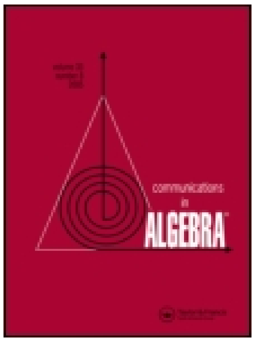


**Communications in Algebra**

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

# Flat Nonunimodular Lorentzian Lie Algebras

**Mohamed Boucetta & Hicham Lebzioui**

**To cite this article:** Mohamed Boucetta & Hicham Lebzioui (2016) Flat Nonunimodular Lorentzian Lie Algebras, Communications in Algebra, 44:10, 4185-4195

**To link to this article:** http://dx.doi.org/10.1080/00927872.2015.1087016

Published online: 03 Jun 2016.

*Communications in Algebra®*, 44: 4185–4195, 2016

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2015.1087016



# FLAT NONUNIMODULAR LORENTZIAN LIE ALGEBRAS

**Mohamed Boucetta[1] and Hicham Lebzioui[2]**
[1]*Université Cadi-Ayyad, Faculté des sciences et techniques, Marrakech, Marocco*
[2]*Université Moulay Smail, Faculté des sciences, Meknès, Marocco*

***A flat Lorentzian Lie algebra is a left symmetric algebra endowed with a symmetric bilinear form of signature $(-, +, \ldots, +)$ such that left multiplications are skew-symmetric. In geometrical terms, a flat Lorentzian Lie algebra is the Lie algebra of a Lie group with a left-invariant Lorentzian metric with vanishing curvature. In this article, we show that any flat nonunimodular Lorentzian Lie algebras can be obtained as a double extension of flat Riemannian Lie algebras. As an application, we give all flat nonunimodular Lorentzian Lie algebras up to dimension 4.***

***Key Words:*** Double extension; Flat Lorentzian Lie algebras; Nonunimodular Lie algebras; Representations of solvable Lie algebras.

***2010 Mathematics Subject Classification:*** 17B60; 17B30; 17B10; 53C50.



## 1. INTRODUCTION AND MAIN RESULTS

A *pseudo-Riemannian Lie group* is a Lie group $G$ with a left invariant pseudo-Riemannian metric $g$. The Lie algebra $\mathfrak{g} = T_eG$ of $G$ endowed with $\langle\ ,\ \rangle = g(e)$ is called *pseudo-Riemannian Lie algebra*. The Levi–Civita connection of $(G, g)$ defines a product $(u, v) \mapsto u.v$ on $\mathfrak{g}$ called *Levi–Civita product* given by Koszul's formula

$$2\langle u.v, w\rangle = \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle.$$

For any $u \in \mathfrak{g}$, we denote by $\mathrm{L}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ and $\mathrm{R}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$, respectively, the left multiplication and the right multiplication by $u$ given by $\mathrm{L}_u v = u.v$ and $\mathrm{R}_u v = v.u$. For any $u \in \mathfrak{g}$, $\mathrm{L}_u$ is skew-symmetric with respect to $\langle\ ,\ \rangle$ and $\mathrm{ad}_u = \mathrm{L}_u - \mathrm{R}_u$, where $\mathrm{ad}_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ is given by $\mathrm{ad}_u v = [u, v]$. The curvature of $g$ at $e$ is given by

$$\mathrm{K}(u, v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v].$$

If K vanishes, then $(G, g)$ is called *flat pseudo-Riemannian Lie group*, and $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called *flat pseudo-Riemannian Lie algebra*. The vanishing of the curvature is

Received March 24, 2015; Revised June 24, 2015. Communicated by A. Elduque.
Address correspondence to Dr. Hicham Lebzioui, Université Moulay Smail, Faculté des Sciences, BP 11201, Zitoune, Meknès, Marocco; E-mail: hlebzioui@gmail.com

equivalent to the fact that $\mathfrak{g}$ endowed with the Levi–Civita product is a left symmetric algebra, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}(u, v, w) = \mathrm{ass}(v, u, w),$$

where $\mathrm{ass}(u, v, w) = (u.v).w - u.(v.w)$. This relation is equivalent to

$$\mathrm{R}_{u.v} - \mathrm{R}_v \circ \mathrm{R}_u = [\mathrm{L}_u, \mathrm{R}_v], \tag{1}$$

for any $u, v \in \mathfrak{g}$. So a flat pseudo-Riemannian Lie algebra can be viewed as a left symmetric algebra with a bilinear symmetric nondegenerate form for which the left multiplications are skew-symmetric. If $g$ is geodesically complete, then $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called *complete*. A flat pseudo-Riemannian Lie algebra is complete if and only if it is unimodular (see [2]). A Riemannian (resp. Lorentzian) Lie group is a pseudo-Riemannian Lie group for which the metric is definite positive (resp. of signature $(-, + \cdots +)$). In [7], Milnor showed that a Riemannian Lie group is flat if and only if its Lie algebra is a semidirect product of an abelian algebra $\mathfrak{b}$ with an abelian ideal $\mathfrak{u}$ and, for any $u \in \mathfrak{b}$, $\mathrm{ad}_u$ is skew-symmetric. The determination of flat Lorentzian Lie groups is an open problem. A flat Lorentzian Lie algebra must be solvable (see [3]). In [2], Aubert and Medina showed that nilpotent flat Lorentzian Lie algebras are obtained by the double extension process from Riemannian abelian Lie algebras. In [4], Guediri studied Lie groups which may act isometrically and simply transitively on Minkowski space and get a precise description of nilpotent flat Lorentzian Lie groups. In [1], the authors showed that flat Lorentzian Lie algebras with degenerate center can be obtained by the double extension process from flat Riemannian Lie algebras. In the first part of this article, we show that any flat nonunimodular Lorentzian Lie algebra is obtained by the double extension process from a flat Riemannian Lie algebra. In the second part, as application of this result, we determine all nonunimodular flat Lorentzian Lie algebras up to dimension 4.

Let us state our main result in a more precise way. To do so, we need to recall some basic material.

- The double extension process was described in [2]. In particular, Propositions 3.1 and 3.2 of [2] are essential in this process. Let $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ be a pseudo-Riemannian flat Lie algebra, $\xi, D : B \longrightarrow B$ two endomorphisms of $B$, $b_0 \in B$ and $\mu \in \mathbb{R}$ such that the following statements hold:

  1. $\xi$ is a 1-cocycle of $(B, [\ ,\ ]_0)$ with respect to the representation $\mathrm{L} : B \longrightarrow \mathrm{End}(B)$ defined by the left multiplication associated to the Levi–Civita product, i.e., for any $a, b \in B$,

  $$\xi([a, b]) = \mathrm{L}_a \xi(b) - \mathrm{L}_b \xi(a); \tag{2}$$

  2. $D - \xi$ is skew-symmetric with respect to $\langle\ ,\ \rangle_0$,

  $$[D, \xi] = \xi^2 - \mu\xi - \mathrm{R}_{b_0}, \tag{3}$$

and for any $a, b \in B$

$$a.\xi(b) - \xi(a.b) = D(a).b + a.D(b) - D(a.b). \tag{4}$$

We call $(\xi, D, \mu, b_0)$ satisfying the two conditions above *admissible*.
Given $(\xi, D, \mu, b_0)$ admissible, we endow the vector space $\mathfrak{g} = \mathbb{R}e \oplus B \oplus \mathbb{R}\bar{e}$ with the inner product $\langle\ ,\ \rangle$ which extends $\langle\ ,\ \rangle_0$, for which span$\{e, \bar{e}\}$ and $B$ are orthogonal, $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$ and $\langle e, \bar{e}\rangle = 1$. We define also on $\mathfrak{g}$ the brackets

$$[\bar{e}, e] = \mu e,\ [\bar{e}, a] = D(a) - \langle b_0, a\rangle_0 e \quad \text{and} \quad [a, b] = [a, b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 e, \tag{5}$$

where $a, b \in B$ and $\xi^*$ is the adjoint of $\xi$ with respect to $\langle\ ,\ \rangle_0$. Then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a flat pseudo-Riemannian Lie algebra called *double extension* of $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to $(\xi, D, \mu, b_0)$.

- The *modular vector* of a pseudo-Riemannian Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$ is the vector $\mathbf{h} \in \mathfrak{g}$ given by

$$\langle \mathbf{h}, u\rangle = \mathrm{tr}(\mathrm{ad}_u) = -\mathrm{tr}(\mathrm{R}_u), \qquad \forall u \in \mathfrak{g}. \tag{6}$$

The Lie algebra $\mathfrak{g}$ is unimodular if and only if $\mathbf{h} = 0$. Denote by $H = \mathrm{span}\{\mathbf{h}\}$ and $H^\perp$ its orthogonal with respect to $\langle\ ,\ \rangle$.

We can now state our main result.

**Theorem 1.1.** *Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat nonunimodular Lorentzian Lie algebra.*

(i) *The left multiplication by $\mathbf{h}$ vanishes, i.e., $\mathrm{L}_\mathbf{h} = 0$ and both $H$ and $H^\perp$ are two-sided ideals with respect to the Levi–Civita product.*
(ii) *$(\mathfrak{g}, \langle\ ,\ \rangle)$ is obtained by the double extension process from a flat Riemannian Lie algebra $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to $(\xi, D, \mu, b_0)$ with $\mathrm{tr}(D) \neq -\mu$.*

The proof of Theorem 1.1 is given in Section 3. It is based on a property of the modular vector given in Proposition 3.1, and on the fact that a Lorentzian representation of a solvable Lie algebra can be reduced in an useful way by virtue of Lie's Theorem (see [5] Theorem 1.25 pp. 42). Section 2 is devoted to the study of Lorentzian representations of solvable Lie algebras. In Section 4 we give all flat nonunimodular Lorentzian Lie algebras up to dimension 4 (see Tables 1 and 2).

**Table 1** Three-dimensional flat nonunimodular Lorentzian Lie algebras

| The Lie algebra | The non vanishing brackets | The metric ($\mu \neq 0$) |
|---|---|---|
| $A_2 \oplus A_1$ | $[\bar{e}, e] = e.$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^*$ |
| $A_{3,3}$ | $[\bar{e}, e] = e,\ [\bar{e}, e_1] = e_1.$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^*$ |
| $A_{3,2}$ | $[\bar{e}, e] = e,\ [\bar{e}, e_1] = e_1 + e.$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^*$ |

**Table 2** Four-dimensional flat nonunimodular Lorentzian Lie algebras

| $\mathfrak{g}$ | The non vanishing brackets | The metric ($\mu \neq 0, \gamma \neq 0, \alpha, \beta \in \mathbb{R}$) |
|---|---|---|
| $A_{3,3} \oplus A_1$ | $[\bar{e}, e] = e, [\bar{e}, e_1] = e_1$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^* + \beta\bar{e}^* \odot e_2^*$ |
| $A_{3,2} \oplus A_1$ | $[\bar{e}, e] = e, [\bar{e}, e_1] = e_1 + e$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^* + \beta\bar{e}^* \odot e_2^*$ |
| $A_{4,9}^0$ | $[\bar{e}, e] = e, [\bar{e}, e_1] = e_1,$ $[e_1, e_2] = e$ | $\mu\bar{e}^* \odot e^* + (\gamma + 1)e_1^* \odot e_1^* + \gamma e_2^* \odot e_2^* + \beta\bar{e}^* \odot e_2^*$ $+\alpha(\bar{e}^* \odot \bar{e}^* + \bar{e}^* \odot e_1^*),\ \gamma \neq 0$ |
| $A_2 \oplus 2A_1$ | $[\bar{e}, e] = e$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^* + \alpha\bar{e}^* \odot e_1^*$ $+\beta\bar{e}^* \odot e_2^*$ |
| $A_{4,6}^{\gamma^{-1},0}$ | $[\bar{e}, e] = e,\ [\bar{e}, e_1] = \gamma e_2,$ $[\bar{e}, e_2] = -\gamma e_1,\ \gamma \neq 0$ | $\mu\bar{e}^* \odot e^* + (1 + \gamma)(e_1^* \odot e_1^* + e_2^* \odot e_2^*)$ $+\beta\bar{e}^* \odot e_2^* + \alpha\bar{e}^* \odot e_1^*$ |
| $A_{4,6}^{\gamma^{-1},\gamma^{-1}}$ | $[\bar{e}, e] = e\ ,\ [\bar{e}, e_1] = e_1 + \gamma e_2\ ,$ $[\bar{e}, e_2] = -\gamma e_1 + e_2,\ \gamma \neq 0$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^*$ $+\alpha\bar{e}^* \odot e_1^* + \beta\bar{e}^* \odot e_2^*$ |
| $A_{4,5}^{1,1}$ | $[\bar{e}, e] = e\ ,\ [\bar{e}, e_1] = e_1\ ,$ $[\bar{e}, e_2] = e_2$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^*$ |
| $A_{4,2}^1$ | $[\bar{e}, e] = e\ ,\ [\bar{e}, e_1] = e_1\ ,$ $[\bar{e}, e_2] = e_2 + e$ | $\mu\bar{e}^* \odot e^* + e_1^* \odot e_1^* + (1 + \beta^2)e_2^* \odot e_2^* + \beta e_1^* \odot e_2^*$ |



## 2. LORENTZIAN REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS

In this section, by using Lie's Theorem (see [5] Theorem 1.25, p. 42), we derive some interesting results on Euclidean and Lorentzian representations of solvable Lie algebras. Through this section, $\mathfrak{g}$ is a real solvable Lie algebra. We fix an ordering on $\mathfrak{g}^*$ and, for any $\lambda \in \mathfrak{g}^*$, we denote by $d\lambda$ the element of $\wedge^2\mathfrak{g}^*$ given by $d\lambda(u, v) = -\lambda([u, v])$.

A *pseudo-Euclidean vector space* is a real vector space of finite dimension $n$ endowed with a nondegenerate symmetric inner product of signature $(q, n - q) = (-, \ldots, -, +, \ldots, +)$. When the signature is $(0, n)$ (resp. $(1, n - 1)$) the space is called *Euclidean* (resp. *Lorentzian*). Let $(V, \langle\ ,\ \rangle)$ be a pseudo-Euclidean vector space whose signature is $(q, n - q)$, we denote by $\mathrm{so}(V)$ the Lie algebra of skew-symmetric endomorphisms of $(V, \langle\ ,\ \rangle)$. It is a well-known fact that the dimension of a totally isotropic subspace of $V$ is less or equal to $\min(q, n - q)$. In particular, if $V$ is Lorentzian, then any nontrivial totally isotropic space has dimension 1. This fact will be used frequently in this article.

Let $\rho : \mathfrak{g} \longrightarrow \mathrm{so}(V)$ be a representation of $\mathfrak{g}$. For any $\lambda \in \mathfrak{g}^*$, put

$$V_\lambda = \{x \in V,\ \rho(u)x = \lambda(u)x \text{ for all } u \in \mathfrak{g}\}.$$

The representation $\rho$ is called *indecomposable* if $V$ does not contain any nondegenerate invariant vector subspace.

**Proposition 2.1.** *Let $\mathfrak{g}$ be a real solvable Lie algebra, and let $\rho : \mathfrak{g} \longrightarrow \mathrm{so}(V)$ be an indecomposable Lorentzian representation. Then one of the following cases occurs:*

1. $\dim V = 1$ *and* $V = V_0$.
2. $\dim V = 2$, *there exists* $\lambda > 0$ *such that* $d\lambda = 0$ *and a basis* $(e, \bar{e})$ *of* $V$ *such that* $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$, $\langle e, \bar{e}\rangle = 1$ *and, for any* $u \in \mathfrak{g}$,

$$\rho(u)e = \lambda(u)e \quad \textit{and} \quad \rho(u)\bar{e} = -\lambda(u)\bar{e}.$$

3. $\dim V \geq 3$, *there exists* $\lambda \in \mathfrak{g}^*$ *such that* $d\lambda = 0$ *and* $V_\lambda$ *is a totally isotropic one dimensional vector space. Moreover, for any* $\mu \neq \lambda$, $V_\mu = \{0\}$.

*Proof.* We consider the complexification $V^{\mathbb{C}}$ of $V$. Then $\rho$ extends to a representation $\rho^{\mathbb{C}} : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}}(V^{\mathbb{C}})$ by putting

$$\rho^{\mathbb{C}}(u)(a + \imath b) = \rho(u)(a) + \imath\rho(u)(b).$$

Since $\mathfrak{g}$ is solvable then, by virtue of Lie's Theorem, there exists $\lambda = \lambda_1 + \imath\lambda_2 : \mathfrak{g} \longrightarrow \mathbb{C}$ and $z = x + \imath y \neq 0$ such that, for any $u \in \mathfrak{g}$, $\rho^{\mathbb{C}}(u)(z) = \lambda(u)z$. This is equivalent to

$$\rho(u)(x) = \lambda_1(u)x - \lambda_2(u)y \quad \text{and} \quad \rho(u)(y) = \lambda_2(u)x + \lambda_1(u)y. \tag{7}$$

From

$$\langle \rho(u)x, x\rangle = \langle \rho(u)y, y\rangle = 0 \quad \text{and} \quad \langle \rho(u)x, y\rangle = -\langle \rho(u)y, x\rangle,$$

we get

$$\begin{pmatrix} \lambda_1(u) & -\lambda_2(u) & 0 \\ 0 & \lambda_2(u) & \lambda_1(u) \\ \lambda_2(u) & 2\lambda_1(u) & -\lambda_2(u) \end{pmatrix} \begin{pmatrix} \langle x, x\rangle \\ \langle y, x\rangle \\ \langle y, y\rangle \end{pmatrix} = 0. \tag{8}$$

We distinguish two cases.

(a) The vectors $x, y$ are linearly dependent say $y = ax$ with $x \neq 0$. From (7) and (8), we get $\lambda_2 = 0$ and, for any $u \in \mathfrak{g}$, $\lambda_1(u)\langle x, x\rangle = 0$. If $\langle x, x\rangle \neq 0$, then $\dim V = 1$, $V = V_0$, and we are in the first case.

Suppose now that $\langle x, x\rangle = 0$. If $V_{\lambda_1}$ is not totally isotropic, then it contains a nonisotropic vector $z$ and hence $V = \mathrm{span}\{z\}$, which is impossible since $x \in V$. So $V_{\lambda_1}$ must be totally isotropic and hence $V_{\lambda_1} = \mathrm{span}\{x\}$. We then have two situations. The first one is that there exists $\mu \neq \lambda_1$ such that $V_\mu = \mathrm{span}\{z\}$ is a totally isotropic one dimensional vector space. From the relation $\langle \rho(u)x, z\rangle = -\langle \rho(u)z, x\rangle$ and $\langle x, z\rangle \neq 0$, we deduce that $\mu = -\lambda_1$. Then $\lambda_1 \neq 0$ and hence $V = V_{\lambda_1} \oplus V_\mu$, and we are in the second case. The second situation is that, for any $\mu \neq \lambda_1$, $V_\mu = \{0\}$. In this case $\dim V \geq 3$, and we are in the third case. Indeed, if $\dim V = 2$, choose an isotropic vector $\bar{x}$ such that $\langle x, \bar{x}\rangle = 1$. It is easy to check that $\bar{x} \in V_{-\lambda_1}$, which is impossible.

(b) The vectors $x, y$ are linearly independent. Since $\mathrm{span}\{x, y\}$ cannot be totally isotropic, we can deduce from (8) that $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\langle x, y\rangle = 0$, and $\langle x, x\rangle = \langle y, y\rangle \neq 0$. So $\mathrm{span}\{x, y\}$ is Euclidean nondegenerate invariant, which is impossible. □

*Complement to Proposition 2.1.* Let us study the third case in Proposition 2.1 more deeply. Let $\rho : \mathfrak{g} \longrightarrow \mathrm{so}(V)$ be an indecomposable Lorentzian representation with $\dim V \geq 3$. Then there exists $\lambda \in \mathfrak{g}^*$ such that $d\lambda = 0$, $V_\lambda$ is a one dimensional totally isotropic subspace and, for any $\mu \neq \lambda$, $V_\mu = \{0\}$. The quotient $\widetilde{V} = V_\lambda^\perp / V_\lambda$ is an

Euclidean vector space and $\rho$ induces a representation $\tilde{\rho} : \mathfrak{g} \longrightarrow \mathrm{so}(\widetilde{V})$. So, it is a well-known fact that

$$\widetilde{V} = \bigoplus_{i=1}^{q} \widetilde{E}_i \oplus \widetilde{V}_0,$$

where, for any $i = 1, \ldots, q$, $\dim \widetilde{E}_i = 2$, and there exists $\lambda_i \in \mathfrak{g}^* \setminus \{0\}$ and an orthonormal basis $(\bar{e}_i, \bar{f}_i)$ of $\widetilde{E}_i$ such that $d\lambda_i = 0$ and, for any $u \in \mathfrak{g}$,

$$\tilde{\rho}(u)\bar{e}_i = \lambda_i(u)\bar{f}_i \quad \text{and} \quad \tilde{\rho}(u)\bar{f}_i = -\lambda_i(u)\bar{e}_i. \tag{9}$$

Denote by $\pi : V_\lambda^\perp \longrightarrow \widetilde{V}$ the natural projection, and choose a generator $e$ of $V_\lambda$. Put $E_0 = \pi^{-1}(\widetilde{V}_0)$ and, for any $i \in \{1, \ldots, q\}$, choose $(e_i, f_i)$ such that $\pi(e_i) = \bar{e}_i$ and $\pi(f_i) = \bar{f}_i$.

For any $x \in E_0$ there exists $a_x \in \mathfrak{g}^*$ such that, for any $u \in \mathfrak{g}$,

$$\rho(u)(x) = a_x(u)e. \tag{10}$$

For any $u, v \in \mathfrak{g}$, we have

$$\begin{aligned}\rho([u, v])(x) &= a_x([u, v])e \\ &= (\lambda(u)a_x(v) - \lambda(v)a_x(u))e.\end{aligned}$$

Thus, for any $x \in E_0$,

$$da_x = a_x \wedge \lambda. \tag{11}$$

On the other hand, for any $i \in \{1, \ldots, q\}$, by virtue of (9), there exists $b_i, c_i \in \mathfrak{g}^*$ such that, for any $u \in \mathfrak{g}$,

$$\rho(u)e_i = b_i(u)e + \lambda_i(u)f_i \quad \text{and} \quad \rho(u)f_i = c_i(u)e - \lambda_i(u)e_i. \tag{12}$$

We have, for any $u, v \in \mathfrak{g}$,

$$\begin{aligned}\rho([u, v])e_i &= b_i([u, v])e + \lambda_i([u, v])f_i \\ &= \rho(u)(b_i(v)e + \lambda_i(v)f_i) - \rho(v)(b_i(u)e + \lambda_i(u)f_i) \\ &= (\lambda(u)b_i(v) - \lambda(v)b_i(u) + \lambda_i(v)c_i(u) - \lambda_i(u)c_i(v))e,\end{aligned}$$

so

$$b_i([u, v]) = \lambda(u)b_i(v) - \lambda(v)b_i(u) + \lambda_i(v)c_i(u) - \lambda_i(u)c_i(v).$$

In the same way, by computing $\rho([u, v])f_i$, we get

$$c_i([u, v]) = \lambda(u)c_i(v) - \lambda(v)c_i(u) - \lambda_i(v)b_i(u) + \lambda_i(u)b_i(v).$$

Thus

$$db_i = b_i \wedge \lambda + \lambda_i \wedge c_i \quad \text{and} \quad dc_i = c_i \wedge \lambda + b_i \wedge \lambda_i. \tag{13}$$

## 3. PROOF OF THEOREM 1.1

Before proving Theorem 1.1, we establish an important property of the modular vector of a flat pseudo-Riemannian Lie algebra which will be crucial in the proof.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat pseudo-Riemannian Lie algebra. One can see easily that the orthogonal of the derived ideal of $\mathfrak{g}$ is given by

$$[\mathfrak{g}, \mathfrak{g}]^\perp = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\}. \tag{14}$$

The first assertion of the following proposition appeared in [1]. The second one has been pointed out to us by the referee.

**Proposition 3.1.** *Let* $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ *be a flat pseudo-Riemannian Lie algebra.*

1. *The modular vector satisfies* $\mathbf{h} \in [\mathfrak{g}, \mathfrak{g}] \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ *and* $\mathrm{R}_{\mathbf{h}}$ *is symmetric with respect to* $\langle\ ,\ \rangle$. *In particular, if* $\mathfrak{g}$ *is nonunimodular then* $[\mathfrak{g}, \mathfrak{g}]$ *is degenerate,* $\langle \mathbf{h}, \mathbf{h}\rangle = 0$ *and* $\mathbf{h}.\mathbf{h} = 0$.
2. *If* $\langle\ ,\ \rangle$ *is Lorentzian, then* $H = \mathrm{span}\{\mathbf{h}\}$ *is a two-sided ideal (with respect to the Levi–Civita product) in* $H^\perp$.

*Proof.* 1. For any $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$ and any $v \in \mathfrak{g}$, since $\mathrm{R}_u$ is symmetric, we have $\langle u.u, v\rangle = \langle u, v.u\rangle = 0$, and hence $u.u = 0$. So, by virtue of (1), we get $[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2$. One can deduce by induction that, for any $k \in \mathbb{N}^*$, $[\mathrm{R}_u^k, \mathrm{L}_u] = k\mathrm{R}_u^{k+1}$, and hence $\mathrm{tr}(\mathrm{R}_u^k) = 0$ for any $k \geq 2$, which implies that $\mathrm{R}_u$ is nilpotent. Since, for any $u, v \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_{[u,v]}) = 0$, we deduce that $\mathbf{h} \in [\mathfrak{g}, \mathfrak{g}]^\perp$. Now, for any $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$, $\mathrm{R}_u$ is nilpotent and hence

$$\mathrm{tr}(\mathrm{ad}_u) = -\mathrm{tr}(\mathrm{R}_u) = \langle \mathbf{h}, u\rangle = 0,$$

which implies $\mathbf{h} \in [\mathfrak{g}, \mathfrak{g}]$. This implies that $\langle \mathbf{h}, \mathbf{h}\rangle = 0$. From the relation $\langle \mathbf{h}.\mathbf{h}, u\rangle = \langle [u, \mathbf{h}], \mathbf{h}\rangle = 0$, we deduce also that $\mathbf{h}.\mathbf{h} = 0$.

2. Suppose now that $\langle\ ,\ \rangle$ is Lorentzian. According to [3] Corollary 3.6, $\mathfrak{g}$ must be solvable. From the relation $\mathbf{h}.\mathbf{h} = 0$ and the fact that $\mathrm{L}_{\mathbf{h}}$ is skew-symmetric and $\mathrm{R}_{\mathbf{h}}$ is symmetric, we deduce that $\mathrm{L}_{\mathbf{h}}(H^\perp) \subset H^\perp$ and $\mathrm{R}_{\mathbf{h}}(H^\perp) \subset H^\perp$. Hence we get two endomorphisms $\overline{\mathrm{L}}_{\mathbf{h}}$ and $\overline{\mathrm{R}}_{\mathbf{h}}$ of $H^\perp/H$. Since $\langle\ ,\ \rangle$ is Lorentzian and $H$ is totally isotropic, $H^\perp/H$ carries an Euclidean product for which $\overline{\mathrm{L}}_{\mathbf{h}}$ is skew-symmetric and $\overline{\mathrm{R}}_{\mathbf{h}}$ is symmetric. We have seen that, for any $u \in [\mathfrak{g}, \mathfrak{g}]^\perp$, $\mathrm{R}_u$ is nilpotent so $\mathrm{R}_{\mathbf{h}}$ is nilpotent, and hence $\overline{\mathrm{R}}_{\mathbf{h}}$ is a symmetric nilpotent endomorphism in an Euclidean vector space; thus $\overline{\mathrm{R}}_{\mathbf{h}} = 0$ and hence $\mathrm{R}_{\mathbf{h}}(H^\perp) \subset H$. On the other hand, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent and $\mathbf{h} \in [\mathfrak{g}, \mathfrak{g}]$ thus $\mathrm{ad}_{\mathbf{h}}$ is nilpotent in restriction to $[\mathfrak{g}, \mathfrak{g}]$ and, actually, on all $\mathfrak{g}$. Now $\overline{\mathrm{ad}}_{\mathbf{h}} = \overline{\mathrm{L}}_{\mathbf{h}} - \overline{\mathrm{R}}_{\mathbf{h}} = \overline{\mathrm{L}}_{\mathbf{h}}$, and hence $\overline{\mathrm{L}}_{\mathbf{h}}$ is nilpotent, and being skew-symmetric, it must vanish, so $\mathrm{L}_{\mathbf{h}}(H^\perp) \subset H$. This achieves the proof. $\square$

### 3.1. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 based on Proposition 2.1, its complement, and Proposition 3.1.

*Proof.* Remark first that, since the left multiplications are skew-symmetric and $\mathrm{R}_{\mathbf{h}}$ is symmetric, we can deduce that if $H$ is a two-sided ideal then $H^\perp$ is also a two-sided ideal. With this remark in mind, we begin by proving the first assertion of the theorem, i.e., $\mathrm{L}_{\mathbf{h}} = 0$ and $\mathfrak{g}.H \subset H$.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a flat nonunimodular Lorentzian Lie algebra. According to [3] Corollary 3.6, $\mathfrak{g}$ must be solvable. The left multiplication $\mathrm{L} : \mathfrak{g} \longrightarrow \mathrm{so}(\mathfrak{g})$ is a representation and hence $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, where $\mathfrak{h}$ is $L$-invariant Lorentzian nondegenerate indecomposable and $\mathfrak{k}$ is $L$-invariant Euclidean nondegenerate. It is obvious that $\mathfrak{k}$ is a flat Riemannian Lie algebra and hence it is unimodular. This implies that $\mathbf{h} \in \mathfrak{h}$, it coincides with the modular vector of $\mathfrak{h}$ and $\dim \mathfrak{h} \geq 2$. We have

$$\mathrm{L}_{\mathbf{h}}(\mathfrak{k}) = \mathrm{R}_{\mathbf{h}}(\mathfrak{k}) = 0. \tag{15}$$

Indeed, $\mathfrak{k} \subset H^\perp$, $\mathrm{L}_{\mathbf{h}}(\mathfrak{k}) \subset \mathfrak{k}$ and, according to Proposition 3.1, $\mathrm{L}_{\mathbf{h}}(H^\perp) \subset H \subset \mathfrak{h}$, so $\mathrm{L}_{\mathbf{h}}(\mathfrak{k}) = 0$. On the other hand, according to Proposition 3.1, $\mathrm{R}_{\mathbf{h}}$ is symmetric and hence, for any $u \in \mathfrak{k}$ and $v \in \mathfrak{g}$,

$$\langle u.\mathbf{h}, v\rangle = \langle v.\mathbf{h}, u\rangle = -\langle v.u, \mathbf{h}\rangle = 0,$$

so $\mathrm{R}_{\mathbf{h}}(\mathfrak{k}) = 0$.

According to (15), to reach our goal we need to prove that $\mathrm{L}_{\mathbf{h}}(\mathfrak{h}) = 0$ and $\mathfrak{h}.H \subset H$.

Since $\dim \mathfrak{h} \geq 2$ and according to Proposition 2.1, we have two situations.

(i) There exists $\lambda > 0$ with $d\lambda = 0$, a basis $(e, \bar{e})$ of $\mathfrak{h}$ such that $\langle e, e\rangle = \langle \bar{e}, \bar{e}\rangle = 0$, $\langle e, \bar{e}\rangle = 1$, and, for any $u \in \mathfrak{g}$,

$$u.e = \lambda(u)e \qquad \text{and} \qquad u.\bar{e} = -\lambda(u)\bar{e}.$$

We have $[e, \bar{e}] = -\lambda(e)\bar{e} - \lambda(\bar{e})e$. Since $\lambda([e, \bar{e}]) = 0$, we get $\lambda(e)\lambda(\bar{e}) = 0$. We can suppose without loss of generality that $\lambda(e) = 0$. So

$$e.e = e.\bar{e} = 0,\ \bar{e}.e = \lambda(\bar{e})e \qquad \text{and} \qquad \bar{e}.\bar{e} = -\lambda(\bar{e})\bar{e}.$$

So $\mathbf{h} = \lambda(\bar{e})e$ and hence $\lambda(\bar{e}) \neq 0$. Thus $\mathrm{L}_{\mathbf{h}}(\mathfrak{h}) = 0$ and $\mathfrak{h}.H \subset H$, and the first assertion of the theorem holds in this case.

(ii) In this case, $\dim \mathfrak{h} \geq 3$, and there exists $\lambda \in \mathfrak{g}^*$ such that $d\lambda = 0$ and $\mathfrak{h}_\lambda$ is a totally isotropic one dimensional vector space. Moreover, for any $\mu \neq \lambda$, $\mathfrak{h}_\mu = \{0\}$.

Choose a generator $e$ of $\mathfrak{h}_\lambda$. Since $\mathrm{tr}(\mathrm{R}_e) = \lambda(e)$, we deduce from (6) that $\langle \mathbf{h}, e\rangle = -\lambda(e)$. We will show first that $\lambda(e) = 0$.

Suppose by contradiction that $\lambda(e) \neq 0$. Consider the Lorentzian nondegenerate vector space $V = \mathrm{span}\{e, \mathbf{h}\}$. We claim that $V$ is $L$-invariant. Indeed, for any $u \in \mathfrak{g}$, $u.e = \lambda(u)e$, $\mathbf{h}.\mathbf{h} = 0$, and $V^\perp$ is contained in $H^\perp$ so, by virtue of Proposition 3.1, $V^\perp.\mathbf{h} \subset H$. Let us show now that $e.\mathbf{h} \in V$. Remark first that $\mathbf{h} \in [\mathfrak{g}, \mathfrak{g}]$ and hence $\lambda(\mathbf{h}) = 0$. Write $e.\mathbf{h} = ae + b\mathbf{h} + v_0$ with $v_0 \in V^\perp$. Since $\langle e.\mathbf{h}, \mathbf{h}\rangle = 0$, then $a = 0$. So $e.\mathbf{h} = b\mathbf{h} + v_0$. Since $d\lambda = 0$ and $\lambda(\mathbf{h}) = 0$, we have $\lambda(v_0) = \lambda(e.\mathbf{h}) = \lambda(\mathbf{h}.e) = 0$. Now, $\mathrm{R}_\mathbf{h}$ is symmetric and then

$$\langle v_0, v_0\rangle = \langle e.\mathbf{h}, v_0\rangle = \langle e, v_0.\mathbf{h}\rangle = -\langle v_0.e, \mathbf{h}\rangle = -\lambda(v_0)\lambda(e) = 0.$$

Since $V^\perp$ is Euclidean, then $v_0 = 0$, thus $V$ is $L$-invariant. This is impossible since $\mathfrak{h}$ is indecomposable and $\dim \mathfrak{h} \geq 3$. This shows that $\lambda(e) = 0$ and $\langle e, \mathbf{h}\rangle = 0$. Since both $e$ and $\mathbf{h}$ are isotropic, we deduce that $H = \mathfrak{h}_\lambda$ and then $\mathfrak{h}.H \subset H$. We need now to show that $\mathrm{L}_\mathbf{h}(\mathfrak{h}) = 0$.

Denote by $\mathfrak{h}_\lambda^\perp$ the orthogonal of $\mathfrak{h}_\lambda$ in $\mathfrak{h}$ so that $H^\perp = \mathfrak{h}_\lambda^\perp \oplus \mathfrak{k}$. Since $\mathrm{R}_\mathbf{h}$ is symmetric, for any $u, v \in \mathfrak{g}$,

$$\lambda(u)\langle \mathbf{h}, v\rangle = \lambda(v)\langle \mathbf{h}, u\rangle.$$

So if $u \in H^\perp$, then $\lambda(u) = 0$, and hence $H^\perp.H = 0$.

Let $\pi : \mathfrak{h}_\lambda^\perp \longrightarrow \tilde{\mathfrak{h}}$ be the canonical projection where $\tilde{\mathfrak{h}}$ is the quotient of $\mathfrak{h}_\lambda^\perp$ by $\mathfrak{h}_\lambda$. The representation L induces a representation $\widetilde{\mathrm{L}}$ of $\mathfrak{g}$ on $\tilde{\mathfrak{h}}$. As in the complement to Proposition 2.1,

$$\tilde{\mathfrak{h}} = \bigoplus_{i=1}^{q} \tilde{\mathfrak{h}}_i \oplus \tilde{\mathfrak{h}}_0,$$

and, for any $i \in \{1, \ldots, q\}$, there exists $\lambda_i \in \mathfrak{g}^*\backslash\{0\}$ satisfying $d\lambda_i = 0$, $b_i, c_i \in \mathfrak{g}^*$ satisfying (13) and $(e_i, f_i)$ an orthonormal couple in $\mathfrak{h}_\lambda^\perp$ such that

$$u.e_i = b_i(u)\mathbf{h} + \lambda_i(u)f_i \qquad \text{and} \qquad u.f_i = c_i(u)\mathbf{h} - \lambda_i(u)e_i.$$

Moreover, for any $x \in \pi^{-1}(\tilde{\mathfrak{h}}_0)$, there exists $a_x \in \mathfrak{g}^*$ satisfying (11) such that, for any $u \in \mathfrak{g}$,

$$u.x = a_x(u)\mathbf{h}.$$

According to Proposition 3.1, $H.H^\perp \subset H$ and hence $\lambda_i(\mathbf{h}) = 0$. Moreover, $H^\perp.H = 0$ so $[\mathbf{h}, e_i] = b_i(\mathbf{h})\mathbf{h}$, $[\mathbf{h}, f_i] = c_i(\mathbf{h})\mathbf{h}$, and

$$[e_i, f_i] = (c_i(e_i) - b_i(f_i))\mathbf{h} - \lambda_i(e_i)e_i - \lambda_i(f_i)f_i.$$

Since $d\lambda_i = 0$, by applying $\lambda_i$ to $[e_i, f_i]$, we get $\lambda_i(e_i) = \lambda_i(f_i) = 0$. By applying $db_i$ and $dc_i$ to $(\mathbf{h}, e_i)$ and $(\mathbf{h}, f_i)$ and by using (13), we get $b_i(\mathbf{h}) = c_i(\mathbf{h}) = 0$. On the other hand, for any $x \in \pi^{-1}(\tilde{\mathfrak{h}}_0)$, we have $[\mathbf{h}, x] = a_x(\mathbf{h})\mathbf{h}$. So by applying $da_x$ to $(\mathbf{h}, x)$ and by using (11), we get $a_x(\mathbf{h}) = 0$. These relations show that $H.\mathfrak{h}_\lambda^\perp = 0$. Now, for any $u \in \mathfrak{h}\backslash\mathfrak{h}_\lambda^\perp$, $\langle \mathbf{h}.u, u\rangle = 0$ and, for any $v \in \mathfrak{h}_\lambda^\perp$, $\langle \mathbf{h}.u, v\rangle = -\langle \mathbf{h}.v, u\rangle = 0$, and hence $\mathbf{h}.u = 0$. Finally, $\mathrm{L}_\mathbf{h}(\mathfrak{h}) = 0$, and the first assertion of the theorem holds also in this case.

To complete the proof, we will show the second assertion of the theorem. Since both $H$ and $H^{\perp}$ are two-sided ideals, according to Proposition 3.1 of [2], $\mathfrak{g}$ is a double extension of a flat Riemannian Lie algebra $B$ according to $(\xi, D, \mu, b_0)$. The Lie brackets are given by

$$[\bar{e}, e] = \mu e, \ [\bar{e}, a] = D(a) - \langle b_0, a\rangle_0 e \quad \text{and} \quad [a, b] = [a, b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 e.$$

It is easy to see that $\mathbf{h} = (\mu + \mathrm{tr}(D))e$. So $\mathfrak{g}$ is nonunimodular iff $\mathrm{tr}(D) \neq -\mu$. □

## 4. FLAT NONUNIMODULAR LORENTZIAN LIE ALGEBRAS UP TO DIMENSION 4

According to Theorem 1.1, one can determine entirely all flat nonunimodular Lorentzian Lie algebras if one can find all admissible $(\xi, D, \mu, b_0)$ on flat Riemannian Lie algebras with $\mathrm{tr}(D) \neq -\mu$. In this section, we give all families of flat nonunimodular Lorentzian Lie algebras up to dimension 4 (see Tables 1 and 2). The notations $A_{i,j}$ appearing in the first column of these tables are those used in the classification of four-dimensional nonunimodular Lie algebras given in [6]. So we get a classification up to an isomorphism of Lie algebras.

*The Two-Dimensional Case.* There is one flat nonunimodular Lorentzian Lie algebra $(A_2, \langle\ ,\ \rangle)$ such that $A_2 = span\{e, \bar{e}\}$, $[\bar{e}, e] = e$, and $\langle\ ,\ \rangle = \mu e \odot \bar{e}$ with $\mu \neq 0$.

*The Three-Dimensional Case.* Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a three-dimensional flat nonunimodular Lorentzian Lie algebra. It is easy to show that $(\xi, D, \mu, b_0)$ is admissible in a one-dimensional Riemannian Lie algebra $B = \mathbb{R}e_1$ if and only if $D = \xi = 0$ or $D = \xi = \mu Id_B$. Put $b_0 = \alpha e_1$, and by using (5), we get that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3$ or $\mathfrak{g}_3'$, where we have as follows:

(i) $\mathfrak{g}_3 = span\{e, \bar{e}, e_1\}$, where the only nonvanishing brackets are

$$[\bar{e}, e] = \mu e, \ [\bar{e}, e_1] = \alpha e, \ \text{with } \mu \neq 0 \text{ and } \alpha \in \mathbb{R}.$$

(ii) $\mathfrak{g}_3' = span\{e, \bar{e}, e_1\}$, where the only nonvanishing brackets are

$$[\bar{e}, e] = \mu e, \ [\bar{e}, e_1] = \mu e_1 + \alpha e, \ \text{with } \mu \neq 0 \text{ and } \alpha \in \mathbb{R}.$$

In both cases, the metric is given by $\langle\ ,\ \rangle = \bar{e}^* \odot e^* + e_1^* \odot e_1^*$. By replacing $\bar{e}$ by $\mu^{-1}\bar{e}$ and $e$ by $\alpha e$ if $\alpha \neq 0$, we get Table 1 which gives, up to an isomorphism which preserves both the Lie brackets and the metrics, all three-dimensional flat nonunimodular Lorentzian Lie algebras.

*The Four-Dimensional Case.* Any flat Riemannian Lie algebra $(B, \langle\ ,\ \rangle)$ of dimension 2 must be abelian. Then $(D, \xi, \mu, b_0)$ is admissible if and only if $A = D - \xi$ is skew-symmetric and

$$[A, \xi] = \xi^2 - \mu\xi. \tag{16}$$

It is easy to solve this equation so we skip the details of the computation. By putting $b_0 = \alpha e_1 + \beta e_2$, using (5), and replacing $\bar{e}$ by $\mu^{-1}\bar{e}$, we get that any four-dimensional flat nonunimodular Lorentzian Lie algebra is isomorphic to span$\{e, \bar{e}, e_1, e_2\}$ with the nonvanishing brackets one of the following ones:

(i) $[\bar{e}, e] = e,\ [\bar{e}, e_1] = e_1 + \lambda e_2 + \alpha e,\ [\bar{e}, e_2] = \beta e,\ [e_1, e_2] = \lambda e$ with $\alpha, \beta, \lambda \in \mathbb{R}$.
(ii) $[\bar{e}, e] = e,\ [\bar{e}, e_1] = \epsilon e_1 + \gamma e_2 + \alpha e,\ [\bar{e}, e_2] = -\gamma e_1 + \epsilon e_2 + \beta e$, with $\epsilon = 0$ or 1 and $\alpha, \beta, \gamma \in \mathbb{R}$.

In both cases, the metric is given by $\langle\ ,\ \rangle = \mu \bar{e}^* \odot e^* + e_1^* \odot e_1^* + e_2^* \odot e_2^*$ with $\mu \neq 0$.

By studying carefully these two cases, we get eight classes of Lie algebras as follows:

1. Three flat Lorentzian Lie algebras obtained from the case ($i$) by taking $(\lambda, \alpha) = (0, 0)$, $(\lambda = 0, \alpha \neq 0)$, or $\lambda \neq 0$.
2. Two flat Lorentzian Lie algebras obtained from the case ($ii$) by taking $(\epsilon, \gamma) = (0, 0)$ or $(\epsilon = 0, \gamma \neq 0)$.
3. Three flat Lorentzian Lie algebras obtained from the case ($ii$) by taking $(\epsilon = 1, \gamma \neq 0)$, $(\epsilon = 1, \gamma = \alpha = \beta = 0)$, or $(\epsilon = 1, \gamma = 0, \alpha \neq 0)$.

For any of these eight Lie algebras, we find the corresponding Lie algebra in the list classifying four-dimensional nonunimodular Lie algebras in [6].

Table 2 gives, up to an isomorphism which preserves both the Lie brackets and the metrics, all the four-dimensional flat nonunimodular Lorentzian Lie algebras. In all cases, $\mathfrak{g} = \text{span}\{\bar{e}, e, e_1, e_2\}$. The notations $A_{3,3} \oplus A_1$ etc. are those used in the classification of four-dimensional nonunimodular Lie algebras given in [6].

## ACKNOWLEDGMENT

We think the referee for many useful suggestions and, in particular, for pointing out the second assertion in Proposition 3.1 which has shortened drastically the proof of Theorem 1.1.

## REFERENCES

[1] Ait Ben Haddou, M., Boucetta, M., Lebzioui, H. (2012). Left-invariant Lorentzian flat metrics on Lie groups. *Journal of Lie Theory* 22:269–289.
[2] Aubert, A., Medina, A. (2003). Groupes de Lie Pseudo-riemanniens plats. *Tohoku Math. J.* 55:487–506.
[3] Duncan, D. C., Ihrig, E. C. (1989). Homogeneous spacetimes of zero curvature. *Proceedings of the American Mathematical Society* 107(3):785–795.
[4] Guediri M. (2004). Compact flat spacetimes. *Differential Geometry and Its Applications* 21:283–295.
[5] Knapp A. W. (2002). *Lie Groups Beyond an Introduction*. Progress in Mathematics, Vol. 140, p. 42.
[6] Kremlev, A. G., Nikonorov, Yu. G. (2010). The signature of the Ricci curvature of left-invariant Riemannian metrics on four-dimensional Lie groups. The nonunimodular case. *Siberian Advances in Mathematics* 20(1):1–57.
[7] Milnor, J. (1976). Curvatures of left invariant metrics on Lie groups. *Advances in Mathematics* 21:293–329.