_{\alpha^\#}\beta, X \succ, \tag{2.4}$$

for any $\alpha, \beta \in \Omega^1(M)$ and $X \in \Gamma(TM)$. This bracket is skew-symmetric and satisfies obviously

$$[\alpha, \beta]_h = \mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha \quad \text{and} \quad [\alpha, f\beta]_h = f[\alpha, \beta]_h + \alpha^\#(f)\beta,$$

where $f \in C^\infty(M), \alpha, \beta \in \Omega^1(M)$.

Theorem 2.4. *With the hypothesis and notations above, the following assertions are equivalent:*

- *(i) h is a pseudo-Hessian bivector field.*
- *(ii) $(T^*M, h_\#, [\ ,\ ]_h)$ is a Lie algebroid.*

In this case, $\mathcal{D}$ is a connection for the Lie algebroid structure $(T^*M, h_\#, [\ ,\ ]_h)$ satisfying

$$(\mathcal{D}_\alpha\beta)^\# = \nabla_{\alpha^\#}\beta^\# \quad \textit{and} \quad R_{\mathcal{D}}(\alpha, \beta) := \mathcal{D}_{[\alpha,\beta]_h} - \mathcal{D}_\alpha\mathcal{D}_\beta + \mathcal{D}_\beta\mathcal{D}_\alpha = 0,$$

for any $\alpha, \beta \in \Omega^1(M)$.

Proof. According to [2, Proposition 2.1], $(T^*M, h_\#, [\ ,\ ]_h)$ is a Lie algebroid if and only if, for any affine coordinates system $(x_1, \ldots, x_n)$,

$$([dx_i, dx_j]_h)^\# = [(dx_i)^\#, (dx_j)^\#] \quad \text{and} \quad \oint_{i,j,k} [dx_i, [dx_j, dx_k]_h]_h = 0,$$

for $1 \leq i < j < k \leq n$. Since $[dx_i, dx_j]_h = 0$ this is equivalent to $[(dx_i)^\#, (dx_j)^\#] = 0$ for any $1 \leq i < j \leq n$ which is equivalent to (2.2).

Suppose now that (i) or (ii) holds. For any, $\alpha, \beta, \gamma \in \Omega^1(M)$,

$$\begin{aligned}
\prec \mathcal{D}_\alpha\beta, \gamma^\# \succ &= \nabla_{\gamma^\#}h(\alpha, \beta) + h(\nabla^*_{\alpha^\#}\beta, \gamma) \\
&= \nabla_{\alpha^\#}h(\gamma, \beta) + h(\nabla^*_{\alpha^\#}\beta, \gamma) \\
&= \alpha^\#.h(\beta, \gamma) - h(\nabla^*_{\alpha^\#}\gamma, \beta) \\
&= \prec \gamma, \nabla_{\alpha^\#}\beta^\# \succ .
\end{aligned}$$

This shows that $(\mathcal{D}_\alpha\beta)^\# = \nabla_{\alpha^\#}\beta^\#$.

Let us show now that the curvature of $\mathcal{D}$ vanishes. Since $[dx_i, dx_j]_h = 0$, it suffices to show that, for any $i, j, k \in \{1, \ldots, n\}$ with $i < j$, $\mathcal{D}_{dx_i}\mathcal{D}_{dx_j}dx_k = \mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k$. We have

$$\prec \mathcal{D}_{dx_i}dx_k, \frac{\partial}{\partial x_l} \succ = \frac{\partial h_{ik}}{\partial x_l}$$

and hence

$$\mathcal{D}_{dx_i}dx_k = \sum_{l=1}^{n} \frac{\partial h_{ik}}{\partial x_l} dx_l$$

and then

$$\begin{aligned}
\mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k &= \sum_{l=1}^{n} \mathcal{D}_{dx_j}\left(\frac{\partial h_{ik}}{\partial x_l} dx_l\right) \\
&= \sum_{l=1}^{n} \left((dx_j)^{\#}\left(\frac{\partial h_{ik}}{\partial x_l}\right) dx_l + \frac{\partial h_{ik}}{\partial x_l}\left(\sum_{s=1}^{n} \frac{\partial h_{jl}}{\partial x_s} dx_s\right)\right) \\
&= \sum_{l,r} h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right) dx_l + \sum_{s,l} \frac{\partial h_{ik}}{\partial x_l}\frac{\partial h_{jl}}{\partial x_s} dx_s \\
&= \sum_{l,r} h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right) dx_l + \sum_{l,r} \frac{\partial h_{ik}}{\partial x_r}\frac{\partial h_{jr}}{\partial x_l} dx_l \\
&= \sum_{l,r} \left(h_{jr}\left(\frac{\partial^2 h_{ik}}{\partial x_r \partial x_l}\right) + \frac{\partial h_{ik}}{\partial x_r}\frac{\partial h_{jr}}{\partial x_l}\right) dx_l \\
&= \sum_{l} \frac{\partial}{\partial x_l}\left(\sum_r h_{jr}\frac{\partial h_{ik}}{\partial x_r}\right) dx_l.
\end{aligned}$$

So

$$\mathcal{D}_{dx_i}\mathcal{D}_{dx_j}dx_k - \mathcal{D}_{dx_j}\mathcal{D}_{dx_i}dx_k = d\left(\sum_r \left(h_{jr}\frac{\partial h_{ik}}{\partial x_r} - h_{ir}\frac{\partial h_{jk}}{\partial x_r}\right)\right) \overset{(2.2)}{=} d(0) = 0. \quad \square$$

The following result is an important consequence of Theorem 2.4.

Proposition 2.5 *([3, Theorem 6.7]). Let (M, ∇, h) be a contravariant pseudo-Hessian manifold. Then:*

1. *The distribution $\mathrm{Im}h_{\#}$ is integrable and defines a singular foliation $\mathcal{L}$ on M.*
2. *For any leaf L of $\mathcal{L}$, $(L, \nabla_{|L}, g_L)$ is a pseudo-Hessian manifold where g_L is given by $g_L(\alpha^{\#}, \beta^{\#}) = h(\alpha, \beta)$.*

We will call the foliation defined by $\mathrm{Im}h_{\#}$ the affine foliation associated to (M, ∇, h).

Remark 2.6. This proposition shows that pseudo-Hessian bivector fields can be used either to build examples of affine foliations on affine manifolds or to build examples of pseudo-Hessian manifolds.

For the reader familiar with Poisson manifolds what we have established so far shows the similarities between Poisson manifolds and contravariant pseudo-Hessian manifolds. One can consult [7] for more details on Poisson geometry. Poisson manifolds have many relations with Lie algebras and we will see now and in Section 4 that contravariant pseudo-Hessian manifolds are related to commutative associative algebras.

Let (M, ∇, h) be a contravariant pseudo-Hessian manifold and $\mathcal{D}$ the connection given in (2.4). Let $x \in M$ and $\mathfrak{g}_x = \ker h_{\#}(x)$. For any $\alpha, \beta \in \Omega^1(M)$, $(\mathcal{D}_\alpha\beta)^{\#} = \nabla_{\alpha^{\#}}\beta^{\#}$. This shows that if $\alpha_x^{\#} = 0$ then $(\mathcal{D}_\alpha\beta)_x^{\#} = 0$. Moreover, $\mathcal{D}_\alpha\beta - \mathcal{D}_\beta\alpha = \nabla_{\alpha^{\#}}\beta - \nabla_{\beta^{\#}}\alpha$. This implies that if $\alpha_x^{\#} = \beta_x^{\#} = 0$ then $\mathcal{D}_\alpha\beta(x) = \mathcal{D}_\beta\alpha(x)$. For any $a, b \in \mathfrak{g}_x$ put

$$a \bullet b = (\mathcal{D}_\alpha\beta)(x),$$

where α, β are two differential 1-forms satisfying $\alpha(x) = a$ and $\beta(x) = b$. This defines a commutative product on $\mathfrak{g}_x$ and moreover, by using the vanishing of the curvature of $\mathcal{D}$, we get:

Proposition 2.7. *$(\mathfrak{g}_x, \bullet)$ is a commutative associative algebra.*

Near a point where h vanishes, the algebra structure of $\mathfrak{g}_x$ can be made explicit.

Proposition 2.8. *We consider $\mathbb{R}^n$ endowed with its canonical affine connection, h a symmetric bivector field on $\mathbb{R}^n$ such that $h(0) = 0$ and $(\mathbb{R}^n, \nabla, h)$ is a contravariant pseudo-Hessian manifold. Then the product on $(\mathbb{R}^n)^*$ given by*

$$e_i^* \bullet e_j^* = \sum_{k=1}^{n} \frac{\partial h_{ij}}{\partial x_k}(0) e_k^*$$

is associative and commutative.

Proof. It is a consequence of the relation $\mathcal{D}_{dx_i} dx_j = dh_{ij}$ true by virtue of (2.4). $\square$

2.3. The product of contravariant pseudo-Hessian manifolds and the splitting theorem

As the product of two Poisson manifolds is a Poisson manifold [15], the product of two contravariant pseudo-Hessian manifolds is a contravariant pseudo-Hessian manifold.

Let (M_1, ∇^1, h^1) and (M_2, ∇^2, h^2) be two contravariant pseudo-Hessian manifolds. We denote by $p_i : M = M_1 \times M_2 \to M_i, i = 1, 2$ the canonical projections. For any $X \in \Gamma(TM_1)$ and $Y \in \Gamma(TM_2)$, we denote by $X + Y$ the vector field on M given by $(X + Y)(m_1, m_2) = (X(m_1), Y(m_2))$. The product of the affine atlases on M_1 and M_2 is an affine atlas on M and the corresponding affine connection is the unique flat connection ∇ on M satisfying $\nabla_{X_1+Y_1}(X_2 + Y_2) = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2$, for any $X_1, X_2 \in \Gamma(TM_1)$ and $Y_1, Y_2 \in \Gamma(TM_2)$. Moreover, the product of h_1 and h_2 is the unique symmetric bivector field h satisfying

$$h(p_1^*\alpha_1, p_1^*\alpha_2) = h^1(\alpha_1, \alpha_2) \circ p_1,\ h(p_2^*\beta_1, p_2^*\beta_2) = h^2(\beta_1, \beta_2) \circ p_2 \quad \text{and} \quad h(p_1^*\alpha_1, p_2^*\beta_1) = 0,$$

for any $\alpha_1, \beta_1 \in \Omega^1(M_1)$, $\alpha_2, \beta_2 \in \Omega^1(M_2)$,

Proposition 2.9. *(M, ∇, h) is a contravariant pseudo-Hessian manifold.*

Proof. Let $(m_1, m_2) \in M$. Choose an affine coordinates system $(x_1, \ldots, x_{n_1})$ near m_1 and an affine coordinates system $(y_1, \ldots, y_{n_2})$ near m_2. Then

$$h = \sum_{i,j} h^1_{ij} \circ p_1 \partial_{x_i} \otimes \partial_{x_j} + \sum_{l,k} h^2_{lk} \circ p_2 \partial_{y_l} \otimes \partial_{y_k}$$

and one can check easily that h satisfies (2.2). $\square$

If we pursue the exploration of the analogies between Poisson manifolds and contravariant pseudo-Hessian manifolds we can ask naturally if there is an analog of the Darboux-Weinstein's theorem (see [15]) in the context of contravariant pseudo-Hessian manifolds. More precisely, let (M, ∇, h) be a contravariant pseudo-Hessian manifold and $m \in M$ where $\mathrm{rank} h_\#(m) = r$. One can ask if there exists an affine coordinates system $(x_1, \ldots, x_r, y_1, \ldots, y_{n-r})$ such that

$$h = \sum_{i,j=1}^{r} h_{ij}(x_1, \ldots, x_r)\partial_{x_i} \otimes \partial_{x_j} + \sum_{i,j=1}^{n-r} f_{ij}(y_1, \ldots, y_{n-r})\partial_{y_i} \otimes \partial_{y_j},$$

where $(h_{ij})_{1\leq i,j\leq r}$ is invertible and its the inverse of $\left(\frac{\partial^2\phi}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq r}$ and $f_{ij}(m) = 0$ for any i, j. Moreover, if the rank of $h_\#$ is constant near m then the functions f_{ij} vanish.

The answer is no in general for a geometric reason. Suppose that m is regular, i.e., the rank of h is constant near m and suppose that there exists an affine coordinates system $(x_1, \ldots, x_r, y_1, \ldots, y_{n-r})$ such that

$$h = \sum_{i,j=1}^{r} h_{ij}(x_1, \ldots, x_r)\partial_{x_i} \otimes \partial_{x_j}.$$

This will have a strong geometric consequence, namely that $\mathrm{Im}h_\# = \mathrm{span}(\partial_{x_1}, \ldots, \partial_{x_r})$ and the associated affine foliation is parallel, i.e., if X is a local vector field and Y is tangent to the foliation then $\nabla_X Y$ is tangent to the foliation. We give now an example of a regular contravariant pseudo-Hessian manifold whose associated affine foliation is not parallel which shows that the analog of Darboux-Weinstein is not true in general.

Example 2.10. We consider $M = \mathbb{R}^4$ endowed with its canonical affine connection ∇, denote by (x, y, z, t) its canonical coordinates and consider

$$X = \cos(t)\partial_x + \sin(t)\partial_y + \partial_z,\ Y = -\sin(t)\partial_x + \cos(t)\partial_y \quad \text{and} \quad h = X \otimes Y + Y \otimes X.$$

We have $\nabla_X X = \nabla_Y X = \nabla_X Y = \nabla_Y Y = 0$ and hence h is a pseudo-Hessian bivector field, $\mathrm{Im}h_\# = \mathrm{span}\{X, Y\}$ and the rank of h is constant equal to 2. However, the foliation associated to $\mathrm{Im}h_\#$ is not parallel since $\nabla_{\partial_t} Y = -X + \partial_z \notin \mathrm{Im}h_\#$.

However, when h has constant rank equal to $\dim M - 1$, we have the following result and its important corollary.

Theorem 2.11. *Let (M, ∇, h) be a contravariant pseudo-Hessian manifold and $m \in M$ such that m is a regular point and the rank of $h_\#(m)$ is equal to $n - 1$. Then there exists an affine coordinates system $(x_1, \ldots, x_n)$ around m and a function $f(x_1, \ldots, x_n)$ such that*

$$h = \sum_{i,j=1}^{n-1} h_{ij}\partial_{x_i} \otimes \partial_{x_j},$$

and the matrix $(h_{ij})_{1\leq i,j\leq n-1}$ is invertible and its inverse is the matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq n-1}$.

Corollary 2.12. *Let (M, ∇, h) be a contravariant pseudo-Hessian manifold with h of constant rank equal to $\dim M - 1$. Then the affine foliation associated to $\mathrm{Im}h_\#$ is ∇-parallel.*

In order to prove this theorem, we need the following lemma.

Lemma 2.13. *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a differentiable function such that $\partial_x(f) + f\partial_y(f) = 0$. Then f is a constant.*

Proof. Let f be a solution of the equation above. We consider the vector field $X_f = \partial_x + f\partial_y$. The integral curve $(x(t), y(t))$ of X_f passing through $(a, b) \in \mathbb{R}^2$ satisfies

$$x'(t) = 1, \quad y'(t) = f(x(t), y(t)) \quad \text{and} \quad (x(0), y(0)) = (a, b).$$

Now

$$y''(t) = \partial_x(f)(x(t), y(t)) + y'(t)\partial_y(f)(x(t), y(t)) = 0$$

and hence, the flow of X_f is given by $\phi(t, (x, y)) = (t + x, f(x, y)t + y)$. The relation $\phi(t + s, (x, y)) = \phi(t, \phi(s, (x, y)))$ implies that the map $F(x, y) = (1, f(x, y))$ satisfies

$$F(u + tF(u)) = F(u), \ u \in \mathbb{R}^2, t \in \mathbb{R}.$$

Let $u, v \in \mathbb{R}^2$ such that $F(u)$ and $F(v)$ are linearly independent. Then there exists $s, t \in \mathbb{R}$ such that $u - v = tF(u) + sF(v)$ and hence $F(u) = F(v)$ which is a contradiction. So $F(x, y) = \alpha(x, y)(a, b)$, i.e., $(1, f(x, y)) = (\alpha(x, y)a, \alpha(x, y)b)$ and α must be constant and hence f is constant. $\square$

Proof of Theorem 2.11. Let $(x_1, \dots, x_n)$ be an affine coordinates system near m such that $(X_1, \dots, X_{n-1})$ are linearly independent in a neighborhood of m, where $X_i = (dx_i)^\#$, $X_n = \sum_{j=1}^{n-1} f_j X_j$ and, by virtue of the proof of Theorem 2.4, for any $1 \le i < j \le n$, $[X_i, X_j] = 0$. For any $i = 1, \dots, n-1$, the relation $[X_i, X_n] = 0$ is equivalent to

$$X_i(f_j) = h_{in}\partial_{x_n}(f_j) + \sum_{l=1}^{n-1} h_{il}\partial_{x_l}(f_j) = 0, \qquad j = 1, \dots, n-1.$$

But $h_{in} = X_n(x_i) = \sum_{l=1}^{n-1} f_l h_{il}$ and hence, for any $i, j = 1, \dots n-1$,

$$\sum_{l=1}^{n-1} h_{il}(f_l\partial_{x_n}(f_j) + \partial_{x_l}(f_j)) = 0.$$

Or the matrix $(h_{ij})_{1 \le i,j \le n-1}$ is invertible so we get

$$f_l\partial_{x_n}(f_j) + \partial_{x_l}(f_j) = 0, \quad l, j = 1, \dots, n-1. \tag{2.5}$$

For $l = j$ we get that f_j satisfies $f_j\partial_{x_n}(f_j) + \partial_{x_j}(f_j) = 0$ so, according to Lemma 2.13, $\partial_{x_n}(f_j) = \partial_{x_j}(f_j) = 0$ and, from (2.5), $f_j = \text{constant}$. We consider $y = f_1x_1 + \ldots + f_{n-1}x_{n-1} - x_n$, we have $(dy)^\# = 0$ and $(x_1, \dots, x_{n-1}, y)$ is an affine coordinates system around m.

On the other hand, there exists a coordinates system $(z_1, \dots, z_n)$ such that

$$(dx_i)^\# = \partial_{z_i}, \ i = 1, \dots, n-1.$$

We deduce that

$$\partial_{x_i} = \sum_{j=1}^{n-1} h^{ij}\partial_{z_j}, \ i = 1, \dots, n-1,$$

with $h^{ij} = \frac{\partial z_j}{\partial x_i}$. We consider $\sigma = \sum_{j=1}^{n-1} z_j dx_j$. We have $d\sigma = 0$ so according to the foliated Poincaré Lemma (see [5, p. 56]) there exists a function f such that $h^{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. $\square$

2.4. The divergence and the modular class of a contravariant pseudo-Hessian manifold

We define now the divergence of a contravariant pseudo-Hessian structure. We recall first the definition of the divergence of multivector fields associated to a connection on a manifold.

Let (M, ∇) be a manifold endowed with a connection. We define $\mathrm{div}_\nabla : \Gamma(\otimes^p TM) \to \Gamma(\otimes^{p-1}TM)$ by

$$\mathrm{div}_\nabla(T)(\alpha_1, \dots, \alpha_{p-1}) = \sum_{i=1}^{n} \nabla_{e_i}(T)(e_i^*, \alpha_1, \dots, \alpha_{p-1}),$$

where $\alpha_1, \dots, \alpha_{p-1} \in T_x^*M$, $(e_1, \dots, e_n)$ a basis of T_xM and $(e_1^*, \dots, e_n^*)$ its dual basis. This operator respects the symmetries of tensor fields.

Suppose now that (M, ∇, h) is a contravariant pseudo-Hessian manifold. The divergence of this structure is the vector field $\mathrm{div}_\nabla(h)$. This vector field is an invariant of the pseudo-Hessian structure and has an important property. Indeed, let $\mathbf{d}_h : \Gamma(\wedge^\bullet TM) \longrightarrow \Gamma(\wedge^{\bullet+1}TM)$ be the differential associated to the Lie algebroid structure $(T^*M, h_\#, [\ ,\ ]_h)$ and given by

$$\begin{aligned}\mathbf{d}_h Q(\alpha_1, \dots, \alpha_p) &= \sum_{j=1}^{p} (-1)^{j+1} \alpha_j^{\#}.Q(\alpha_1, \dots, \hat{\alpha_j}, \dots \alpha_p)\\ &+ \sum_{1 \le i < j \le p} (-1)^{i+j} Q([\alpha_i, \alpha_j]_h, \alpha_1, \dots, \hat{\alpha_i}, \dots, \hat{\alpha_j}, \dots, \alpha_p).\end{aligned}$$

Proposition 2.14. $\mathbf{d}_h(\mathrm{div}_\nabla(h)) = 0$.

Proof. Let $(x_1, \dots, x_n)$ be an affine coordinates system. We have

$$\begin{aligned}&\mathbf{d}_h \mathrm{div}_\nabla(h)(\alpha, \beta)\\ &= \sum_{i=1}^{n} \left(\alpha^{\#}.\nabla_{\partial_{x_i}}(h)(dx_i, \beta) - \beta^{\#}.\nabla_{\partial_{x_i}}(h)(dx_i, \alpha) - \nabla_{\partial_{x_i}}(h)(dx_i, \nabla_{\alpha^\#}\beta) + \nabla_{\partial_{x_i}}(h)(dx_i, \nabla_{\beta^\#}\alpha)\right)\\ &= \sum_{i=1}^{n} \left(\nabla_{\alpha^\#}\nabla_{\partial_{x_i}}(h)(dx_i, \beta) - \nabla_{\beta^\#}\nabla_{\partial_{x_i}}(h)(dx_i, \alpha)\right)\\ &\overset{(2.1)}{=} \sum_{i=1}^{n} \left(\nabla_{[\alpha^\#, \partial_{x_i}]}(h)(dx_i, \beta) - \nabla_{[\beta^\#, \partial_{x_i}]}(h)(dx_i, \alpha)\right).\end{aligned}$$

If we take $\alpha = dx_l$ and $\beta = dx_k$, we have

$$[\partial_{x_i}, (dx_l)^\#] = \sum_{m=1}^{n} \partial_{x_i}(h_{ml})\partial_{x_m}$$

and hence

$$\mathbf{d}_h \mathrm{div}_\nabla(h)(\alpha, \beta) = \sum_{i,m=1}^{n} \left(\partial_{x_i}(h_{ml})\partial_{x_m}(h_{ik}) - \partial_{x_i}(h_{mk})\partial_{x_m}(h_{il})\right) = 0. \quad \square$$

Let (M, ∇, h) be an orientable contravariant pseudo-Hessian manifold and Ω a volume form on M. For any f we denote by $X_f = h_\#(df)$ and we define $\mathbf{M}_\Omega : C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ by putting for any $f \in C^\infty(M, \mathbb{R})$,

$$\nabla_{X_f}\Omega = \mathbf{M}_\Omega(f)\Omega.$$

It is obvious that $\mathbf{M}_\Omega$ is a derivation and hence a vector field and $\mathbf{M}_{e^f\Omega} = X_f + \mathbf{M}_\Omega$. Moreover, if $(x_1, \ldots, x_n)$ is an affine coordinates system and $\mu = \Omega(\partial_{x_1}, \ldots, \partial_{x_n})$ then

$$\nabla_{X_f}\Omega(\partial_{x_1}, \ldots, \partial_{x_n}) = X_f(\mu) = X_{\ln|\mu|}(f)\mu.$$

So in the coordinates system $(x_1, \ldots, x_n)$, we have $\mathbf{M}_\Omega = X_{\ln|\mu|}$. This implies $\mathbf{d}_h\mathbf{M}_\Omega = 0$. The cohomology class of $\mathbf{M}_\Omega$ doesn't depend on Ω and we call it the *modular class* of (M, ∇, h).

Proposition 2.15. *The modular class of (M, ∇, h) vanishes if and only if there exists a volume form Ω such that $\nabla_{X_f}\Omega = 0$ for any $f \in C^\infty(M, \mathbb{R})$.*

By analogy with the case of Poisson manifolds, one can ask if it is possible to find a volume form Ω such that $\mathcal{L}_{X_f}\Omega = 0$ for any $f \in C^\infty(M, \mathbb{R})$. The following proposition gives a negative answer to this question unless $h = 0$.

Proposition 2.16. *Let (M, ∇, h) be an orientable contravariant pseudo-Hessian manifold. Then:*

1. *For any volume form Ω and any $f \in C^\infty(M, \mathbb{R})$,*

$$\mathcal{L}_{X_f}\Omega = [\mathbf{M}_\Omega(f) + \mathrm{div}_\nabla(h)(f) + \prec h, \mathrm{Hess}(f) \succ]\,\Omega,$$

   where $\mathrm{Hess}(f)(X, Y) = \nabla_X(df)(Y)$ *and* $\prec h, \mathrm{Hess}(f) \succ$ *is the pairing between the bivector field h and the 2-form* $\mathrm{Hess}(f)$.
2. *There exists a volume form Ω such that $\mathcal{L}_{X_f}\Omega = 0$ for any $f \in C^\infty(M, \mathbb{R})$ if and only if $h = 0$.*

Proof. 1. Let $(x_1, \ldots, x_n)$ be an affine coordinates system. Then:

$$\begin{aligned}[X_f, \partial_{x_i}] &= \sum_{l,j=1}^{n} [\partial_{x_j}(f)h_{jl}\partial_{x_l}, \partial_{x_i}] \\ &= -\sum_{l,j=1}^{n} \left(h_{jl}\partial_{x_i}\partial_{x_j}(f) + \partial_{x_j}(f)\partial_{x_i}(h_{jl})\right)\partial_{x_l},\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{X_f}\Omega(\partial_{x_1}, \ldots, \partial_{x_n}) &= (\nabla_{X_f}\Omega)(\partial_{x_1}, \ldots, \partial_{x_n}) - \sum_{i=1}^{n} \Omega((\partial_{x_1}, \ldots, [X_f, \partial_{x_i}], \ldots, \partial_{x_n})) \\ &= (\nabla_{X_f}\Omega)(\partial_{x_1}, \ldots, \partial_{x_n}) + \sum_{i,j=1}^{n} \left(h_{ji}\partial_{x_i}\partial_{x_j}(f) + \partial_{x_j}(f)\partial_{x_i}(h_{ji})\right)\Omega(\partial_{x_1}, \ldots, \partial_{x_n})\end{aligned}$$

   and the formula follows since $\mathrm{div}_\nabla(h) = \sum_{i,j=1}^{n} \partial_{x_i}(h_{ji})\partial_{x_j}$.
2. This is a consequence of the fact that $\mathbf{M}_\Omega$ and $\mathrm{div}_\nabla(h)$ are derivation and

$$\prec h, \mathrm{Hess}(fg) \succ = f \prec h, \mathrm{Hess}(g) \succ + g \prec h, \mathrm{Hess}(f) \succ + \prec h, df \odot dg \succ. \quad \square$$

3. The tangent bundle of a contravariant pseudo-Hessian manifold

In this section, we define and study the associated Poisson tensor on the tangent bundle of a contravariant pseudo-Hessian manifold. We will start this paragraph by recalling some useful results concerning the geometry of the tangent bundle.

Let (M, ∇) be an n-dimensional smooth manifold with a connection and denote by $p : TM \to M$ the canonical projection of the tangent bundle. It is a well known fact that one can define on TM the so called *Sasaki connection* $\overline{\nabla}$ associated to ∇, and also the *Sasaki almost complex structure* $J : T(TM) \to T(TM)$. For more details one can see [6,16,11]. Indeed, associated to ∇ there exists a splitting

$$TTM = V(M) \oplus H(M)$$

such that for any $u \in TM$, $T_u p : H_u(M) \longrightarrow T_{p(x)}M$ is an isomorphism. For any $X \in \Gamma(TM)$ we denote by $X^v \in \Gamma(V(M))$ its vertical lift and by $X^h \in \Gamma(H(M))$ its horizontal lift. There are given, for any $u \in TM$, by

$$X_u^v = \frac{d}{dt}_{|t=0}(u + tX_{p(u)}), \quad \text{and } Tp(X_u^h) = X_{p(u)}.$$

The *Sasaki almost complex structure* $J : TTM \longrightarrow TTM$ determined by ∇ is defined by

$$J(X^h) = X^v \quad \text{and} \quad J(X^v) = -X^h.$$

It is integrable to a complex structure on TM if and only if ∇ is flat.

Suppose now that (M, ∇) is an affine manifold. Since the curvature of ∇ vanishes, for any $X, Y \in \Gamma(TM)$,

$$[X^h, Y^h] = [X, Y]^h, \ [X^h, Y^v] = (\nabla_X Y)^v \quad \text{and} \quad [X^v, Y^v] = 0. \tag{3.1}$$

As for the vector fields, for any $\alpha \in \Omega^1(M)$, we define $\alpha^v, \alpha^h \in \Omega^1(TM)$ by

$$\begin{cases} \alpha^v(X^v) = \alpha(X) \circ p, \\ \alpha^v(X^h) = 0, \end{cases} \quad \text{and} \quad \begin{cases} \alpha^h(X^h) = \alpha(X) \circ p, \\ \alpha^h(X^v) = 0. \end{cases}$$

The *Sasaki connection* $\overline{\nabla}$ on TM is defined by

$$\overline{\nabla}_{X^h} Y^h = (\nabla_X Y)^h, \ \overline{\nabla}_{X^h} Y^v = (\nabla_X Y)^v \quad \text{and} \quad \overline{\nabla}_{X^v} Y^h = \overline{\nabla}_{X^v} Y^v = 0, \tag{3.2}$$

where $X, Y \in \Gamma(TM)$. This connection is torsionless and flat and hence defines an affine structure on TM. Moreover, the endomorphism vector field $J : TTM \to TTM$ is parallel with respect to $\overline{\nabla}$.

Remark 3.1. All the above geometrical structures on TM could be described locally in an easy way. In fact, let $(x_1, \cdots, x_n)$ be an affine coordinates system on an open set $U \subset M$. Then we can easily see that the connection $\overline{\nabla}$ is the canonical one for which the associated canonical coordinate system $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ on TU is affine (where $x_i(u) := x_i(p(u))$ and $y_j(u) := dx_i(u)$ for any $u \in TU$). The complex structure is given by $J(\partial_{x_i}) = \partial_{y_i}$, $J(\partial_{y_i}) = -\partial_{x_i}$.

Now, let h be a symmetric bivector field on M. We associate to h a skew-symmetric bivector field Π on TM by putting

$$\Pi(\alpha^v, \beta^v) = \Pi(\alpha^h, \beta^h) = 0 \quad \text{and} \quad \Pi(\alpha^h, \beta^v) = -\Pi(\beta^v, \alpha^h) = h(\alpha, \beta) \circ p,$$

for any $\alpha, \beta \in \Omega^1(M)$. For any $\alpha \in \Omega^1(M)$,

$$\Pi_\#(\alpha^v) = -(\alpha^\#)^h \quad \text{and} \quad \Pi_\#(\alpha^h) = (\alpha^\#)^v. \tag{3.3}$$

To prove one of our main result in this section, we need the following proposition which is a part of the folklore.

Proposition 3.2. *Let (P, ∇) be a manifold endowed with a torsionless connection and π is a bivector field on P. Then the Nijenhuis-Schouten bracket $[\pi, \pi]$ is given by*

$$[\pi, \pi](\alpha, \beta, \gamma) = 2\left(\nabla_{\pi_\#(\alpha)}\pi(\beta, \gamma) + \nabla_{\pi_\#(\beta)}\pi(\gamma, \alpha) + \nabla_{\pi_\#(\gamma)}\pi(\alpha, \beta)\right).$$

Theorem 3.3. *The following assertions are equivalent:*

(*i*) *(M, ∇, h) is a contravariant pseudo-Hessian manifold.*
(*ii*) *(TM, Π) is a Poisson manifold.*

In this case, if L is a leaf of $\mathrm{Im}h_\#$ then $TL \subset TM$ is a symplectic leaf of Π which is also a complex submanifold of TM. Moreover, if ω_L is the symplectic form of TL induced by Π and g_L is the pseudo-Riemannian metric given by $g_L(U, V) = \omega(JU, V)$ then (TL, g_L, ω_L, J) is a pseudo-Kähler manifold.

Proof. We will use Proposition 3.2 to prove the equivalence. Indeed, by a direct computation one can establish easily, for any $\alpha, \beta, \gamma \in \Omega^1(M)$, the following relations

$$\overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^v, \gamma^v) = \overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^h, \gamma^h) = \overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^v, \gamma^v) = \overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^h, \gamma^h) = \overline{\nabla}_{\Pi_\#(\alpha^h)}\Pi(\beta^h, \gamma^v) = 0,$$
$$\overline{\nabla}_{\Pi_\#(\alpha^v)}\Pi(\beta^h, \gamma^v) = \nabla_{\alpha^\#}(h)(\beta, \gamma) \circ p,$$

and the equivalence follows. The second part of the theorem is obvious and the only point which need to be checked is that g_L is nondegenerate. □

Remark 3.4.

1. The total space of the dual of a Lie algebroid carries a Poisson tensor (see [12]). If (M, ∇, h) is a contravariant pseudo-Hessian manifold then, according to Theorem 2.4, T^*M carries a Lie algebroid structure and one can see easily that Π is the corresponding Poisson tensor on TM.
2. The equivalence of (i) and (ii) in Theorem 3.3 deserves to be stated explicitly in the case of $\mathbb{R}^n$ endowed with its canonical affine structure ∇. Indeed, let $(h_{ij})_{1 \le i,j \le n}$ be a symmetric matrix where $h_{ij} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and h the associated symmetric bivector field on $\mathbb{R}^n$. The associated bivector field Π_h on $T\mathbb{R}^n = \mathbb{C}^n$ is

$$\Pi_h = \sum_{i,j=1}^n h_{ij}(x)\partial_{x_i} \wedge \partial_{y_j},$$

where $(x_1 + iy_1, \dots, x_n + iy_n)$ are the canonical coordinates of $\mathbb{C}^n$. Then, according to Theorem 2.4, $(\mathbb{R}^n, \nabla, h)$ is a contravariant pseudo-Hessian manifold if and only if $(\mathbb{C}^n, \Pi_h)$ is a Poisson manifold.

We explore now some relations between some invariants of (M, ∇, h) and some invariants of (TM, Π).

Proposition 3.5. *Let (M, ∇, h) be a contravariant pseudo-Hessian manifold. Then $(\mathrm{div}_\nabla h)^v = \mathrm{div}_{\overline{\nabla}}\Pi$.*

Proof. Fix $(x,u) \in TM$ and choose a basis $(e_1,\ldots,e_n)$ of T_xM. Then $(e_1^v,\ldots,e_n^v,e_1^h,\ldots,e_n^h)$ is a basis of $T_{(x,u)}TM$ with $((e_1^*)^v,\ldots,(e_n^*)^v,(e_1^*)^h,\ldots,(e_n^*)^h)$ as a dual basis. For any $\alpha \in T_x^*M$, we have

$$\begin{aligned}\prec \alpha^v, \mathrm{div}_{\overline{\nabla}}\Pi \succ &= \sum_{i=1}^n \left(\overline{\nabla}_{e_i^v}(\Pi)((e_i^*)^v,\alpha^v) + \overline{\nabla}_{e_i^h}(\Pi)((e_i^*)^h,\alpha^v)\right)\\ &\overset{(3.2)}{=} \prec \alpha, \mathrm{div}_\nabla(h) \succ \circ p = \prec \alpha^v, (\mathrm{div}_\nabla(h))^v \succ .\end{aligned}$$

In the same way we get that $\prec \alpha^h, \mathrm{div}_{\overline{\nabla}}\Pi \succ = 0$ and the result follows. □

Let (M,∇,h) be a contravariant pseudo-Hessian manifold. For any multivector field Q on M we define its vertical lift Q^v on TM by

$$i_{\alpha^h}Q^v = 0 \quad \text{and} \quad Q^v(\alpha_1^v,\ldots,\alpha_q^v) = Q(\alpha_1,\ldots,\alpha_q)\circ p.$$

Recall that h defines a Lie algebroid structure on T^*M whose anchor is $h_\#$ and the Lie bracket is given by (2.3). The Poisson tensor Π defines a Lie algebroid structure on T^*TM whose anchor is $\Pi_\#$ and the Lie bracket is the Koszul bracket

$$[\phi_1,\phi_2]_\Pi = \mathcal{L}_{\Pi_\#(\phi_1)}\phi_2 - \mathcal{L}_{\Pi_\#(\phi_2)}\phi_1 - d\Pi(\phi_1,\phi_2), \quad \phi_1,\phi_2 \in \Omega^1(TM).$$

We denote by $\mathbf{d}_h$ (resp. $\mathbf{d}_\Pi$) the differential associated to the Lie algebroid structure on T^*M (resp. T^*TM) defined by h (resp. Π).

Proposition 3.6.

(*i*) *For any $\alpha,\beta \in \Omega^1(M)$ and $X \in \Gamma(TM)$,*

$$\begin{cases}\mathcal{L}_{X^h}\alpha^h = (\mathcal{L}_X\alpha)^h,\ \mathcal{L}_{X^h}\alpha^v = (\nabla_X\alpha)^v,\ \mathcal{L}_{X^v}\alpha^h = 0 \quad \textit{and} \quad \mathcal{L}_{X^v}\alpha^v = (\mathcal{L}_X\alpha)^h - (\nabla_X\alpha)^h,\\ [\alpha^h,\beta^h]_\Pi = 0,\ [\alpha^v,\beta^v]_\Pi = -[\alpha,\beta]_h^v \quad \textit{and} \quad [\alpha^h,\beta^v]_\Pi = (\mathcal{D}_\beta\alpha)^h,\end{cases}$$

where $\mathcal{D}$ is the connection given by (2.4).

(*ii*) $(\mathbf{d}_hQ)^v = -\mathbf{d}_\Pi(Q^v)$.

Proof. The relations in (i) can be established by a straightforward computation. From these relations and the fact that $\Pi_\#(\alpha^h) = (\alpha^\#)^v$ one can deduce easily that $i_{\alpha^h}\mathbf{d}_\Pi(Q^v) = 0$. On the other hand, since $\Pi_\#(\alpha^v) = -(\alpha^\#)^h$ and $[\alpha^v,\beta^v]_\Pi = -[\alpha,\beta]^v$ we can conclude. □

Remark 3.7. From Propositions 2.14 and Proposition 3.6, we can deduce that $\mathbf{d}_\Pi(\mathrm{div}_{\overline{\nabla}}\Pi) = 0$. This is not a surprising result because $\overline{\nabla}$ is flat and $\mathrm{div}_{\overline{\nabla}}\Pi$ is a representative of the modular class of Π.

As a consequence of Proposition 3.6 we can define a linear map from the cohomology of $(T^*M, h_\#, [\ ,\ ]_h)$ to the cohomology of $(T^*TM, \Pi_\#, [\ ,\ ]_\Pi)$ by

$$V : H^*(M,h) \longrightarrow H^*(TM,\Pi),\ [Q] \mapsto [Q^v].$$

Proposition 3.8. *V is injective.*

Proof. An element $P \in \Gamma(\wedge^d TTM)$ is of type $(r, d-r)$ if for any $q \neq r$

$$P(\alpha_1^v, \dots, \alpha_q^v, \beta_1^h, \dots, \beta_{d-q}^h) = 0,$$

for any $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_{d-q} \in \Omega^1(M)$. We have

$$\begin{cases} \Gamma(\wedge^d TTM) = \bigoplus_{r=0}^d \Gamma_{(r,d-r)}(\wedge^d TTM), \\ \mathbf{d}_\Pi(\Gamma_{(r,d-r)}(\wedge^d TTM)) \subset \Gamma_{(r+1,d-r)}(\wedge^{d+1} TTM) \oplus \Gamma_{(r,d+1-r)}(\wedge^{d+1} TTM). \end{cases}$$

Let $Q \in \Gamma(\wedge^d TM)$ such that $\mathbf{d}_h Q = 0$ and there exists $P \in \Gamma(\wedge^{d-1} TTM)$ such that $\mathbf{d}_\Pi P = Q^v$. Since $Q^v \in \Gamma_{(d,0)}(\wedge^d TTM)$ then $P \in \Gamma_{(d-1,0)}(\wedge^{d-1} TTM)$. Let us show that $P = T^v$. For $\alpha_1, \dots, \alpha_{d-1}, \beta \in \Omega^1(M)$, we have

$$0 = \mathbf{d}_\Pi P(\beta^h, \alpha_1^v, \dots, \alpha_{d-1}^v) = (\beta^\#)^v.P(\alpha_1^v, \dots, \alpha_{d-1}^v).$$

So the function $P(\alpha_1^v, \dots, \alpha_{d-1}^v)$ is constant on the fibers of TM and hence there exists $T \in \Gamma(\wedge^{d-1} TM)$ such that $P(\alpha_1^v, \dots, \alpha_{d-1}^v) = T(\alpha_1, \dots, \alpha_{d-1}) \circ p$. So $[Q] = 0$ which completes the proof. $\square$

4. Linear, affine and multiplicative contravariant pseudo-Hessian structures

4.1. Linear and affine contravariant pseudo-Hessian structures

As in the Poisson geometry context, we have the notions of linear and affine contravariant pseudo-Hessian structures. One can see [10] for the notion of cocycle in associative algebras.

Let (V, ∇) be a finite dimensional real vector space endowed with its canonical affine structure. A symmetric bivector field h on V is called affine if there exists a commutative product $\bullet$ on V^* and a symmetric bilinear form B on V^* such that, for any $\alpha, \beta \in V^* \subset \Omega^1(V)$ and $u \in V$,

$$h(\alpha, \beta)(u) = \prec \alpha \bullet \beta, u \succ + B(\alpha, \beta).$$

One can see easily that if $\alpha, \beta \in \Omega^1(V) = C^\infty(V, V^*)$ then

$$h(\alpha, \beta)(u) = \prec \alpha(u) \bullet \beta(u), u \succ + B(\alpha(u), \beta(u)).$$

If $B = 0$, h is called linear.

If $(x_1, \dots, x_n)$ is a linear coordinates system on V^* associated to a basis $(e_1, \dots, e_n)$ then

$$h(dx_i, dx_j) = b_{ij} + \sum_{k=1}^n C_{ij}^k x_k,$$

where $e_i \bullet e_j = \sum_{k=1}^n C_{ij}^k e_k$ and $b_{ij} = B(e_i, e_j)$.

Proposition 4.1. *(V, ∇, h) is a contravariant pseudo-Hessian manifold if and only if $\bullet$ is associative and B is a scalar 2-cocycle of $(V^*, \bullet)$, i.e.,*

$$B(\alpha \bullet \beta, \gamma) = B(\alpha, \beta \bullet \gamma)$$

for any $\alpha, \beta, \gamma \in V^*$.

Proof. For any $\alpha \in V^*$ and $u \in V$, $\alpha^\#(u) = \mathrm{L}_\alpha^* u + i_\alpha B$ where $\mathrm{L}_\alpha(\beta) = \alpha \bullet \beta$ and $i_\alpha B \in V^{**} = V$. We denote by $\phi^{\alpha^\#}$ the flow of the vector field $\alpha^\#$. Then, for any $\alpha, \beta, \gamma \in V^*$,

$$\begin{aligned}\nabla_{\alpha^\#}(h)(\beta,\gamma)(u) &= \frac{d}{dt}_{|t=0}\Big(\prec \beta\bullet\gamma, \phi^{\alpha^\#}(t,u)\succ + B(\beta,\gamma)\Big)\\ &= \prec \beta\bullet\gamma, \mathrm{L}_\alpha^* u + i_\alpha B\succ\\ &= \prec \alpha\bullet(\beta\bullet\gamma), u\succ + B(\alpha,\beta\bullet\gamma)\end{aligned}$$

and the result follows. □

Conversely, we have the following result.

Proposition 4.2. *Let $(\mathcal{A}, \bullet, B)$ be a commutative and associative algebra endowed with a symmetric scalar 2-cocycle. Then:*

1. *$\mathcal{A}^*$ carries a structure of a contravariant pseudo-Hessian structure (∇, h) where ∇ is the canonical affine structure of $\mathcal{A}^*$ and h is given by*

$$h(u,v)(\alpha) = \prec \alpha, u(\alpha)\bullet v(\alpha)\succ + B(u(\alpha), v(\alpha)), \quad \alpha\in\mathcal{A}^*, u, v\in\Omega^1(\mathcal{A}^*).$$

2. *When $B = 0$, the leaves of the affine foliation associated to $\mathrm{Im} h_\#$ are the orbits of the action Φ of $(\mathcal{A}, +)$ on $\mathcal{A}^*$ given by $\Phi(u,\alpha) = \exp(\mathrm{L}_u^*)(\alpha)$.*
3. *The associated Poisson tensor Π on $T\mathcal{A}^* = \mathcal{A}^*\times\mathcal{A}^*$ is the affine Poisson tensor dual associated to the Lie algebra $(\mathcal{A}\times\mathcal{A}, [\ ,\ ])$ endowed with the 2-cocycle B_0 where*

$$[(a,b),(c,d)] = (a\bullet d - b\bullet c, 0) \quad \textit{and} \quad B_0((a,b),(c,d)) = B(a,d) - B(c,b).$$

Proof. It is only the third point which need to be checked. One can see easily that $[\ ,\ ]$ is a Lie bracket on $\mathcal{A}\times\mathcal{A}$ and B_0 is a scalar 2-cocycle for this Lie bracket. For any $a\in\mathcal{A}\subset\Omega^1(\mathcal{A}^*)$, $a^v = (0,a)\in\mathcal{A}\times\mathcal{A}\subset\Omega^1(\mathcal{A}^*\times\mathcal{A}^*)$ and $a^h = (a,0)$. So

$$\Pi(a^h, b^v)(\alpha,\beta) = h(a,b)(\alpha) = \prec\alpha, a\bullet b\succ + B(a,b).$$

On the other hand, if Π^* is the Poisson tensor dual, then

$$\begin{aligned}\Pi^*(a^h,b^v)(\alpha,\beta) &= \Pi^*((a,0),(0,b))(\alpha,\beta)\\ &= \prec(\alpha,\beta), [(a,0),(0,b)]\succ + B_0((a,0),(0,b))\\ &= \prec\alpha, a\bullet b\succ + B(a,b)\\ &= \Pi(a^h,b^v)(\alpha,\beta).\end{aligned}$$

In the same way one can check the other equalities. □

This proposition can be used as a machinery to build examples of pseudo-Hessian manifolds. Indeed, by virtue of Proposition 2.5, any orbit L of the action Φ has an affine structure ∇_L and a pseudo-Riemannian metric g_L such that (L, ∇_L, g_L) is a pseudo-Hessian manifold.

Example 4.3. Let $\mathcal{A}$ be the algebra generated by one element x satisfying the relation $x^5 = 0$, and setting $e_k := x^k$. It is clear that $\mathcal{A}$ is a commutative associative algebra. We endow $\mathcal{A}^*$ with the linear contravariant pseudo-Hessian structure associated to product of $\mathcal{A}$. We denote by (a,b,c,d) the linear coordinates on $\mathcal{A}$ and (x,y,z,t) the dual coordinates on $\mathcal{A}^*$. We have

$$\Phi(ae_1+be_2+ce_3+de_4^*, xe_1^*+ye_2^*+ze_3^*+te_4^*) = (x+ay+(\frac{1}{2}a^2+b)z+(\frac{1}{6}a^3+ab+c)t, y+az+(\frac{1}{2}a^2+b)t, z+at, t)$$

and

$$X_{e_1} = y\partial_x + z\partial_y + t\partial_z,\ X_{e_2} = z\partial_x + t\partial_y,\ X_{e_3} = t\partial_x \quad \text{and} \quad X_{e_4} = 0.$$

Let us describe the pseudo-Hessian structure of the hyperplane $M_c = \{t = c, c \neq 0\}$ endowed with the coordinates (x,y,z). We denote by g_c the pseudo-Riemannian of M_c. We have, for instance,

$$g_c(X_{e_1}, X_{e_1})(x,y,z,c) = h(e_1,e_1)(x,y,z,c) = \prec e_1 \bullet e_1, (x,y,z,c) \succ = y.$$

So, one can see that the matrix of g_c in $(X_{e_1}, X_{e_2}, X_{e_3})$ is the passage matrix P from $(X_{e_1}, X_{e_2}, X_{e_3})$ to $(\partial_x, \partial_y, \partial_z)$ and hence

$$g_c = \frac{1}{c}\left(2dxdz + dy^2 - \frac{2z}{c}dydz + \frac{(z^2-yc)}{c^2}dz^2\right).$$

The signature of this metric is $(+,+,-)$ if $c > 0$ and $(+,-,-)$ if $c < 0$. One can check easily that g_c is the restriction of $\nabla d\phi$ to M_c, where

$$\phi(x,y,z,t) = \frac{z^4}{12t^3} + \frac{y^2}{2t} - \frac{z^2y}{2t} + \frac{xz}{t}.$$

4.2. Multiplicative contravariant pseudo-Hessian structures

A contravariant pseudo-Hessian structure (∇, h) on a Lie group G is called multiplicative if the multiplication $m : (G \times G, \nabla \oplus \nabla, h \oplus h) \longrightarrow (G, \nabla, h)$ is affine and sends $h \oplus h$ to h.

Lemma 4.4. *Let G be a connected Lie group and ∇ a connection on G such that the multiplication m : $(G \times G, \nabla \oplus \nabla) \longrightarrow (G, \nabla)$ preserves the connections. Then G is abelian and ∇ is bi-invariant.*

Proof. We will denote by $\chi^r(G)$ (resp. $\chi^l(G)$) the space of right invariant vector fields (resp. the left invariant vector fields) on G. It is clear that for any $X \in \chi^r(G)$ and $Y \in \chi^l(G)$, the vector field (X,Y) on $G \times G$ is m-related to the vector field $X + Y$ on G:

$$Tm(X_a, Y_b) = X_a.b + a.Y_b = X_{ab} + Y_{ab} = (X+Y)_{ab}$$

It follows that for any $X_1, X_2 \in \chi^r(G)$ and $Y_1, Y_2 \in \chi^l(G)$, the vector field $(\nabla \oplus \nabla)_{(X_1,Y_1)}(X_2,Y_2)$ is m-related to $\nabla_{X_1+Y_1}(X_2+Y_2)$, hence:

$$Tm((\nabla_{X_1}X_2)_a, (\nabla_{Y_1}Y_2)_b) = (\nabla_{(X_1+Y_1)}(X_2+Y_2))_{ab}$$

So we get

$$(\nabla_{X_1}X_2)_a.b + a.(\nabla_{Y_1}Y_2)_b = (\nabla_{X_1}X_2 + \nabla_{X_1}Y_2 + \nabla_{Y_1}X_2 + \nabla_{Y_1}Y_2)_{ab} \tag{4.1}$$

If we take $Y_1 = 0 = Y_2$ we obtain that ∇ is right invariant. In the same way we get that ∇ is left invariant. Now, if we return back to the equation (4.1) we obtain that for any $X \in \chi^r(G)$ and $Y \in \chi^l(G)$

we have $\nabla X = 0 = \nabla Y$. This implies that any left invariant vector field is also right invariant; indeed, if $Y = \sum_{i=1}^{n} f_i X_i$ with $Y \in \chi^l(G)$ and $X_i \in \chi^r(G)$ then $X_j f_i = 0$ for all $i, j = 1, \cdot, n$. Hence the adjoint representation is trivial and hence G must be abelian. $\square$

At the end of the paper, we give another proof of this Lemma based on parallel transport.

Corollary 4.5. *Let (∇, h) be multiplicative contravariant pseudo-Hessian structure on a simply connected Lie group G. Then G is a vector space, ∇ its canonical affine connection and h is linear.*

Example 4.6. Based on the classification of complex associative commutative algebras given in [14], we can give a list of examples of affine contravariant pseudo-Hessian structures up to dimension 4.

1. On $\mathbb{R}^2$:

$$h_1 = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} x_1 & x_2 \\ x_2 & 0 \end{pmatrix} \quad \text{and} \quad h_3 = \begin{pmatrix} x_2 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. On $\mathbb{R}^3$:

$$h_1 = \begin{pmatrix} a & 0 & x_2 \\ 0 & 0 & 0 \\ x_2 & 0 & b \end{pmatrix}, \; h_2 = \begin{pmatrix} x_2 & x_3 & a \\ x_3 & a & 0 \\ a & 0 & 0 \end{pmatrix}, \; h_3 = \begin{pmatrix} a & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} x_2 & 0 & x_2 \\ 0 & 0 & x_2 + a \\ x_2 & x_2 + a & x_3 \end{pmatrix} \quad \text{and} \quad h_5 = \begin{pmatrix} x_2 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

3. On $\mathbb{R}^4$:

$$h_1 = \begin{pmatrix} x_3 & a & x_4 + b & 0 \\ a & -x_4 + c & 0 & 0 \\ x_4 + b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} x_2 & x_3 & x_4 & a \\ x_3 & x_4 & a & 0 \\ x_4 & a & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}, \; h_3 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_4 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_5 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & 0 \\ x_3 & x_4 & 0 & 0 \\ x_4 & 0 & 0 & 0 \end{pmatrix}.$$

5. Quadratic contravariant pseudo-Hessian structures

Let V be a vector space of dimension n. Denote by ∇ its canonical affine connection. A symmetric bivector field h on V is quadratic if there exists a basis $\mathbb{B}$ of V such that, for any $i, j = 1, \ldots, n$,

$$h(dx_i, dx_j) = \sum_{l,k=1}^{n} a_{l,k}^{i,j} x_l x_k,$$

where the $a_{k,l}^{i,j}$ are real constants and $(x_1, \ldots, x_n)$ are the linear coordinates associated to $\mathbb{B}$.

For any linear endomorphism A on V we denote by $\widetilde{A}$ the associated linear vector field on V.

The key point is that if h is a quadratic contravariant pseudo-Hessian bivector field on V then its divergence is a linear vector field, i.e., $\operatorname{div}_\nabla(h) = \widetilde{L^h}$ where L^h is a linear endomorphism of V. Moreover, if $F = (A, u)$ is an affine transformation of V then $\operatorname{div}_\nabla(F_* h) = \widetilde{A^{-1} L^h A}$. So the Jordan form of L_h is an invariant of the quadratic contravariant pseudo-Hessian structure. By using Maple we can classify quadratic contravariant pseudo-Hessian structures on $\mathbb{R}^2$. The same approach has been used by [8] to classify quadratic Poisson structures on $\mathbb{R}^4$. Note that if h is a quadratic contravariant pseudo-Hessian tensor on $\mathbb{R}^n$ then its associated Poisson tensor on $\mathbb{C}^n$ is also quadratic.

Theorem 5.1.

1. *Up to an affine isomorphism, there is two quadratic contravariant pseudo-Hessian structures on* $\mathbb{R}^2$ *which are divergence free*

$$h_1 = \begin{pmatrix} 0 & 0 \\ 0 & ux^2 \end{pmatrix} \quad \textit{and} \quad h_2 = \begin{pmatrix} \frac{r^2x^2}{c} - 2rxy + cy^2 & \frac{r^3x^2}{c^2} - \frac{2r^2xy}{c} + ry^2 \\ \frac{r^3x^2}{c^2} - \frac{2r^2xy}{c} + ry^2 & -\frac{2r^3xy}{c^2} + \frac{r^4x^2}{c^3} + \frac{r^2y^2}{c} \end{pmatrix}.$$

2. *Up to an affine isomorphism, there is two quadratic contravariant pseudo-Hessian structures on* $\mathbb{R}^2$ *with the divergence equivalent to the Jordan form* $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$,

$$h_1 = \begin{pmatrix} cy^2 + xy & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad h_2 = \begin{pmatrix} \frac{1}{2}xy + cy^2 & \frac{y^2}{4} \\ \frac{y^2}{4} & 0 \end{pmatrix}.$$

3. *Up to an affine isomorphism, there is five quadratic contravariant pseudo-Hessian structures on* $\mathbb{R}^2$ *with diagonalizable divergence*

$$h_1 = \begin{pmatrix} ax^2 & 0 \\ 0 & by^2 \end{pmatrix}, \ h_2 = \begin{pmatrix} ax^2 + by^2 & 0 \\ 0 & 0 \end{pmatrix}, \ h_3 = \begin{pmatrix} ax^2 & axy \\ axy & ay^2 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} \frac{2r^2x^2}{c} - 2rxy + cy^2 & ry^2 \\ ry^2 & \frac{2r^2y^2}{c} \end{pmatrix} \quad \textit{and}$$

$$h_5 = \begin{pmatrix} (\frac{2p^2}{u} + \frac{q}{2})x^2 + \frac{pqxy}{u} + \frac{q^2y^2}{4u} & px^2 + qxy - \frac{pqy^2}{2u} \\ px^2 + qxy - \frac{pqy^2}{2u} & (\frac{2p^2}{u} + \frac{q}{2})y^2 + ux^2 - 2pxy \end{pmatrix}.$$

4. *Up to an affine isomorphism, there is a unique quadratic pseudo-Hessian structure on* $\mathbb{R}^2$ *with the divergence having non real eigenvalues*

$$h = \begin{pmatrix} -2pxy - ux^2 + uy^2 & px^2 - py^2 - 2uxy \\ px^2 - py^2 - 2uxy & 2pxy + ux^2 - uy^2 \end{pmatrix}.$$

Example 5.2. The study of quadratic contravariant pseudo-Hessian structures on $\mathbb{R}^3$ is more complicated and we give here a class of quadratic pseudo-Hessian structures on $\mathbb{R}^3$ of the form $\widetilde{A} \odot \widetilde{\mathrm{I}}_3$ where $\widetilde{A}$ is linear.

1. A is diagonal:

$$h_1 = \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} x^2 & xy & 0 \\ xy & y^2 & 0 \\ 0 & 0 & -z^2 \end{pmatrix}.$$

2. $A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$:

$$h_3 = \begin{pmatrix} 2x(y-px) & (y-px)y+pyx & pxz+(y-px)z \\ (y-px)y+pyx & 2py^2 & 2pyz \\ pxz+(y-px)z & 2pyz & 2pz^2 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} 2x(y+px) & (y+px)y-pyx & pxz+(y+px)z \\ (y+px)y+pyx & -2py^2 & 0 \\ pxz+(y+px)z & 0 & 2pz^2 \end{pmatrix}.$$

6. Right-invariant contravariant pseudo-Hessian structures on Lie groups

Let $(\mathfrak{g}, \bullet)$ be a left symmetric algebra, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}(u,v,w) = \mathrm{ass}(v,u,w) \quad \text{and} \quad \mathrm{ass}(u,v,w) = (u \bullet v) \bullet w - u \bullet (v \bullet w).$$

This implies that $[u,v] = u \bullet v - v \bullet u$ is a Lie bracket on $\mathfrak{g}$ and $\mathrm{L} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $u \mapsto \mathrm{L}_u$ is a representation of the Lie algebra $(\mathfrak{g}, [\ ,\ ])$. We denote by L_u the left multiplication by u.

We consider a connected Lie group G whose Lie algebra is $(\mathfrak{g}, [\ ,\ ])$ and we define on G a right invariant connection by

$$\nabla_{u^-} v^- = -(u \bullet v)^-, \tag{6.1}$$

where u^- is the right vector field associated to $u \in \mathfrak{g}$. This connection is torsionless and without curvature and hence (G, ∇) is an affine manifold. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ which is symmetric and let r^- be the associated right invariant symmetric bivector field.

Proposition 6.1. *(G, ∇, r^-) is a contravariant pseudo-Hessian manifold if and only if, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,*

$$[[r,r]](\alpha,\beta,\gamma) := \prec \gamma, [\alpha,\beta]_r^\# - [\alpha^\#, \beta^\#] \succ = 0, \tag{6.2}$$

where

$$[\alpha,\beta]_r = \mathrm{L}^*_{\alpha^\#}\beta - \mathrm{L}^*_{\beta^\#}\alpha \quad \text{and} \quad \prec \mathrm{L}^*_u \alpha, v \succ = - \prec \alpha, u \bullet v \succ .$$

In this case, the product on $\mathfrak{g}^*$ given by $\alpha.\beta = \mathrm{L}^*_{\alpha^\#}\beta$ is left symmetric, $[\ ,\ ]_r$ is a Lie bracket and $r_\#$ is a morphism of Lie algebras.

Proof. Note first that for any $\alpha \in \mathfrak{g}^*$, $(\alpha^-)^\# = (\alpha^\#)^-$ and $\nabla_{u^-}\alpha^- = -(\mathrm{L}^*_u\alpha)^-$ and hence, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\nabla_{(\alpha^-)^\#}(r^-)(\beta^-, \gamma^-) = r(\mathrm{L}^*_{\alpha^\#}\beta, \gamma) + r(\beta, \mathrm{L}^*_{\alpha^\#}\gamma).$$

So, (G, ∇, r^-) is a contravariant pseudo-Hessian manifold if and only if, for any $\alpha, \beta, \gamma \in \mathfrak{g}^*$,

$$\begin{aligned} 0 &= r(\mathrm{L}^*_{\alpha^\#}\beta, \gamma) + r(\beta, \mathrm{L}^*_{\alpha^\#}\gamma) - r(\mathrm{L}^*_{\beta^\#}\alpha, \gamma) - r(\alpha, \mathrm{L}^*_{\beta^\#}\gamma) \\ &= \prec \gamma, [\alpha,\beta]_r^\# - \alpha^\# \bullet \beta^\# + \beta^\# \bullet \alpha^\# \succ \\ &= \prec \gamma, [\alpha,\beta]_r^\# - [\alpha^\#, \beta^\#] \succ \end{aligned}$$

and the first part of the proposition follows. Suppose now that $[\alpha,\beta]_r^{\#} = [\alpha^{\#},\beta^{\#}]$ for any $\alpha,\beta \in \mathfrak{g}^*$. Then, for any $\alpha,\beta,\gamma \in \mathfrak{g}^*$,

$$\mathrm{ass}(\alpha,\beta,\gamma) - \mathrm{ass}(\beta,\alpha,\gamma) = \mathrm{L}^*_{[\alpha,\beta]_r^{\#}}\gamma - \mathrm{L}^*_{\alpha^{\#}}\mathrm{L}^*_{\beta^{\#}}\gamma + \mathrm{L}^*_{\beta^{\#}}\mathrm{L}^*_{\alpha^{\#}}\gamma = 0.$$

This completes the proof. $\square$

Definition 6.2.

1. Let $(\mathfrak{g},\bullet)$ be a left symmetric algebra. A symmetric bivector $r \in \mathfrak{g}\otimes\mathfrak{g}$ satisfying $[[r,r]] = 0$ is called a S-matrix.
2. A left symmetric algebra $(\mathfrak{g},\bullet,r)$ endowed with a S-matrix is called a contravariant pseudo-Hessian algebra.

Let $(\mathfrak{g},\bullet,r)$ be a contravariant pseudo-Hessian algebra, $[u,v] = u\bullet v - v\bullet u$ and G a connected Lie group with $(\mathfrak{g},[\ ,\ ])$ as a Lie algebra. We have shown that G carries a right invariant contravariant pseudo-Hessian structure (∇, r^-). On the other hand, in Section 3, we have associated to (∇, r^-) a flat connection $\overline{\nabla}$, a complex structure J and a Poisson tensor Π on TG. Now we will show that TG carries a structure of Lie group and the triple $(\overline{\nabla}, J, \Pi)$ is right invariant. This structure of Lie group on TG is different from the usual one defined by the adjoint action of G on $\mathfrak{g}$.

Let us start with a general algebraic construction which is interesting on its own. Let $(\mathfrak{g},\bullet)$ be a left symmetric algebra, put $\Phi(\mathfrak{g}) = \mathfrak{g}\times\mathfrak{g}$ and define a product $\star$ and a bracket on $\Phi(\mathfrak{g})$ by

$$(a,b)\star(c,d) = (a\bullet c, a\bullet d) \quad \text{and} \quad [(a,b),(c,d)] = ([a,c], a\bullet d - c\bullet b),$$

for any $(a,b),(c,d) \in \Phi(\mathfrak{g})$. It is easy to check that $\star$ is left symmetric, $[\ ,\ ]$ is the commutator of $\star$ and hence is a Lie bracket. We define also $J_0 : \Phi(\mathfrak{g}) \longrightarrow \Phi(\mathfrak{g})$ by $J_0(a,b) = (b,-a)$. It is also a straightforward computation to check that

$$N_{J_0}((a,b),(c,d)) = [J_0(a,b), J_0(c,d)] - J_0[(a,b), J_0(c,d)] - J_0[J_0(a,b),(c,d)] - [(a,b),(c,d)] = 0.$$

For $r \in \otimes^2\mathfrak{g}$ symmetric, we define $R \in \otimes^2\Phi(\mathfrak{g})$ by

$$R((\alpha_1,\beta_1),(\alpha_2,\beta_2)) = r(\alpha_1,\beta_2) - r(\alpha_2,\beta_1), \tag{6.3}$$

for any $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathfrak{g}^*$. We have obviously that $R_{\#}(\alpha_1,\beta_1) = (-\beta_1^{\#}, \alpha_1^{\#})$.

Proposition 6.3. *$[[r,r]] = 0$ if and only if $[R,R] = 0$, where $[R,R]$ is the Schouten bracket associated to the Lie algebra structure of $\Phi(\mathfrak{g})$ and given by*

$$[R,R](\alpha,\beta,\gamma) = \oint_{\alpha,\beta,\gamma} \prec \gamma, [R_{\#}(\alpha), R_{\#}(\beta)] \succ, \quad \alpha,\beta,\gamma \in \Phi^*(\mathfrak{g}).$$

Proof. For any $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2), \gamma = (\gamma_1,\gamma_2) \in \Phi(\mathfrak{g})^*$,

$$\begin{aligned}\prec \gamma, [R_{\#}(\alpha), R_{\#}(\beta)] &= \prec (\gamma_1,\gamma_2), [(-\alpha_2^{\#},\alpha_1^{\#}),(-\beta_2^{\#},\beta_1^{\#})] \succ \\ &= \prec \gamma_1, [\alpha_2^{\#},\beta_2^{\#}] \succ - \prec \gamma_2, \alpha_2^{\#}\bullet\beta_1^{\#} \succ + \prec \gamma_2, \beta_2^{\#}\bullet\alpha_1^{\#} \succ \\ &= \prec \gamma_1, [\alpha_2^{\#},\beta_2^{\#}] \succ + \prec \beta_1, (\mathrm{L}^*_{\alpha_2^{\#}}\gamma_2)^{\#} \succ - \prec \alpha_1, (\mathrm{L}^*_{\beta_2^{\#}}\gamma_2)^{\#} \succ,\end{aligned}$$

$$\prec \beta, [R_{\#}(\gamma), R_{\#}(\alpha)] \succ = \prec \beta_1, [\gamma_2^{\#}, \alpha_2^{\#}] \succ + \prec \alpha_1, (\mathrm{L}^*_{\gamma_2^{\#}}\beta_2)^{\#} \succ - \prec \gamma_1, (\mathrm{L}^*_{\alpha_2^{\#}}\beta_2)^{\#} \succ$$
$$\prec \alpha, [R_{\#}(\beta), R_{\#}(\gamma)] \succ = \prec \alpha_1, [\beta_2^{\#}, \gamma_2^{\#}] \succ + \prec \gamma_1, (\mathrm{L}^*_{\beta_2^{\#}}\alpha_2)^{\#} \succ - \prec \beta_1, (\mathrm{L}^*_{\gamma_2^{\#}}\alpha_2)^{\#} \succ .$$

So

$$[R, R](\alpha, \beta, \gamma) = -[[r, r]](\beta_2, \gamma_2, \alpha_1) - [[r, r]](\gamma_2, \alpha_2, \beta_1) - [[r, r]](\alpha_2, \beta_2, \gamma_1)$$

and the result follows. □

Let G be a Lie group whose Lie algebra is $(\mathfrak{g}, [\ ,\ ])$ and let $\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the homomorphism of groups such that $d_e\rho = \mathrm{L}$ where $\mathrm{L} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is the representation associated to $\bullet$. Then the product

$$(g, u).(h, v) = (gh, u + \rho(g)(v)), \quad g, h \in G, u, v \in \mathfrak{g}$$

induces a Lie group structure on $G \times \mathfrak{g}$ whose Lie algebra is $(\Phi(\mathfrak{g}), [\ ,\ ])$. The complex endomorphism J_0 and the left symmetric product $\star$ induce a right invariant complex tensor J_0^- and a right invariant connection $\widetilde{\nabla}$ given by

$$J_0^-(a, b)^- = (b, -a)^- \quad \text{and} \quad \widetilde{\nabla}_{(a,b)^-}(c, d)^- = -((a, b) \star (c, d))^-.$$

Let $r \in \otimes^2\mathfrak{g}$ symmetric such that $[[r, r]] = 0$, r^- the associated right invariant symmetric bivector field and ∇ the affine connection given by (6.1). Then (G, ∇, r^-) is a contravariant pseudo-Hessian manifold and let $\overline{\nabla}$, J and Π be the associated structure on TG defined in Section 3.

Theorem 6.4. *If we identify TG with $G \times \mathfrak{g}$ by $u_g \longrightarrow (g, T_gR_{g^{-1}}u_g)$ we denote also by Π, $\overline{\nabla}$ and J the images of Π, $\overline{\nabla}$ and J under this identification then $\Pi = R^-$, $\overline{\nabla} = \widetilde{\nabla}$ and $J = J_0^-$.*

To prove this theorem, we need some preparation.

Proposition 6.5. *Let (G, ∇) be a Lie group endowed with a right invariant connection and $\gamma : [0, 1] \longrightarrow G$ a curve. Let $V : [0, 1] \longrightarrow TG$ be a vector field along γ. We define $\mu : [0, 1] \longrightarrow \mathfrak{g}$ and $W : [0, 1] \longrightarrow \mathfrak{g}$ by*

$$\mu(t) = T_{\gamma(t)}R_{\gamma(t)^{-1}}(\gamma'(t)) \quad \text{and} \quad W(t) = T_{\gamma(t)}R_{\gamma(t)^{-1}}(V(t)).$$

Then V is parallel along γ with respect ∇ if and only if

$$W'(t) - \mu(t) \bullet W(t) = 0,$$

where $u \bullet v = -(\nabla_{u^-}v^-)(e)$.

Proof. We consider $(u_1, \dots, u_n)$ a basis of $\mathfrak{g}$ and $(X_1, \dots, X_n)$ the corresponding right invariant vector fields. Then

$$\begin{cases} \mu(t) = \sum_{i=1}^n \mu_i(t)u_i, \ W(t) = \sum_{i=1}^n W_i(t)u_i, \\ \gamma'(t) = \sum_{i=1}^n \mu_i(t)X_i, \ V(t) = \sum_{i=1}^n W_i(t)X_i. \end{cases}$$

Then

$$\begin{aligned}\nabla_t V(t) &= \sum_{i=1}^{n} W_i'(t)X_i + \sum_{i=1}^{n} W_i(t)\nabla_{\gamma'(t)}X_i \\ &= \sum_{i=1}^{n} W_i'(t)X_i + \sum_{i,j=1}^{n} W_i(t)\mu_j(t)\nabla_{X_j}X_i \\ &= \sum_{i=1}^{n} W_i'(t)X_i - \sum_{i,j=1}^{n} W_i(t)\mu_j(t)(u_j \bullet u_i)^- \\ &= (W'(t) - \mu(t) \bullet W(t))^-\end{aligned}$$

and the result follows having in mind that u^- is the right invariant vector field associated to $u \in \mathfrak{g}$. □

Let (G, ∇) be a Lie group endowed with a right invariant connection. Then ∇ induces a splitting of $TTG = \ker dp \oplus \mathcal{H}$. For any tangent vector $X \in T_gG$, we denote by $X^v, X^h \in T_{(g,u)}TG$ the vertical and the horizontal lift of X.

Proposition 6.6. *If we identify TG to $G \times \mathfrak{g}$ by $X_g \mapsto (g, T_gR_{g^{-1}}(X_g))$ then for any $X \in T_gG$,*

$$X^v(g, u) = (0, T_gR_{g^{-1}}(X)) \quad \textit{and} \quad X^h(g, u) = (X, T_gR_{g^{-1}}(X) \bullet u).$$

Proof. The first relation is obvious. Recall that the horizontal lift of X at $u_g \in TG$ is given by:

$$X^h(u_g) = \frac{d}{dt}_{|t=0} V(t)$$

where $V : [0, 1] \longrightarrow TG$ is the parallel vector field along $\gamma : [0, 1] \longrightarrow G$ a curve such that $\gamma(0) = g$ and $\gamma'(0) = X$. If we denote by $\Theta_R : TG \longrightarrow G \times \mathfrak{g}$ the identification $u_g \mapsto (g, T_gR_{g^{-1}}(u_g))$ then, by virtue of Proposition 6.5,

$$T_{u_g}\Theta_R(X^h) = \frac{d}{dt}_{|t=0}(\gamma(t), W(t)) = (X, T_gR_{g^{-1}}(X) \bullet u). \quad \square$$

We consider now a left symmetric algebra $(\mathfrak{g}, \bullet)$, G a connected Lie group associated to $(\mathfrak{g}, [\ ,\ ])$, ∇ the right invariant affine connection associated to $\bullet$. We have seen that $G \times \mathfrak{g}$ has a structure of Lie group. We identify TG to $G \times \mathfrak{g}$ and, for any vector field X on G, we denote by X^v and X^h the vector fields on $G \times \mathfrak{g}$ obtained by the identification from the horizontal and the vertical lift of X. For $a, b \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$, a^- (resp. α^-) is the right invariant vector field (resp. 1-form) on G associated to a (resp. α), $(a, b)^-$ (resp. $(\alpha, \beta)^-$) the right invariant vector field (resp. 1-form) on $G \times \mathfrak{g}$ associated to (a, b) (resp. (α, β)).

Proposition 6.7. *For any $(a, b) \in \mathfrak{g} \times \mathfrak{g}$ and $(\alpha, \beta) \in \mathfrak{g}^* \times \mathfrak{g}^*$,*

$$(a, b)^- = (a^-)^h + (b^-)^v \quad \textit{and} \quad (\alpha, \beta)^- = (\alpha^-)^h + (\beta^-)^v.$$

Proof. We have

$$\begin{aligned}(a, b)^-(g, u) &= T_{(e,0)}R_{(g,u)}(a, b) \\ &= \frac{d}{dt}_{|t=0}(\exp(ta), tb)(g, u) \\ &= \frac{d}{dt}_{|t=0}(\exp(ta)g, tb + \rho(\exp(ta))(u))\end{aligned}$$

$$\begin{aligned} &= (a^-(g), b + a \bullet u) \\ &= (a^-(g), T_gR_{g^{-1}}(a^-(g)) \bullet u) + (0, T_gR_{g^{-1}}(b^-(g)) \\ &= (a^-)^h(g,u) + (b^-)^v(g,u). \qquad \text{(Proposition 6.5)} \end{aligned}$$

The second relation can be deduced easily from the first one. □

Proof of Theorem 6.4. Let Π be the Poisson tensor on $G \times \mathfrak{g}$ associated to r^-. Then, by using the precedent proposition,

$$\begin{aligned} \Pi((\alpha_1,\beta_1)^-, (\alpha_2,\beta_2)^-) &= \Pi((\alpha_1^-)^h + (\beta_1^-)^v, (\alpha_2^-)^h + (\beta_2^-)^v) \\ &= r^-(\alpha_1^-, \beta_2^-) - r^-(\alpha_2^-, \beta_1^-) \\ &= r(\alpha_1, \beta_2) - r(\alpha_2, \beta_1) \\ &= R^-((\alpha_1,\beta_1)^-, (\alpha_2,\beta_2)^-). \end{aligned}$$

In the same way,

$$\begin{aligned} J_0^-(a,b)^- &= (b,-a)^- = (b^-)^h - (a^-)^v, \\ J(a,b)^- &= (b^-)^h - (a^-)^v, \\ \overline{\nabla}_{(a,b)^-}(c,d)^- &= (\nabla_{a^-}c^-)^h + (\nabla_{a^-}d^-)^v = -((a \bullet c)^-)^h - ((a \bullet d)^-)^v = -((a,b).(c,d))^- \\ &= \widetilde{\nabla}_{(a,b)^-}(c,d)^-. \quad \square \end{aligned}$$

Let $(\mathfrak{g}, \bullet)$ be a left symmetric algebra, (M, ∇) and affine manifold and $\rho : \mathfrak{g} \longrightarrow \Gamma(TM)$ a linear map such that $\rho(u \bullet v) = \nabla_{\rho(u)}\rho(v)$. Then ρ defines an action on M of the Lie algebra $(\mathfrak{g}, [\ ,\ ])$. We consider $\rho^l : \Phi(\mathfrak{g}) \longrightarrow \Gamma(TTM)$, $(u,v) \longrightarrow \rho(u)^h + \rho^v(v)$. It is easy to check that

$$\rho^l([a,b]) = [\rho^l(a), \rho^l(b)].$$

Let $r \in \otimes^2\mathfrak{g}$ satisfying $[[r,r]] = 0$ and $R \in \otimes^2\Phi(\mathfrak{g})$ given by (6.3).

Theorem 6.8. *The bivector field on TM associated to $\rho(r)$ is $\rho^l(R)$ which is a Poisson tensor and $(M, \nabla, \rho(r))$ is a contravariant pseudo-Hessian manifold.*

Proof. Let $(e_1, \dots, e_n)$ a basis of $\mathfrak{g}$ and $E_i = (e_i, 0)$ and $F_i = (0, e_i)$. Then $(E_1, \dots, E_n, F_1, \dots, F_n)$ is a basis of $\Phi(\mathfrak{g})$. Then

$$r = \sum_{i,j} r_{i,j} e_i \otimes e_j \quad \text{and} \quad R = \sum_{i,j} r_{i,j} \left(E_i \otimes F_j - F_i \otimes E_j\right).$$

So

$$\rho(r) = \sum_{i,j=1}^n r_{i,j}\rho(e_i) \otimes \rho(e_j) \quad \text{and} \quad \rho^l(R) = \sum_{i,j=1}^n r_{i,j}\left(\rho(e_i)^h \otimes \rho(e_j)^v - \rho(e_i)^v \otimes \rho(e_j)^h\right).$$

Then for any $\alpha, \beta \in \Omega^1(M)$

$$\rho^l(R)(\alpha^v, \beta^v) = \rho^l(R)(\alpha^h, \beta^h) = 0 \quad \text{and} \quad \rho^l(R)(\alpha^h, \beta^v) = \rho(r)(\alpha, \beta) \circ p.$$

According to Proposition (6.3), R is a solution of the classical Yang-Baxer equation and hence $\rho^l(R)$ is a Poisson tensor. By using Theorem 3.3, we get that $(M, \nabla, \rho(r))$ is a contravariant pseudo-Hessian manifold. $\square$

Example 6.9.

1. Let $\mathfrak{g} = \mathrm{gl}(n, \mathbb{R})$ be the Lie algebra of n-square matrices. It has a structure of left symmetric algebra given by $A \bullet B = BA$. Let $\rho : \mathfrak{g} \longrightarrow \Gamma(T\mathbb{R}^n)$ given by $\rho(A) = A$. Then $\rho(A \bullet B) = \nabla_A B$, where ∇ is the canonical connection of $\mathbb{R}^n$. According to Theorem 6.8, any S-matrix on $\mathfrak{g}$ gives rise to a quadratic contravariant pseudo-Hessian structure on $\mathbb{R}^n$.
2. More generally, let (M, ∇) be an affine manifold and $\mathfrak{g}$ the finite dimensional Lie algebra of affine vector fields. Recall that $X \in \mathfrak{g}$ if for any $Y, Z \in \Gamma(TM)$,

$$[X, \nabla_Y Z] = \nabla_{[X,Y]} Z + \nabla_Y [X, Z].$$

Since the curvature and the torsion of ∇ vanish this is equivalent to

$$\nabla_{\nabla_Y Z} X = \nabla_Y \nabla_Z X.$$

From this relation, one can see easily that, for any $X, Y \in \mathfrak{g}$, $X \bullet Y := \nabla_X Y \in \mathfrak{g}$ and $(\mathfrak{g}, \bullet)$ is an associative finite dimensional Lie algebra which acts on M by $\rho(X) = X$. Moreover, $\rho(X \bullet Y) = \nabla_X Y$. According to Theorem 6.8, any S-matrix on $\mathfrak{g}$ gives rise to a contravariant pseudo-Hessian structure on M.

Classification of two-dimensional contravariant pseudo-Hessian algebras

Using the classification of two-dimensional non-abelian left symmetric algebras given in [4] and the classification of abelian left symmetric algebras given in [14], we give a classification (over the field $\mathbb{R}$) of 2-dimensional contravariant pseudo-Hessian algebras. We proceed as follows:

1. For any left symmetric 2-dimensional algebra $\mathfrak{g}$, we determine its automorphism group $\mathrm{Aut}(\mathfrak{g})$ and the space of S-matrices on $\mathfrak{g}$, we denote by $\mathcal{A}(\mathfrak{g})$.
2. We give the quotient $\mathcal{A}(\mathfrak{g})/\sim$ where $\sim$ is the equivalence relation:

$$r^1 \sim r^2 \Longleftrightarrow \exists\, A \in \mathrm{Aut}(\mathfrak{g}) \;\text{ or }\; \exists\, \lambda \in \mathbb{R} \text{ such that } r^2_\sharp = A \circ r^1_\sharp \circ A^t \text{ or } r^2 = \lambda r^1.$$

We end this paper by giving another proof to Lemma 4.4.

Proof. For any $\gamma : [0,1] \longrightarrow G \times G$, $t \mapsto (\gamma_1(t), \gamma_2(t))$ with $\gamma(0) = (a, b)$ and $\gamma(1) = (c, d)$,

$$\tau_{m(\gamma)}(T_{(a,b)} m(u, v)) = T_{(c,d)} m(\tau_\gamma(u, v)),$$

where $\tau_\gamma : T_{(a,b)}(G \times G) \longrightarrow T_{(c,d)}(G \times G)$ and $\tau_{m\gamma} : T_{ab}G \longrightarrow T_{cd}G$ are the parallel transports. But

$$T_{(a,b)} m(u, v) = T_a R_b(u) + T_b L_a(v) \quad \text{and} \quad \tau_\gamma(u, v) = (\tau_{\gamma_1}(u), \tau_{\gamma_2}(v)).$$

So we get

$$\tau_{\gamma_1\gamma_2}(T_a R_b(u)) + \tau_{\gamma_1\gamma_2}(T_b L_a(v)) = T_c R_d(\tau_{\gamma_1}(u)) + T_d L_c(\tau_{\gamma_2}(v)).$$

$(\mathfrak{g}, .)$	$\mathrm{Aut}(\mathfrak{g})$	$\mathcal{A}(\mathfrak{g})/\sim$
$b_{1,\alpha\neq -1,1}$ $e_2.e_1 = e_1, e_2.e_2 = \alpha e_2$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
$b_{1,\alpha=-1}$ $e_2.e_1 = e_1, e_2.e_2 = -e_2$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};$ $r_\sharp^3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^4 = 0$
$b_{1,\alpha=1}$ $e_2.e_1 = e_1, e_2.e_2 = e_2$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, ab \neq 0$	$r_\sharp^1 = \begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
$b_{2,\beta\neq 0,1,2}$ $e_1.e_2 = \beta e_1, e_2.e_1 = (\beta-1)e_1, e_2.e_2 = \beta e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
$b_{2,\beta=1}$ $e_1.e_2 = e_1, e_2.e_2 = e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
$b_{2,\beta=2}$ $e_1.e_2 = 2e_1,$ $e_2.e_1 = e_1, e_2.e_2 = 2e_2$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
b_3 $e_2.e_1 = e_1, e_2.e_2 = e_1 + e_2$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$r_\sharp^1 = \begin{pmatrix} 1/2 & 1 \\ 1 & 1 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^3 = 0$
b_4 $e_1.e_1 = 2e_1, e_1.e_2 = e_2,$ $e_2.e_2 = e_1$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$r_\sharp^1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^3 = 0$
b_5 $e_1.e_2 = e_1, e_2.e_2 = e_1 + e_2$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$r_\sharp^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^2 = 0$
As_2^1 $e_1.e_1 = e_2$	$\begin{pmatrix} a & 0 \\ b & a^2 \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; r_\sharp^3 = 0$
As_2^4 $e_1.e_1 = e_1, e_1.e_2 = e_2,$ $e_2.e_2 = e_2$	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, a \neq 0$	$r_\sharp^1 = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}; r_\sharp^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$ $r_\sharp^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; r_\sharp^4 = 0$

If we take $v = 0$ and $\gamma_2(t) = b = d$. We get

$$\tau_{\gamma_1 b}(T_a R_b(u)) = T_c R_b(\tau_{\gamma_1}(u))$$

and hence ∇ is right invariant. In the same way we get that ∇ is left invariant. And finally

$$\tau_{\gamma_1\gamma_2}(T_a R_b(u)) = T_c R_d(\tau_{\gamma_1}(u)) \quad \text{and} \quad \tau_{\gamma_1\gamma_2}(T_b L_a(v)) = T_d L_c(\tau_{\gamma_2}(v)).$$

If we take $\gamma_2 = \gamma_1^{-1}$ we get that

$$\tau_{\gamma_1}(u) = T_a R_{a^{-1}c}(u) = T_a L_{ca^{-1}}(u).$$

This implies that the adjoint representation is trivial and hence G must be abelian. $\square$

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References

[1] C. Bai, Left-symmetric bialgebras and an analogue of the classical Yang-Baxter equation, Commun. Contemp. Math. 10 (2) (2008) 221–260.
[2] S. Benayadi, M. Boucetta, On para-Kähler Lie algebroids and contravariant pseudo-Hessian structures, Math. Nachr. (2019) 1–26.
[3] S. Benayadi, M. Boucetta, On para-Kähler and hyper-Kähler Lie algebras, J. Algebra 436 (2015) 61–101.
[4] D. Burde, Simple left-symmetric algebras with solvable Lie algebra, Manuscr. Math. 95 (1998) 397–411.
[5] C. Calvin, L.M.C. Schochet, Global Analysis on Foliated Space, MISRP, 2006.
[6] P. Dombrowski, On the geometry of the tangent bundle, J. Reine Angew. Math. 210 (1962) 73–78.

[7] Jean-Paul Dufour, Nguyen Tien Zung, Poisson Structures and Their Normal Forms, vol. 242, Springer Science and Business Media, 2006.
[8] J.P. Dufour, A. Haraki, Rotationnels et structures de Poisson quadratiques, C. R. Acad. Sci., Sér. 1 Math. 312 (1) (1991) 137–140.
[9] R.L. Fernandes, Lie algebroids, holonomy and characteristic classes, Adv. Math. 170 (2002) 119–179.
[10] Christian Kassel, Homology and cohomology of associative algebras. A concise introduction to cyclic homology, in: Thematic School, August 2004 at ICTP, Trieste (Italy), 2006, <cel-00119891>.
[11] M. De León, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, Elsevier, 2011.
[12] K. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, London Math. Soc. Lecture Notes Ser., vol. 124, Cambridge Univ. Press, Cambridge, 1987.
[13] H. Shima, The Geometry of Hessian Structures, World Scientific Publishing, 2007.
[14] I.S. Rakhimov, I.M. Rikhsiboev, W. Basri, Complete lists of low dimensional complex associative algebras, arXiv:0910.0932v2.
[15] A. Weinstein, The local structure of Poisson manifolds, J. Differ. Geom. 18 (3) (1983) 523–557.
[16] K. Yano, S. Ishihara, Tangent and Cotangent Bundles: Differential Geometry, Dekker, 1973.