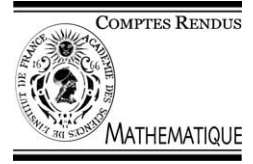




Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

C. R. Acad. Sci. Paris, Ser. I 337 (2003) 61–66



Differential Topology

The modular class of a regular Poisson manifold and the Reeb class of its symplectic foliation

Abdelhak Abouqateb, Mohamed Boucetta

Faculté des sciences et techniques, BP 618, Guéliz-Marrakech, Morocco

Received 7 April 2003; accepted after revision 6 May 2003

Presented by Étienne Ghys

Abstract

We show that, for any regular Poisson manifold, there is an injective natural linear map from the first leafwise cohomology space into the first Poisson cohomology space which maps the Reeb class of the symplectic foliation to the modular class of the Poisson manifold. A Riemannian interpretation of the Reeb class will give some geometric criteria which enables one to tell whether the modular class vanishes or not. It also enables one to construct examples of unimodular Poisson manifolds and others which are not unimodular. Finally, we prove that the first leafwise cohomology space is an invariant of Morita equivalence. **To cite this article:** *A. Abouqateb, M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

La classe modulaire d'une variété de Poisson régulière et la classe de Reeb de son feuilletage symplectique. Pour une variété de Poisson régulière, il existe une application linéaire naturelle de la 1-cohomologie feuilletée vers la 1-cohomologie de Poisson qui envoie la classe de Reeb du feuilletage symplectique sur la classe modulaire de la structure de Poisson. Nous donnons une interprétation riemannienne de la classe de Reeb ; ce qui permettra d'avoir des critères géométriques pour décider de la nullité ou non de la classe modulaire. Finalement, nous prouvons que la 1-cohomologie feuilletée est un invariant de l'équivalence de Morita. **Pour citer cet article :** *A. Abouqateb, M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Version française abrégée

La classe de Reeb d'une variété feuilletée est une obstruction dans la 1-cohomologie feuilletée à l'existence d'une forme volume du fibré normal invariante par les champs de vecteurs tangents au feuilletage. La classe modulaire d'une variété de Poisson est une obstruction dans la 1-cohomologie de Poisson à l'existence d'une forme volume invariante par les flots hamiltoniens. Pour une variété de Poisson régulière, il existe une application linéaire naturelle de la 1-cohomologie feuilletée vers la 1-cohomologie de Poisson qui envoie la classe de Reeb du feuilletage symplectique sur la classe modulaire de la structure de Poisson. D'un autre côté, sur une variété

E-mail addresses: abouqateb@fstg-marrakech.ac.ma (A. Abouqateb), boucetta@fstg-marrakech.ac.ma (M. Boucetta).

1631-073X/03/\$ – see front matter © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

doi:10.1016/S1631-073X(03)00254-1

feuilletée munie d'une métrique riemannienne, la trace par rapport à la métrique riemannienne de la seconde forme fondamentale de la distribution orthogonale au feuilletage définit une 1-forme le long des feuilles qui représente la classe de Reeb. Le fait que la seconde forme fondamentale de la distribution orthogonale s'annule équivaut à ce que le feuilletage soit riemannien, nous permet de déduire de ce qui précède les résultats suivants :

Théorème 0.1. *Soit (P, π) une variété de Poisson. Si le feuilletage symplectique associé à π est riemannien alors la classe modulaire de (P, π) s'annule. En plus, si le feuilletage symplectique est transversalement orienté de codimension 1 alors la classe modulaire s'annule si et seulement si le feuilletage est riemannien.*

Théorème 0.2. *Soit (P, π) une variété de Poisson régulière, simplement connexe et compacte dont le feuilletage symplectique est de codimension 1. Alors, la classe modulaire de P est différente de zéro.*

Corollaire 0.3. *Tout feuilletage orienté de dimension 2 sur la sphère S^3 est le feuilletage symplectique d'une structure de Poisson dont la classe modulaire est non nulle.*

1. Introduction

The Reeb class of a foliated manifold is an obstruction lying in the first leafwise cohomology space to the existence of a volume normal form invariant by the vector fields tangent to the foliation [4]. The modular class of a Poisson manifold is an obstruction lying in the first Poisson cohomology to the existence of a volume form invariant with respect to the Hamiltonian flows [9]. For a regular Poisson manifold, Weinstein [9] pointed out that the two classes are closely related without giving an explicit relation between them. In fact, the two classes represent the same mathematical object. We will show in Section 4 that the first leafwise cohomology space is, in a natural way, a subspace of the first Poisson cohomology space and the Reeb class of the symplectic foliation agrees with the modular class of the Poisson manifold.

In Section 2, we remark that, given a foliated manifold $(M, T\mathcal{F})$ endowed with a Riemannian metric g , the trace with respect to g of the second fundamental form of the orthogonal distribution to $T\mathcal{F}$ gives rise to a tangential 1-form whose leafwise cohomology class is the Reeb class of the foliation. It is known that the second fundamental form of the orthogonal distribution vanishes if and only if g is bundle-like [7]. This remark and the fact that the Reeb class is the same object as the modular class will enable us to deduce the following results in Section 4.

Theorem 1.1. *Let (P, π) be a regular Poisson manifold. If the symplectic foliation is Riemannian then the modular class of P vanishes.*

Furthermore, if the symplectic foliation is transversally oriented of codimension 1, then the modular class of P vanishes if and only if the symplectic foliation is Riemannian.

Theorem 1.2. *Let (P, π) be a simply connected and compact regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. Then $\text{mod}(P) \neq 0$.*

Corollary 1.3. *Any oriented foliation of codimension 1 on the sphere S^3 is the symplectic foliation of a Poisson structure on S^3 with non-vanishing modular class.*

Using these results, we construct many examples of regular Poisson manifolds with vanishing modular class and many examples with non-vanishing modular class.

The first Poisson cohomology spaces of Morita equivalent Poisson manifolds are isomorphic, according to Ginzburg and Lu [3], and Ginzburg and Golubev [2] has shown that the modular classes are compatible with this isomorphism. We will show in Section 5 that the first leafwise cohomology spaces are also compatible with this isomorphism.

2. The Reeb class of a foliation and its Riemannian interpretation

Let M be a differentiable manifold endowed with a transversally oriented foliation \mathcal{F} of dimension p and of codimension q . We denote by $T\mathcal{F}$ the tangent bundle to the foliation, by $\mathcal{X}(\mathcal{F})$ the space of vector fields tangent to \mathcal{F} and by $\mathcal{A}_{\mathcal{F}}^r$ the space of sections of the bundle $\wedge^r T^*\mathcal{F} \rightarrow M$. The elements of $\mathcal{A}_{\mathcal{F}}^r$ are called tangential differential r -forms. The expression

$$d_{\mathcal{F}}\alpha(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \cdot \alpha(X_1, \dots, \widehat{X}_i, \dots, X_{r+1}) + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{r+1}),$$

where $\alpha \in \mathcal{A}_{\mathcal{F}}^r$ and $X_1, \dots, X_{r+1} \in \mathcal{X}(\mathcal{F})$, defines a degree one differential operator $d_{\mathcal{F}}$ that satisfies $d_{\mathcal{F}}^2 = 0$. The induced cohomology denoted by $H_{\mathcal{F}}^*(M)$ is called the leafwise cohomology [1].

An orientation of the normal bundle to \mathcal{F} is a differential q -form ν on M such that $\nu_x \neq 0$ for all $x \in M$ and such that $i_X \nu = 0$ for any $X \in \mathcal{X}(\mathcal{F})$.

For all $X \in \mathcal{X}(\mathcal{F})$, $L_X \nu$ is proportional to ν and one can define a tangential 1-form $\alpha_{\mathcal{F}}$ by the relation $L_X \nu = \alpha_{\mathcal{F}}(X)\nu$. From the relation

$$L_{[X,Y]}\nu = L_X \circ L_Y \nu - L_Y \circ L_X \nu,$$

we have $d_{\mathcal{F}}\alpha_{\mathcal{F}} = 0$. The cohomology class of $\alpha_{\mathcal{F}}$ denoted by $\text{mod}(\mathcal{F})$ is the Reeb class of the foliation.

The normal bundle to \mathcal{F} carries an orientation ν such that $L_X \nu = 0$ for any $X \in \mathcal{X}(\mathcal{F})$ if and only if $\text{mod}(\mathcal{F}) = 0$ (see [4]).

Now, we give a Riemannian interpretation of the Reeb class.

Let g be a Riemannian metric on M and let ∇ be the associated Levi-Civita connection. We denote by $T^{\perp}\mathcal{F}$ the orthogonal distribution to $T\mathcal{F}$ and by $\mathcal{X}(\mathcal{F}^{\perp})$ the space of vector fields tangent to $T^{\perp}\mathcal{F}$. For any vector field X , we denote by $X^{\mathcal{F}}$ its component in $\mathcal{X}(\mathcal{F})$ and by $X^{\mathcal{F}^{\perp}}$ its component in $\mathcal{X}(\mathcal{F}^{\perp})$. The orthogonal volume form to the foliation is the differential q -form η defined by $\eta(Y_1, \dots, Y_q) = 1$ for any orthonormal oriented frame (Y_1, \dots, Y_q) in $\mathcal{X}(\mathcal{F}^{\perp})$ and $i_X \eta = 0$ for any $X \in \mathcal{X}(\mathcal{F})$.

A straightforward calculation gives

$$L_X \eta(Y_1, \dots, Y_q) = \sum_{i=1}^q g(\nabla_{Y_i} X, Y_i) = - \sum_{i=1}^q g(\nabla_{Y_i} Y_i, X). \tag{*}$$

The second fundamental form of $T^{\perp}\mathcal{F}$ is the tensor field $B^{\perp}: \mathcal{X}(\mathcal{F}^{\perp}) \times \mathcal{X}(\mathcal{F}^{\perp}) \rightarrow \mathcal{X}(\mathcal{F})$ given by

$$B^{\perp}(Y_1, Y_2) = \frac{1}{2} [\nabla_{Y_1} Y_2 + \nabla_{Y_2} Y_1]^{\mathcal{F}}.$$

Its trace with respect to g , called the tangent mean curvature, is a vector field H^{\perp} tangent to \mathcal{F} . We define a tangential 1-form $K^{\perp} \in \mathcal{A}_{\mathcal{F}}^1$ by

$$K^{\perp}(X) = g(X, H^{\perp}), \quad X \in \mathcal{X}(\mathcal{F}).$$

The relation (*) can be written

$$L_X \eta(Y_1, \dots, Y_q) = -K^{\perp}(X)$$

and so

$$\text{mod}(\mathcal{F}) = -[K^{\perp}]. \tag{1}$$

Remark 1. This formula can be compared to the metric formula for the Godbillon–Vey invariant (see [6]).

It is known that the second fundamental form B^\perp vanishes if and only if g is bundle-like [7] so we get the following proposition.

Proposition 2.1. *Let M be a differentiable manifold endowed with a transversally oriented foliation \mathcal{F} of codimension 1. The following assertions are equivalent:*

- (1) $\text{mod}(\mathcal{F}) = 0$.
- (2) \mathcal{F} is a Riemannian foliation.
- (3) \mathcal{F} is defined by a closed 1-form.

3. The modular class of a Poisson manifold

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman’s monograph [8].

Let P be a Poisson manifold with Poisson tensor π . We have a bundle map $\#_\pi : T^*P \rightarrow TP$ defined by $\beta(\#_\pi(\alpha)) = \pi(\alpha, \beta)$, for all $\alpha, \beta \in T^*P$. On the space of differential 1-forms $\Omega^1(P)$, the Poisson tensor induces a Lie bracket

$$[\alpha, \beta]_\pi = L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)), \quad \alpha, \beta \in \Omega^1(P).$$

For this Lie bracket and the usual Lie bracket on vector fields, the bundle map $\#_\pi$ induces a Lie algebra homomorphism $\#_\pi : \Omega^1(P) \rightarrow \mathcal{X}(P)$.

The Poisson cohomology of a Poisson manifold (P, π) is the cohomology of the chain complex $(\mathcal{X}^*(P), d_\pi)$ where, for $0 \leq p \leq \dim P$, $\mathcal{X}^p(P)$ is the $C^\infty(P, \mathbb{R})$ -module of p -multi-vector fields and d_π is given by

$$\begin{aligned} d_\pi Q(\alpha_0, \dots, \alpha_p) &= \sum_{j=0}^p (-1)^j \#_\pi(\alpha_j) \cdot Q(\alpha_0, \dots, \hat{\alpha}_j, \dots, \alpha_p) \\ &\quad + \sum_{i < j} (-1)^{i+j} Q([\alpha_i, \alpha_j]_\pi, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_p). \end{aligned}$$

We denote $H_\pi^*(P)$ the corresponding cohomology spaces.

The modular class of (P, π) is the obstruction to the existence of a volume form on P which is invariant with respect to Hamiltonian flows. More explicitly, let μ be a volume form on P . As shown in [9], the operator $\phi_\mu : f \mapsto \text{div}_\mu \#_\pi(df)$ is a derivation and hence a vector field called the modular vector field of (P, π) with respect to the volume form μ .

If we replace μ by $a\mu$, where a is a positive function, the modular vector fields becomes

$$\phi_{a\mu} = \phi_\mu + \#_\pi(d(\text{Log } a)).$$

Thus the first Poisson cohomology class of ϕ_μ is independent of μ , we call it the modular class of (P, π) and we denote it $\text{mod}(P)$. The Poisson manifold is unimodular if its modular class vanishes.

4. Link between the Reeb class and the modular class of a regular Poisson manifold

Let (P, π) be a Poisson manifold whose symplectic foliation, denoted by \mathcal{F} , is a regular foliation transversally oriented of dimension $2p$ and of codimension q . As shown in [9, pp. 385], the modular vector field of P is closely

related to the Reeb tangential 1-form. More explicitly, let $\omega \in \Gamma(\wedge^2 T^* \mathcal{F})$ be the leafwise symplectic form given by $\omega(u, v) = \pi(\#_\pi^{-1}(u), \#_\pi^{-1}(v))$, $u, v \in T\mathcal{F}$, where $\#_\pi^{-1}(u)$ denotes any antecedent of u by $\#_\pi$.

Let ν be a transverse orientation of \mathcal{F} , i.e., a differential q -form on P such that $i_{\#_\pi(df)}\nu = 0$ for every smooth function f and $\nu_x \neq 0$ for all $x \in P$. Choose a distribution F supplementary to $T\mathcal{F}$ and extend ω to a differential 2-form on P by setting $i_X\omega = 0$ for all X tangent to F . The form $\mu = \wedge^p \omega \wedge \nu$ is a volume form on P .

For any $f \in C^\infty(P)$, since $[L_{\#_\pi(df)}(\wedge^p \omega)] \wedge \nu = 0$, we have

$$L_{\#_\pi(df)}\mu = (\wedge^p \omega) \wedge L_{\#_\pi(df)}\nu = \alpha_{\mathcal{F}}(\#_\pi(df))\mu = -\#_\pi(\alpha)(f)\mu,$$

where α is any differential 1-form on P whose restriction to \mathcal{F} is $\alpha_{\mathcal{F}}$.

$\#_\pi(\alpha)$ depends only on $\alpha_{\mathcal{F}}$ and we will denote it by $\#_\pi(\alpha_{\mathcal{F}})$. We get that the modular vector field of (P, π) with respect to the volume form μ is given by $\phi_{(\wedge^p \omega) \wedge \nu} = -\#_\pi(\alpha_{\mathcal{F}})$.

For $1 \leq i \leq \dim P$, we consider the subspace $\mathcal{X}_0^i(P) \subset \mathcal{X}^i(P)$ of i -multi-vector fields Q such that $i_\alpha Q = 0$ for all $\alpha \in \text{Ker } \pi$. It is easy to verify that $d_\pi(\mathcal{X}_0^i) \subset \mathcal{X}_0^{i+1}$. The natural injection $\mathcal{X}_0^i \hookrightarrow \mathcal{X}^i$ induces a linear map $H^*(\mathcal{X}_0^i) \rightarrow H^*(P)$ which is injective for $*$ = 1.

Let $\pi : \mathcal{A}_{\mathcal{F}}^i(P) \rightarrow \mathcal{X}_0^i(P)$ be the map given by $\pi(\gamma)(\alpha_1, \dots, \alpha_p) = \gamma(\#_\pi(\alpha_1), \dots, \#_\pi(\alpha_p))$. It is easy to verify that π is an isomorphism and $\pi(d_{\mathcal{F}}\gamma) = d_\pi\pi(\gamma)$ and hence π induces an isomorphism

$$\pi^* : H_{\mathcal{F}}^i(P) \rightarrow H^i(\mathcal{X}_0^i(P)).$$

So, we have shown the following proposition.

Proposition 4.1. *Let (P, π) be a regular Poisson manifold for which the symplectic foliation is transversally oriented. The tensor π induces a linear injection*

$$\pi^* : H_{\mathcal{F}}^1(P) \hookrightarrow H_\pi^1(P)$$

and we have

$$\pi^*(\text{mod}(\mathcal{F})) = \text{mod}(P).$$

Now, we can give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. If the symplectic foliation is Riemannian, there is a bundle-like metric g for which the second fundamental form of the orthogonal distribution to the foliation vanishes [7]. From (1), we deduce that the Reeb class vanishes and by Proposition 4.1 the modular class vanishes. If the foliation is of codimension 1, the second fundamental form of the orthogonal distribution to the foliation vanishes if and only its trace with respect to the metric vanishes and the same argument as above gives the theorem. \square

Proof of Theorem 1.2. Let (P, π) be a simply connected and compact regular Poisson manifold for which the symplectic foliation is transversally oriented of codimension 1. If $\text{mod}(P) = 0$, the symplectic foliation is Riemannian and there is a closed 1-form α which defines the foliation and such that $\alpha_x \neq 0$ for any $x \in P$. Now α is exact since P is simply connected and must vanish in some point since P is compact. \square

Now we give some examples.

1. Let (G, ω) be a symplectic Lie group (for example the affine group $\text{GA}(n)$) (see [5]). Let $P \times G \rightarrow P$ be a locally free action of G on a differentiable manifold P whose associated foliation will be denoted by \mathcal{F} . The symplectic form ω gives rise to a tangential 2-form on \mathcal{F} which is symplectic on restriction to any leaf of \mathcal{F} . This gives canonically a Poisson structure π on P whose symplectic foliation is \mathcal{F} .

If the action of G leaves invariant a Riemannian metric on P , the foliation is Riemannian and the modular class of (P, π) vanishes. This is the case if P is a Lie group and G is a Lie subgroup which acts by left translations.

Another interesting case is the case where H is a Lie group with G as subgroup and Γ is a discrete subgroup of H such that there is a Riemannian metric on H which is right G -invariant and left Γ -invariant. Then the natural homogenous action of G on $P = \Gamma \backslash H$ is locally free and the associated foliation is Riemannian. For example, the natural action of \mathbb{R}^{2p} on the torus $T^n = \mathbb{Z}^n \backslash \mathbb{R}^n$.

2. The affine group $\text{GA}(2)$ can be considered as a subgroup of two simply connected Lie groups of dimension 3 both having a cocompact discrete subgroup Γ . The first one is $\widetilde{\text{SL}}(2, \mathbb{R})$ the universal covering of $\text{SL}(2, \mathbb{R})$ and the second is G_3 whose Lie algebra is given by the relations $[e_1, e_2] = -e_1$, $[e_1, e_3] = 0$, $[e_2, e_3] = -e_3$. There is a Poisson structure (constructed as above) on the compact 3-manifold $M = \Gamma \backslash H$ (where $H = \widetilde{\text{SL}}(2, \mathbb{R})$ or G_3) whose the symplectic foliation is the foliation given by the homogenous action of $\text{GA}(2)$. This foliation (of codimension one) is not Riemannian and so the modular class of the Poisson structure does not vanish.

5. The first leafwise cohomology space is an invariant of Morita equivalence

Following [10], recall that a full dual pair $P_1 \xleftarrow{\rho_1} W \xrightarrow{\rho_2} P_2$ consists of two Poisson manifolds (P_1, π_1) and (P_2, π_2) , a symplectic manifold W , and two submersions $\rho_1 : W \rightarrow P_1$ and $\rho_2 : W \rightarrow P_2$ such that ρ_1 is Poisson, ρ_2 is anti-Poisson, and the fibers of ρ_1 and ρ_2 are symplectic orthogonal to each other. A Poisson (or anti-Poisson) mapping is said to be complete if the pull-back of a complete Hamiltonian flow under this mapping is complete. A full dual pair is called complete if both ρ_1 and ρ_2 are complete. The Poisson manifolds P_1 and P_2 are Morita equivalent if there exists a complete full dual pair $P_1 \xleftarrow{\rho_1} W \xrightarrow{\rho_2} P_2$ such that ρ_1 and ρ_2 both have connected and simply connected fibers. Morita equivalent Poisson manifolds P_1 and P_2 have isomorphic first Poisson cohomology spaces. More explicitly, there is a natural isomorphism $E : H_{\pi_1}^1(P_1) \xrightarrow{\cong} H_{\pi_2}^1(P_2)$ which is defined by (see [2], Lemma 5.2)

$$E([\xi_1]) = [\xi_2] \iff \exists F \in C^\infty(W), \xi_1 = (\rho_1)_* X_F, \xi_2 = -(\rho_2)_* X_F.$$

Proposition 5.1. *Let (P_1, π_1) and (P_2, π_2) be Morita equivalent regular Poisson manifolds. Then*

$$E(\pi_1^*(H_{\mathcal{F}}^1(P_1))) = \pi_2^*(H_{\mathcal{F}}^1(P_2)).$$

Acknowledgements

The second authors' research was supported by The Third World Academy of Sciences RGA No 01-301 RG/Maths/AC.

References

- [1] A. El Kacimi, Sur la cohomologie feuilletée, *Compositio Math.* 49 (1983) 195–215.
- [2] V.L. Ginzburg, A. Golubev, Holonomy on Poisson Manifolds and the modular class, *Israel J. Math.* 122 (2001) 221–242.
- [3] V.L. Ginzburg, J.H. Lu, Poisson cohomology of Morita equivalent Poisson manifolds, *Internat. Math. Res. Notices* 10 (1992) 199–205.
- [4] S. Hurder, The Godbillon measure of amenable foliations, *J. Differential Geom.* 23 (1986) 347–365.
- [5] A. Medina, P. Revoy, Groupes de Lie à structure symplectique invariante, in: *Symplectic Geometry, Groupoids and integrable systems*, in: *Math. Sci. Res. Inst. Publ.*, Vol. 20, Springer, 1991, pp. 247–266.
- [6] B. Reinhart, A metric formula for the Godbillon–Vey invariant for foliations, *Proc. Amer. Soc.* 38 (April 1973).
- [7] P. Tondeur, *Foliations on Riemannian Manifolds*, Springer-Verlag, 1988.
- [8] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, in: *Progr. Math.*, Vol. 118, Birkhäuser, Berlin, 1994.
- [9] A. Weinstein, The modular automorphism group of a poisson manifold, *J. Geom. Phys.* 23 (1997) 379–394.
- [10] A. Weinstein, The local structure of Poisson manifolds, *J. Differential Geometry* 18 (1983) 523–557.