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CENTRE DE RECHERCHE ET
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Unité de Recherche et de Formation
Doctorale en Mathématiques,
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UNIVERSITY OF YAOUNDE I

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POSTGRADUATE SCHOOL OF SCIENCE
TECHNOLOGY AND GEOSCIENCES

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Research & Training Unit for
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LABORATORY OF ALGEBRA, GEOMETRY AND APPLICATIONS

The Signature of Ricci Curvature of Left-invariant Riemannian Metrics on Nilpotent Lie Groups

THESIS

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DEDICATION

To NGAHA familly.

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Contents

Abstract

Recall that given a n-dimensional Riemannian manifold (M, h) with the curvature tensor R, the Ricci curvature and the one dimensional sectional curvature are given, respectively, by

$$\mathrm{ric}(X, Y) = \mathrm{tr}(Z \longrightarrow \mathrm{R}(X, Z)Y) \quad \text{and} \quad \mathrm{A} = \frac{1}{n-2}(\mathrm{ric} - \frac{s}{2(n-1)}h),$$

where s is the scalar curvature given by $s = \mathrm{tr}_h\mathrm{ric}$. In this work, we give a contribution to the study of the Ricci curvature and the one-dimensional sectional curvature of left-invariant Riemannian metrics on nilpotent Lie groups. It is a well-known fact that the signature of the Ricci curvature and the one-dimensional sectional curvature of a left invariant metric on a Lie group are deeply related to the structure of the Lie group. This leads naturally to the following problems:

Problem 1. *For a connected Lie group G, determine all the signatures of the Ricci operators for all left-invariant Riemannian metrics on G.*

Problem 2. *For a connected Lie group G, determine all the signatures of the one dimensional sectional curvature for all left-invariant Riemannian metrics on G.*

These problems are been studied mainly in the low dimensions. The first one has been solved, respectively, in the case of 3-dimensional Lie groups and 4-dimensional Lie groups. For Lie groups of dimension 5 there are only partial results. In this work, we study these problems when G is nilpotent. We show that, associated to any nilpotent Lie group G, there is a subset $\mathbf{Sign}(\mathfrak{g})$ of $\mathbb{N}^3$ depending only on the Lie algebra $\mathfrak{g}$ of G, easy to compute and such that, for any left invariant Riemannian metric on G, the signature of its Ricci curvature belongs to $\mathbf{Sign}(\mathfrak{g})$. In the case where $\dim G \leq 6$, $\mathbf{Sign}(\mathfrak{g})$ is actually the set of signatures of the Ricci curvature of all left invariant Riemannian metrics on G. We give also some general results which support the conjecture that the last result is true in any dimension. On the other hand, by using $\mathbf{Sign}(\mathfrak{g})$ as a geometrical-algebraic invariant, we gave a classification of all possible signatures of the one-dimensional sectional curvatures of all left-invariant Riemannian metrics on some nilpotent Lie groups.

Key Words: Nilpotent Lie groups, nilpotent Lie algebras, Levi-Civita connection, Ricci curvature, Euclidean Lie algebras, Ricci operators, Ricci signatures, Groebner basis, one-dimensional sectional curvature.

Resumé

Rappelons que pour une variété Riemannienne (M, h) de dimension n et de tenseur de courbure de Riemann R, la courbure de Ricci et la courbure sectionnelle à une dimension sont définies par

$$\mathrm{ric}(X, Y) = \mathrm{tr}(Z \longrightarrow \mathrm{R}(X, Z)Y) \quad \text{et} \quad \mathrm{A} = \frac{1}{n-2}(\mathrm{ric} - \frac{s}{2(n-1)}h),$$

avec s la courbure scalaire définie par $s = \mathrm{tr}_h \mathrm{ric}$. Dans cette thèse, nous apportons une contribution à l'étude de la courbure de Ricci et la courbure de Ricci et la courbure sectionnelle à une dimension des métriques Riemanniennes invariantes à gauche sur les groupes de Lie nilpotents. Il est établi que les signatures de la courbure de Ricci et la courbure de Ricci et la courbure sectionnelle à une dimension des métriques Riemanniennes invariantes à gauche sur un groupe de Lie dépendent de la structure du group. Il se dégage les problèmes suivants:

Problème 1. *Pour un groupe de Lie connexe G, déterminer toutes les signatures possibles de tous les opérateurs de Ricci de toutes les métriques Riemanniennes invariantes à gauche sur G.*

Problème 2. *Pour un groupe de Lie connexe G, déterminer toutes les signatures possibles de tous les opérateurs des courbures sectionnelles à une dimension de toutes les métriques Riemanniennes invariantes à gauche sur G..*

Ces problèmes sont plus étudiés pour les groupes de petite deimension. Le premier est résolu pour les dimensions 3 et 4, partiellement pour la dimension 5. Dans cette thèse, nous étudions ces problèmes lorsque le groupe de Lie G est nilpotent. Nous associans à chaque groupe de Lie nilpotent G, un sous-ensemble $\mathbf{Sign}(\mathfrak{g})$ de $\mathbb{N}^3$ dependant uniquement de la structure de l'algèbre de Lie $\mathfrak{g}$ de G, facile à déterminer tel que, pour toute métrique Riemannienne invariante à gauche sur G, la signature de sa courbure de Ricci curvature appartient à $\mathbf{Sign}(\mathfrak{g})$. Dans le cas où $\dim G \leq 6$, $\mathbf{Sign}(\mathfrak{g})$ constitue l'esemble de toutes les signatures de la courbure de Ricci de toutes les métriques Riemanniennes invariantes à gauche sur G. A travers quelques exemples, nous justifions que ce résultat peut-être vrai en toute dimension. Par ailleurs en utilisant $\mathbf{Sign}(\mathfrak{g})$ comme un invariant geometrico-algebrique , nous donnons une classification de toutes les signatures possibles de la courbure sectionnelle à une dimension de toutes les métriques Riemanniennes invariantes à gauche sur certains groupes de Lie nilpotent.

Mots clés: Groupes de Lie nilpotent , Algèbres de Lie nilpotentes , Connexion de Levi-Civita, Courbure de Ricci ,Algèbres de Lie Euclidiennes , Operateurs de Ricci, Signatures de Ricci, Bases de Groebner, courbure sectionnelle à une dimension.

INTRODUCTION

In what follows, we state the problems tackled in this work, the results obtained before our's and our contribution toward a solution of these problems. The main results of this work constitute the paper [23].

Let (M, h) be a n-dimensional Riemannian manifold. We denote by ∇ its Levi-Civita connection and by R the curvature tensor given by

$$\mathrm{R}(X,Y)Z = \nabla_{[X,Y]}Z - (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z)\,.$$

The Ricci curvature and the one-dimensional sectional curvature tensor are the bilinear symmetric tensors field ric and A given, respectively, by

$$\mathrm{ric}(X,Y) = \mathrm{tr}(Z \longrightarrow \mathrm{R}(X,Z)Y) \quad \text{and} \quad \mathrm{A} = \frac{1}{n-2}(\mathrm{ric} - \frac{s}{2(n-1)}h),$$

where s is the scalar curvature given by $s = \mathrm{tr}_h\mathrm{ric}$. The Ricci operator and the one-dimensional sectional curvature operator are given by

$$h(\mathrm{Ric}X,Y) = \mathrm{ric}(X,Y) \quad \text{and} \quad h(\mathcal{A}X,Y) = \mathrm{A}(X,Y).$$

Both ric and A have a signature which can be computed by the mean of the eigenvalues of Ric and $\mathcal{A}$. It is a well established fact that there are deep relations between the topology and the geometry of M and the signatures of ric and A. These relations were investigated by many authors when M is an homogeneous space, in particular, when (M, h) is a Riemannian Lie group[1] [6, 5]. This leads to the following problems:

Problem 1. *For a connected Lie group G, determine all the signatures of the Ricci curvatures for all left-invariant Riemannian metrics on G.*

Problem 2. *For a connected Lie group G, determine all the signatures of the one-dimensional sectional curvature tensors for all left-invariant Riemannian metrics on G.*

[1] A Lie Riemannian Lie group is a Lie group endowed with a left invariant Riemannian metric.

These problems has been studied mainly in the low dimensions and, before enumerating the different results obtained, let say a word on the difficulties one may face when tackling such problems. The different curvatures of a left invariant metric on a Lie group $\mathfrak{g}$ are completely determined at the level of the Lie algebra $\mathfrak{g}$ by the restriction $\langle\ ,\ \rangle$ of the metric to $\mathfrak{g}$ and the Lie bracket. The main difficulty here is to find an orthonormal basis with respect to $\langle\ ,\ \rangle$ in which the Lie bracket are expressed with the less amount of parameters possibles and, if it is possible, the matrix of the the Ricci curvature in this basis is diagonal. It was Milnor first in [68] who exhibited such bases called since Milnor's frame. The existence of Milnor's frame in high dimension has been proved by [42]. Having this in mind, let pursue.

In [68] and [55, 56], Problem 1 has been solved, respectively, in the case of 3-dimensional Lie groups and 4-dimensional Lie groups. For Lie groups of dimension 5 there are only partial results. In [58], A.G. Kremlev, solved Problem 1 in the case of five-dimensional nilpotent Lie groups. The method used in all these cases is based on Milnor's frame.

Problem 2 is recent problem. It was first studied in 2011 by Voronov, Gladunova, Rodionov, and Slavskiĭ [102, 101, 36]. They used Milnor's frames and symbolic computation packages to solve the problem for three-dimensional connected Lie groups. Then Oskorbin in [79], gave a necessary and sufficient conditions for three real numbers to be the eigenvalues values of the one-dimensional curvature operator of a three dimensional connected Riemannian Lie group. His approach used once more Milnor's frames to obtain a system of equation which can be solved under some mild conditions. One the other hand, Klepikov, Oskerbin, and Rodionov in [52] gave an formula which permits the computation of the spectrum of the one-dimensional curvature operators of some four-dimensional Riemannian Lie groups.

In this work, we study Problems 1 and 2 when G is nilpotent by introducing a new method not based on Milnor's frames. We show that, associated to any nilpotent Lie group G, there is a subset $\mathbf{Sign}(\mathfrak{g})$ of $\mathbb{N}^3$ depending only on the Lie algebra $\mathfrak{g}$ of G, easy to compute and such that, for any left invariant Riemannian metric on G, the signature of its Ricci operator belongs to $\mathbf{Sign}(\mathfrak{g})$. In the case where $\dim G \leq 6$, $\mathbf{Sign}(\mathfrak{g})$ is actually the set of signatures of the Ricci curvatures of all left invariant Riemannian metrics on G. We give also some general results which support the conjecture that the last result is true in any dimension. On the other hand, by using $\mathbf{Sign}(\mathfrak{g})$ as a geometrical-algebraic invariant, we gave a classification of all possible signatures of the one-dimensional curvatures of all left-invariant Riemannian metrics on some nilpotent Lie groups.

Now, we introduce $\mathbf{Sign}(\mathfrak{g})$ and state our main results. Throughout this work, we will use the following convention. The signature of a symmetric operator J on an Euclidean vector space V is the sequence (s^-, s^0, s^+) where $s^+ = \sum_{\lambda_i>0} \dim\ker(J - \lambda_i \mathrm{I}_V)$, $s^- = \sum_{\lambda_i<0} \dim\ker(J - \lambda_i \mathrm{I}_V)$ and $s^0 = \dim\ker J$, where $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of J.

Let $\mathfrak{g}$ be a nilpotent n-dimensional Lie algebra, $Z(\mathfrak{g})$ its center and $[\mathfrak{g},\mathfrak{g}]$ its derived ideal. Put $d = \dim[\mathfrak{g},\mathfrak{g}]$, $k = \dim Z(\mathfrak{g})$ and $\ell = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}])$. We associate to $\mathfrak{g}$ the subset of $\mathbb{N}^3$

$$\mathbf{Sign}(\mathfrak{g}) = \left\{ (n-d-p+m^-, p+m^0, \ell+m^+) : \begin{cases} \max(k-d,0) \leq p \leq k-\ell \\ \\ m^- + m^0 + m^+ = d-\ell \end{cases} \right\}.$$

For instance, if $\mathfrak{g}$ is 2-step nilpotent then $[\mathfrak{g},\mathfrak{g}] \subset Z(\mathfrak{g})$ and hence $\mathbf{Sign}(\mathfrak{g}) = \{(n-k, k-d, d)\}$. If $\mathfrak{g}$ is a filiform nilpotent Lie algebra then $Z(\mathfrak{g}) \subset [\mathfrak{g},\mathfrak{g}]$, $\dim Z(\mathfrak{g}) = 1$, $\dim[\mathfrak{g},\mathfrak{g}] = n-2$ and hence

$$\mathbf{Sign}(\mathfrak{g}) = \left\{ (2+m^-, m^0, 1+m^+),\ m^- + m^0 + m^+ = n-3 \right\}.$$

The signature of the Ricci operator of a left invariant Riemannian metric on Lie group of dimension n belongs to $\{(n^-, n^0, n^+) :\ n^- + n^0 + n^+ = n\}$ whose cardinal is $\frac{(n+1)(n+2)}{2}$. Our first main result reduces drastically the set of possibilities for a nilpotent Lie group.

Theorem 0.0.1. *Let (G,h) be a nilpotent Lie group endowed with a left invariant Riemannian metric and $\mathfrak{g}$ its Lie algebra. Then the signature of the Ricci operator of (G,h) belongs to* $\mathbf{Sign}(\mathfrak{g})$.

As an immediate consequence of this result, if G is 2-step nilpotent then any left invariant Riemannian metric on G has the signature of its Ricci operator equal to $(\dim\mathfrak{g} - \dim Z(\mathfrak{g}), \dim Z(\mathfrak{g}) - \dim[\mathfrak{g},\mathfrak{g}], \dim[\mathfrak{g},\mathfrak{g}])$. On the other hand, Theorem 3.2.1 has the following corollary which gives a new proof to a result proved first in [68].

Corollary 0.0.2. *Let (G,h) be a noncommutative nilpotent Lie group endowed with a left invariant Riemannian metric and $\mathfrak{g}$ its Lie algebra. Then the Ricci operator of (G,h) has at least two negative eigenvalues.*

Theorem 3.2.1 gives a candidate to be the set of the signatures of the Ricci operators of all left invariant Riemannian metrics on a nilpotent Lie group. Indeed, our second main result together with Theorem 3.2.1 solve Problem 1 completely for nilpotent Lie groups up to dimension 6.

Theorem 0.0.3. *Let G be a nilpotent Lie group of dimension ≤ 6 and $\mathfrak{g}$ its Lie algebra. Then, for any $(s^-, s^0, s^+) \in$ $\mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+).*

Our third main result involves the notion of nice basis. Recall that a basis $(X_1, \dots, X_n)$ of a nilpotent Lie algebra $\mathfrak{g}$ is called nice if:

1. For any i, j with $i \neq j$, $[X_i, X_j] = 0$ or there exists k such that $[X_i, X_j] = C_{ij}^k X_k$ with $C_{ij}^k \neq 0$,
2. If $[X_i, X_j] = C_{ij}^k X_k$ and $[X_s, X_r] = C_{sr}^k X_k$ with $C_{ij}^k \neq 0$ and $C_{sr}^k \neq 0$ then $\{i, j\} \cap \{s, r\} = \emptyset$.

This notion appeared first in [64]. One of the most important property of a nice basis $\mathbb{B}$ is that any Euclidean inner product on $\mathfrak{g}$ for which $\mathbb{B}$ is orthogonal has its Ricci curvature diagonal in $\mathbb{B}$. The proof of Theorem 3.5.1 is based mainly on the fact that all the nilpotent Lie algebras of dimension less or equal to 6 have a nice basis except one. It is also known (see [72]) that any filiform Lie algebra has a nice basis.

Theorem 0.0.4. *Let G be a nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a nice basis and $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ with $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) = 1$. Then, for any $(s^-, s^0, s^+) \in \mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+).*

This theorem together with Theorem 3.2.1 solve Problem 1 for a large class of nilpotent Lie groups. Indeed, in the list of indecomposable seven-dimensional nilpotent Lie algebras given in [38] there are more than 35 ones satisfying the hypothesis of Theorem 3.3.1. On the other hand, we will point out the difficulty one can face when trying to generalize Theorem 3.3.1 when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. We will also give a method using the inverse function theorem to overcome this difficulty. Although, wa have not succeeded yet to show that this method works in the general case, we will use it successfully in the proof of Theorem 3.5.1. We will refer to this method as inverse function theorem trick.

The results above, the tools we will use to establish them and the examples we will give support the following conjecture.

Conjecture 1. *Let G be a nilpotent Lie group and $\mathfrak{g}$ its Lie algebra. Then, for any $(s^-, s^0, s^+) \in \mathbf{Sign}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+).*

Finally, concerning Problem 2, we tackle it for nilpotent Lie groups in dimension ≤ 5. We solve it in dimension ≤ 4 for all nilpotent Lie groups except one. In dimension 5, we give some partial result. Moreover, we use the method of construction of Milnor's frames to simplify drastically Nikitenko'list of five-dimensional Euclidean nilpotent Lie algebras [71] and we solve Problem 2 for some of them.

This thesis is arranged as follows :

In **Chapter 1.** This chapter state the main properties of left invariant Riemannian metrics on Lie groups, in particular, nilpotent Lie groups.

In **Chapter 2.** We describe the existing different approaches to obtain Ricci signatures on Riemannian Lie groups. Namely, we recall Milnor's results for unimodular three dimensional Lie groups, Chebarikov's results for nonunimodular three dimensional Lie groups, Kremlev and Nikonorov's results for four dimensional Lie groups, Boucetta's results for two step nilpotent Lie groups and Kremlev's results for five dimensional nilpotent Lie groups.

In **Chapter 3.** We study Problem 2 when G is nilpotent. Namely, we will show Theorems 3.2.1, 3.5.1, 3.3.1 and Corollary 3.2.2.

In **Chapter 4.** In this chapter, we give a description of the method of construction of Milnor's frames and we apply it to W.De Graaf's list of real nilpotent Lie algebras of dimension ≤ 4 and we use these basis to solve Problem 2 for these Lie algebras except one. Moreover, we use the method of construction of Milnor's frames to simplify drastically Nikitenko'list of five-dimensional Euclidean nilpotent Lie algebras [71] and we solve Problem 2 for some of them.

In this thesis, **Notation** like $[8, 20, 30]$ means references $([8], [20], [30])$.

CHAPTER ONE

RICCI CURVATURE OF RIEMANNIAN LIE GROUPS

We introduce in this Chapter, notations, definitions, theorems and preliminary facts which are used throughout this thesis. For Riemannian Geometry, Kobayashi-Nomizu [59, 60], Do Carmo [26], Helgason [44], Chavel [14], Klingerberg [53], Kühnel [62], Lee [66] are good references. Bourbaki [12], Jacobson [48], Vinberg[99, 100], Goze- Khakimdjanov [37], and Humphreys [47] are good references for algebraic properties of Lie algebras. Chevalley [19], Duistermaat and Kolk [24], Varadarajan [98], Warner [103], Knapp [54] are good references for geometric properties of Lie groups.

1.1 Main concepts on Riemannian geometry

We recall some elementary notions of Riemannian Geometry that can be found in most classical references.

1.1.1 Levi-Civita connection

A Riemannian metric on a $n-$dimensional smooth manifold M is a map wchich associate to any point $p \in M$ a scalar product $h(p)$ such that, for any local coordinates system $(x^1, ..., x^n)$ on an open set U, the local functions $h_{ij} : U \longrightarrow \mathbb{R}$ given by

$$h_{ij} = h(p)(\partial_{x^i}, \partial_{x^j})$$

are smooth for any $i, j = 1, ..., n$. A smooth manifold with a Riemannian metric is called a Riemanian manifold. Examples of Riemannian metrics are given in [49, 32, 26].

Linear connection in one of most important geometric structures on a manifold(see [29]). The following theorem is a fundamental result in Riemannian geometry which asserts that on every Riemannian manifold there is a uniquely determined linear connection.

Theorem 1.1.1. *For any Riemannian manifold (M, h), there exists a unique linear connection ∇ torsion-free and compatible with the metric. This connection is called Levi-Civita connection.*

In the proof of this theorem, the following key formula is using:

$$2\nabla_X h(Y,Z) = X.h(Y,Z)+Y.h(Z,X)-Z.h(X,Y)-h(X,[Y,Z])+h(Y,[Z,X]) + h(Z,[X,Y]). \quad (1.1)$$

which determines the Levi-Civita connection. It is called Koszul formula, first introduced by J-L. Koszul and developed by K. Nomizu who used this formula to express torsion, curvature, etc.... In a local coordinates, from (1.1) the coefficients Γ_{ij}^k of this connection called the Christoffel symbols are (see [82, 32, 96]):

$$\Gamma_{ij}^k = \frac{1}{2}\sum_l g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

1.1.2 Curvature of Riemannian metrics

From J.M. Lee [65], the basic local isometry invariant in Riemannian manifold is the Riemann Curvature tensor, its qualitative geometric meaning is precisely the obstruction to being locally isometric to Euclidean space.

From [10, 65, 53] it is proved that:

Theorem 1.1.2. *Let (M, h) be a Riemannian manifold and ∇ its Levi-Civita connection. Then the map:*

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

given by

$$R(X,Y)Z := \nabla_{[X,Y]}Z - (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z),$$

is the unique tensor field of type $(1, 3)$ satisfying for any parametrized surface $\Gamma : (s,t) \longrightarrow \Gamma(s,t) \in M$ and any vector field $Y \in \mathcal{X}(M)$:

$$D_t D_s Y - D_s D_t Y = R\left(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\right).$$

R is called the Riemann curvature tensor of the metric h.

In a local coordinates, the curvature tensor coefficients R_{ikj}^s are:

$$R_{ikj}^s := \partial_k\Gamma_{ij}^s - \partial_j\Gamma_{ik}^s + \sum_r \left(\Gamma_{ij}^r\Gamma_{rk}^s - \Gamma_{ik}^r\Gamma_{rj}^s\right).$$

1.1.3 Ricci curvature

The second fundamental form of an hypersurface of Euclidean space played a key role in understanding its geometric properties (see [65]). In an attempt to generalise this concept, Ricci-Curbastro extracted a symmetric 2−form field from the Riemann curvature tensor, nowadays called the Ricci curvature(see [4]) which turn out as evidenced by numerous works of Mathematicians and Physicists in [6, 104] an extremely important invariant.

Let (M, h) be a Riemannian manifold and R its Riemann curvature tensor. The Ricci curvature is the trace of R, we have:

$$\mathrm{ric}(u, v) = tr(x \longmapsto R(u, x)v), \quad u, v \in T_pM.$$

ric is a symmetric tensor, then can be seing as an endomorphism,

$\mathrm{Ric}_p : T_pM \longrightarrow T_pM$ via the formula

$$\mathrm{ric}(u, v) = h(\mathrm{Ric}_p(u), v) = h(u, \mathrm{Ric}_p(v)) \quad \text{for all} \quad u, v \in T_pM.$$

From [10], if $(e_1, e_2, ..., e_n)$ is an orthonormal basis of $T_pM(p \in M)$, we have

$$\mathrm{Ric}_p(u) = \sum_{i=1}^{n} R(e_i, u)e_i.$$

1.1.4 One-dimensional sectional curvature

Definition 1.1.3. *Let (M, h) be a Riemannian manifold with $n := \dim M \geq 3$. The one-dimensional sectional curvature is the bilinear tensor fields denote* A *with*

$$\mathrm{A} := \frac{1}{n-2}\left(\mathrm{ric} - \frac{sh}{2(n-1)}\right),$$

where ric *is the Ricci tensor and s the scalar curvature.*

Then it can be seing as an endomorphism, $\mathcal{A}_p : T_pM \longrightarrow T_pM$ via the formula

$$\mathrm{A}_p(u, v) = h(\mathcal{A}_p(u), v) = h(u, \mathcal{A}_p(v)) \quad \text{for all} u, v \in T_pM.$$

The one-dimensional curvature operator denoted $\mathcal{A}$ is defined by

$$\mathcal{A}_p = \frac{1}{n-2}\left(\mathrm{Ric}_p - \frac{s(p)I_n}{2(n-1)}\right). \tag{1.2}$$

Where Ric_p is the Ricci endomorphism(Ricci operator), $s(p)$ the scalar curvature and I_n the identity endomorphism of T_pM.

1.2 Left-invariant Riemannian metrics on Lie Groups

Since a Lie group G is a smooth manifold, we can endow G with Riemannian metrics. Among all the Riemannian metrics on G, those for which the left translations are isometries are of particular interest because they take the group structure of G into account.

Definition 1.2.1. *A left-invariant Riemannian metric on a Lie group G is a Riemannian metric h on G such that for any $a \in G$, the left multiplication $\mathcal{L}_a$ is an isometry of h. This means that, for any $a, b \in G$ and any $u, v \in T_bG$,*

$$h(T_e\mathcal{L}_a(u), T_e\mathcal{L}_a(v)) = h(u, v).$$

We call Riemannian Lie group, a Lie group equiped with a left-invariant Riemannian metric. From [68], any Riemannian Lie group is complete. As a consequence of this definition, we have:

Lemma 1.2.2 ([10])**.** *Let h be a Riemannian metric on Lie group G. Then the following assertions are equivalent:*

(i) The metric h is left-invariant.

(ii) For any couple of left invariant fields (X, Y), $h(X, Y)$ is a constant function.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, we denote by $\mathfrak{M}^l(G)$ the set of left-invariant Riemannian metrics on G and $\mathfrak{M}$ the set of definite positive inner products on $\mathfrak{g}$. From lemma above, the map

$$\begin{array}{ccc} \mathfrak{M}^l(G) & \longrightarrow & \mathfrak{M} \\ h & \longmapsto & h(e) \end{array}$$

is a bijection. In [61, 42], it is showed that $\mathfrak{M}$ is a noncompact symmetric space, using the equivalence relation "isometric up to scaling" and a process to determine the associate quotient space is given and was applied for some cases.

Moreover, we have,

Lemma 1.2.3 ([10])**.** *Let (G, h) be a Riemannian Lie group. Then the Riemann curvature tensor R of (G, h) is left invariant. Thus for any left invariant vector field X, Y, Z and for any $a \in G$:*

$$R(X(a), Y(a))Z(a) = T_e\mathcal{L}_a(R(X(e), Y(e))Z(e)).$$

This result shows that the Riemann curvature tensor is entirely determined by its value on the neutral element, thus we can talk of curvature of the Euclidean Lie algebra.

1.3 Curvature of an Euclidean Lie algebra of a Riemannian Lie group

Let (G,h) be a Riemannian Lie group, $(\mathfrak{g},[.,.]_{\mathfrak{g}},\langle\ ,\ \rangle = h(e))$ its Euclidean Lie algebra ([10, 11]). For any endomorphism $F:\mathfrak{g}\longrightarrow\mathfrak{g}$, we denote by $F^*:\mathfrak{g}\longrightarrow\mathfrak{g}$ the adjoint of F with respect to $\langle\ ,\ \rangle$ given by

$$\langle\ F(u),v\ \rangle = \langle\ u,\ F^*(v)\rangle,\quad \text{for all}\quad u,v\in\mathfrak{g}$$

It is well known that the set $Kill(h)$ of Killing vector fields of (G,h) is a subalgebra of vector fields on G. Since that not all the left invariant vector fields are Killing vector fields. So we consider this is a subalgebra of $\mathfrak{g}$

$$\begin{aligned} K(\langle\ ,\ \rangle) &= \{u\in\mathfrak{g},\quad u^l\in Kill(h)\} \\ &= \{u\in\mathfrak{g},\quad ad_u + ad_u^* = 0\} \end{aligned}$$

Note that $Z(\mathfrak{g})\subset K(\langle\ ,\ \rangle)$.

Definition 1.3.1. *A Riemannian metric on a Lie group is bi-invariant if it is both left and right invariant.*

We have:

Lemma 1.3.2 ([10])**.** *Let (G,h) be a connected Riemannian Lie group. If h is bi-invariant then:*

$$K(\langle\ ,\ \rangle) = \mathfrak{g}\quad \textit{i.e. for any,}\quad u\in\mathfrak{g},\quad ad_u + ad_u^* = 0.$$

If G is connected the converse is also true.

1.3.1 Riemann curvature of the Euclidean Lie algebra of a Riemannian Lie group

The Levi-Civita product on $\mathfrak{g}$ is the product

$$\begin{array}{rccc} L: & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & (u,v) & \longmapsto & L_u v \end{array}$$

given by

$$L_u v := (\nabla^l_u v^l)(e).$$

Where ∇ is the Levi-Civita connection associated to (G,h). by using Koszul formula [10, 15], we get for any $u,v,w\in\mathfrak{g}$,

$$2\langle\ L_u v, w\ \rangle = \langle\ [u,v]_{\mathfrak{g}}, w\ \rangle + \langle\ [w,u]_{\mathfrak{g}}, v\ \rangle + \langle\ [w,v]_{\mathfrak{g}}, u\ \rangle.$$

For any $u \in \mathfrak{g}$, we denote by $R_u, J_u : \mathfrak{g} \longrightarrow \mathfrak{g}$ the endomorphisms of $\mathfrak{g}$ given by:

$$R_u v = L_v u \quad \text{and} \quad J_u v = ad_v^* u$$

It is easy to check that J_u is a skew-symmetric endomorphism and $J_u = 0$ if and only if $u \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$.

It follows:

Theorem 1.3.3 ([10]). *We have:*

(i) For any $u, v \in \mathfrak{g}$, $L_u v - L_v u = [u, v]_{\mathfrak{g}}$, i.e. $L_u - R_u = ad_u$

(ii) For any $u, v, w \in \mathfrak{g}$, $\langle\ L_u v, w\ \rangle + \langle\ v, L_u w\ \rangle = 0$. This means that, for any $u \in \mathfrak{g}$, L_u is skew- symmetric i.e. , $L_u^* = -L_u$.

(iii) For any $u \in \mathfrak{g}$, $L_u = \frac{1}{2}(ad_u - ad_u^*) - \frac{1}{2}J_u$

(iv) For any $u \in \mathfrak{g}$, $R_u = -\frac{1}{2}(ad_u + ad_u^*) - \frac{1}{2}J_u$

We denote by $K : \mathfrak{g} \times \mathfrak{g} \longrightarrow End(\mathfrak{g})$, the curvature of (G, h) at e. We have, for any $u, v \in \mathfrak{g}$,

$$K(u, v) = L_{[u,v]_{\mathfrak{g}}} - [L_u, L_v]_{\mathfrak{g}}.$$

For any u, v, w, z, K satisfies:

1. $K(u, v) = -K(v, u)$
2. $K(u, v)w + K(v, w) + K(w, u)v = 0$ (Bianchi's identity)
3. $\langle\ K(u, v)w, z\ \rangle = -\langle\ K(u, v)z, w\ \rangle$
4. $\langle\ K(u, v)w, z\ \rangle = -\langle\ K(w, z)u, v\ \rangle$

Then we have:

Proposition 1.3.4. *For any $u, v \in \mathfrak{g}$,*

$$\langle\ K(u, v)u, v\ \rangle = -\frac{3}{4}|ad_u v|^2 + \frac{1}{4}|ad_u^* v + ad_v^* u|^2 - \frac{1}{2}\langle\ ad_u v, ad_u^* v - ad_v^* u\ \rangle - \langle\ ad_u^* u, ad_v^* v\ \rangle \quad (1.3)$$

1.3.2 Ricci curvature of the Euclidean Lie algebra of a Riemannian Lie group

Lemma 1.3.5. *Let (G, h) be a Riemannian Lie group. Then the Ricci curvature* ric *of (G, h) is left invariant. Thus for any left invariant vector field X, Y and for $a \in G$,*

$$\mathrm{ric}(X(a), Y(a)) = \mathrm{ric}(X(e), Y(e)).$$

It follows that the Ricci curvature of a Riemannian Lie group is entirely determined by its value at the neutral element.

Definition 1.3.6. *The Ricci signature of a Riemannian Lie group is the signature of the Ricci curvature of the associated inner product on its Lie algebra.*

Let (G, h) be a Riemannian Lie group and $(\mathfrak{g}, \langle\ ,\ \rangle)$ its Euclidean Lie algebra. The Ricci curvature at e(neutral element of G) is defined by (see [10])

$$\mathrm{ric}(u, v) = tr(w \longmapsto K(u, w)v)) \quad , u, v, w \in \mathfrak{g}.$$

In order to give an useful formula of ric, we introduce the mean curvature vector H on $\mathfrak{g}$, which is the vector given by:

$$\langle\ H, u\ \rangle = tr(ad_u), \quad \text{for all} \quad u \in \mathfrak{g}.$$

We denote also $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the Killing form given by:

$$B(u, v) = tr(ad_u \circ ad_v) \quad \text{for all} \quad u, v \in \mathfrak{g}.$$

It follows:

Lemma 1.3.7. *We have:*

1. *If $\tau(u, v) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the endomorphism given by $\tau(u, v) = K(u, w)v$, then*
$$\tau(u, v) = -R_v \circ R_u + R_{u.v} - [L_u, R_v], \quad u.v = L_u v,$$

2. $$\mathrm{ric}(u, v) = -\frac{1}{2}B(u, v) - \frac{1}{2}tr(ad_u \circ ad_v^*) - \frac{1}{4}tr(J_u \circ J_v) - \frac{1}{2}(\langle\ ad_H u, v\ \rangle + \langle\ ad_H v, u\ \rangle), \text{ for any } u, v \in \mathfrak{g}, \quad (1.4)$$

1.3.3 Unimodular Lie algebras

Definition 1.3.8. *A Lie algebra is called unimodular if $trace(\mathrm{ad}_X) = 0$, for all $X \in \mathfrak{g}$.*

Proposition 1.3.9 (see [68])**.** *Let G be a Lie group and $\mathfrak{g}$ the corresponding Lie algebra. Then G is unimodular if and only if $\mathfrak{g}$ is unimodular.*

Examples of unimodular Lie algebras are given in [68, 59, 37, 11].

Corollary 1.3.10. *Let (G, h) be a Riemannian unimodular Lie group. Then its Ricci curvature at e is given by:*

$$\mathrm{ric}(u, v) = -\frac{1}{2}B(u, v) - \frac{1}{2}tr(ad_u \circ ad_v^*) - \frac{1}{4}tr(J_u \circ J_v), \text{ for any } u, v \in \mathfrak{g}. \quad (1.5)$$

1.3.4 Nilpotent Lie algebras

The lower(descending) central series of $\mathfrak{g}$,

$$\mathfrak{g} = \mathcal{C}^0(\mathfrak{g}) \supset \mathcal{C}^1(\mathfrak{g}) \supset ... \supset \mathcal{C}^k(\mathfrak{g}) \supset ...,$$

is defined by the following ideals

$$\begin{cases} \mathcal{C}^0(\mathfrak{g}) = \mathfrak{g} \\ \\ \mathcal{C}^k(\mathfrak{g}) = [\mathcal{C}^{k-1}(\mathfrak{g}), \mathfrak{g}], \quad \text{for all} \quad k \geq 1. \end{cases}$$

Definition 1.3.11. *A Lie algebra $\mathfrak{g}$ is called nilpotent if there is an interger k such that $\mathcal{C}^k(\mathfrak{g}) = 0$. The smallest interger k such that $\mathcal{C}^k(\mathfrak{g}) = 0$ is called the nilindex(or the nilpotency index) of $\mathfrak{g}$.*

The examples of real nilpotent Lie algebras are giving in [22, 69, 37, 81].

The following important properties of nilpotent Lie algebras are proved using the famous Engel's theorem.

Theorem 1.3.12 (see [93, 48, 12, 47])**.** *For a given nilpotent Lie algebra $\mathfrak{g}$, we have:*

(a) $Z(\mathfrak{g}) \neq \{0\}$,

(b) If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{h} \cap Z(\mathfrak{g}) \neq \{0\}$,

(c) $\operatorname{codim}[\mathfrak{g}, \mathfrak{g}] \geq 2$,

(d) For all $X \in \mathfrak{g}$, ad_X is nilpotent.

From this follows

Proposition 1.3.13 (see [78])**.** *A connected Lie group G is nilpotent if and only if its Lie algebra is nilpotent.*

Corollary 1.3.14. *Let (G, h) be a nilpotent Riemannian Lie group. The formula* (1.5) *becomes in this case quite simple. For any $u, v \in \mathfrak{g}$:*

$$\operatorname{ric}(u, v) = -\frac{1}{2}\operatorname{tr}(\operatorname{ad}_u \circ \operatorname{ad}_v^*) - \frac{1}{4}\operatorname{tr}(J_u \circ J_v) = -\frac{1}{2}\langle \operatorname{ad}_u, \operatorname{ad}_v\rangle_1 + \frac{1}{4}\langle J_u, J_v\rangle_1.$$

Where $\langle\ ,\ \rangle_1$ is the Euclidean product on $\operatorname{End}(\mathfrak{g})$ associated to $\langle\ ,\ \rangle$. In particular, if $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathfrak{g}$ then:

$$\operatorname{ric}(u, v) = -\frac{1}{2}\sum_{i,j}\langle [u, e_i], e_j\rangle\langle [v, e_i], e_j\rangle + \frac{1}{2}\sum_{i<j}\langle [e_i, e_j], u\rangle\langle [e_i, e_j], v\rangle. \quad (1.6)$$

1.3.5 Curvatures of bi-invariant Riemannian metrics on Lie groups

When the Riemannian metric is bi-invariant, much nicer formulae are obtained in [10].

Theorem 1.3.15 (see [10]). *Let (G, h) be a Lie goup with a bi-invariant Riemannian metric. Then,*

(a) The Levi-Civita product on $\mathfrak{g}$ is given by

$$L_u = \frac{1}{2} ad_u, \quad \textit{for all} \quad u \in \mathfrak{g},$$

(b) The curvature of (G, h) at e is given by

$$K(u, v) = \frac{1}{2} ad_{[u,v]}, \quad \textit{for all} \quad u, v \in \mathfrak{g},$$

(c) The Ricci curvature of (G, h) at e is given by

$$\mathrm{ric}(u, v) = -\frac{1}{4} B(u, v), \quad \textit{for all} \quad u, v \in \mathfrak{g}.$$

1.4 Survey on Ricci curvature on Lie groups

In the sequel we give some facts about the signs of Ricci curvature of left-invariant Riemannian metrics on a Lie group, to show how the choice of left-invariant Riemannian metrics on a Lie group it is not arbitrary at all.

J. Milnor in [68], gave many topogical and algebraic obstructions for a Lie group to admit a left-invariant Riemannian metric with a prescribe Ricci curvature. For example, there is a topolocal obstruction for a Lie group to admit a left-invariant metric with positive Ricci curvature. In fact he proved that such Lie group must be compact with the finite fundamental group. Theses results were generalized to homogeneous Riemannian manifolds by Berestovskii in [80] and Bergery in [5]. In [2], D.V. Alekseevsky and B.N. Kimelfeld showed that the flatness of Ricci curvature of a left-invariant Riemannian metric on a Lie group is equivalent to the flatness of this metric. Boucetta, in [10] gave an algebraic obstruction for a Lie group to admit left-invariant Riemannian metric with nonnegative Ricci curvature. In fact he showed that such Lie group must be unimodular. Dotti in [27], gave an algebraic obstruction for an unimodular solvable Lie group to admit a left-invariant Riemannian metric with nonpositive Ricci curvature.

In [73], Y. Nikolayevsky and Yu.G. Nikonorov gave some topological and algebraic obstructions(some mild conditions) on a solvable Lie group to admit left-invariant Riemannian metric with negative Ricci curvature. We also have this general result, formulated in [68] and Boucetta in [10] give a another version of its proof:

Theorem 1.4.1 (Milnor,[68]). *Suppose that the Lie algebra of a Lie group G is nilpotent but not abelian. Then for any left-invariant Riemannian metric on G, there exists a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature.*

In [27], Dotti showed that for any unimodular solvable Lie group such that its derived Lie algebra is not abelian, the Ricci curvaure of any left invariant Riemannian metric has mixed sign. She obtained the set of all possible Ricci signatures on the set of Riemannian $2-$step solvable unimodular Lie group.

In [68, 55, 56, 20], it is established that for any left-invariant Riemannian metric of a connected nonunimodular Lie group of dimension ≤ 4, the Ricci operator has at least two negative eigenvalues. Then this conjecture was formulated in [56]:

Conjecture 2. *For any left-invariant Riemannian metric on a nonunimodular solvable Lie group, its Ricci operator has at least two negative eigenvalues.*

This conjecture was confirmed for nonunimodular matabelian Lie groups in [56], for all nonunimodular solvable Lie groups of dimension ≤ 6 in [17], for all nonunimodular completely solvable Lie groups and for all non abelian nilpotent Lie groups in [16], for all nonunimodular Lie groups such that the derived Lie algebra is $2-$step nilpotent of dimension ≤ 6 in [1]. The full generality proof was obtained in [74] for all nonunimodular solvable Lie groups.

1.5 Lists of nilpotent Lie algebras

We give here the lists of all the noncommutative nilpotent Lie algebras, obtained in [22], [38] and [69] used in this thesis.

Nilpotent Lie algebra of dimension 3

$L_{3,2} = \mathrm{span}\{e_1, e_2, e_3\}$ with

$$[e_1, e_2] = e_3.$$

Nilpotent Lie algebras of dimension 4

$L_{4,2} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3.$$

$L_{4,3} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4.$$

Nilpotent Lie algebras of dimension 5

Lie algebra $\mathfrak{g}$	Nonzero commutators
$L_{5,2} = L_{3,2} \oplus \mathbb{R}^2$	$[e_1, e_2] = e_3$
$L_{5,3} = L_{4,3} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$L_{5,4}$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$
$L_{5,5}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$
$L_{5,6}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$
$L_{5,7}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$
$L_{5,8}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$
$L_{5,9}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$

Table.1: List of five-dimensional nilpotent Lie algebras.

Nilpotent Lie algebras of dimension 6

Lie algebra $\mathfrak{g}$	Nonzero commutators
$L_{6,2} = L_{5,2} \oplus \mathbb{R}$	$[e_1, e_2] = e_3$
$L_{6,3} = L_{5,3} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$L_{6,4} = L_{5,4} \oplus \mathbb{R}$	$[e_1, e_2] = e_5, [e_3, e_4] = e_5$
$L_{6,5} = L_{5,5} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$
$L_{6,6} = L_{5,6} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$
$L_{6,7} = L_{5,7} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$
$L_{6,8} = L_{5,8} \oplus \mathbb{R}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5$
$L_{6,9} = L_{5,9} \oplus \mathbb{R}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5$
$L_{6,10}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_6, [e_4, e_5] = e_6$
$L_{6,11}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_3] = e_6, [e_2, e_5] = e_6$
$L_{6,12}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6, [e_2, e_5] = e_6$
$L_{6,13}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5, [e_1, e_5] = e_6, [e_3, e_4] = e_6$
$L_{6,14}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5,$ $[e_2, e_5] = e_6, [e_3, e_4] = -e_6$
$L_{6,15}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5, [e_2, e_4] = e_6$ $[e_1, e_5] = e_6$
$L_{6,16}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_5] = e_6, [e_3, e_4] = -e_6$
$L_{6,17}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_2, e_3] = e_6$
$L_{6,18}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6$

Table.2 : $\epsilon \in \{-1, 0, 1\}$**:** List of six-dimensional nilpotent Lie algebras.

Lie algebra $\mathfrak{g}$	Nonzero commutators
$L_{6,19}(\epsilon)$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_4] = e_6, [e_3, e_5] = \epsilon e_6$
$L_{6,20}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_5] = e_6, [e_2, e_4] = e_6$
$L_{6,21}(\epsilon)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5, [e_1, e_4] = e_6,\ [e_2, e_5] = \epsilon e_6$
$L_{6,22}(\epsilon)$	$[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = \epsilon e_6, [e_3, e_4] = e_5$
$L_{6,23}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_2, e_4] = e_5$
$L_{6,24}(\epsilon)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = \epsilon e_6, [e_2, e_3] = e_6,\ [e_2, e_4] = e_5$
$L_{6,25}$	$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_1, e_4] = e_6$
$L_{6,26}$	$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_2, e_3] = e_6$

Table.2 : $\epsilon \in \{-1, 0, 1\}$**:** List of six-dimensional nilpotent Lie algebras(continued).

Nilpotent Lie algebra $(12457L1)$

The 7-dimensional nilpotent Lie algebra labelled $(12457L1)$ in [38] is given by:

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = -e_6,\ [e_1, e_6] = e_7,\ [e_2, e_3] = e_5,\ [e_2, e_5] = -e_6,$$

$$[e_3, e_5] = -e_7.$$

Nilpotent Lie algebra $\mathfrak{m}_0(n)$

The $\mathbb{N}$-graded filiform n-dimensional Lie algebra $\mathfrak{m}_0(n) = \text{span}\{X_1, \ldots, X_n\}$ with the non vanishing Lie brackets

$$[X_1, X_i] = X_{i+1}, \quad i = 2, \ldots, n-1.$$

Chapter Two

Ricci signatures: Some approaches

In this chapter, we enumerate in a detailed way the results obtained by many authors to determine all possible Ricci signatures on Lie groups of dimension 3 and 4 [68, 18, 55, 56], on nilpotent Lie groups of dimension 5 [58] and on 2-step nilpotent Lie groups [9]. Except the last case, the method used goes as follows: using a classification list of Lie algebras, for every metric on a picked Lie algebra, one built an orthonormal basis in which the constants structure are described by few parameters (three in dimension 3, six in dimension 4 and seven in dimension 5). Then one computes the matrix of the Ricci curvature in this basis and determine its possible signatures using a case-by-case analysis. In the case of 2-step nilpotent Lie algebras the method is different. It was at the origin of the approach we develop in the next chapter.

2.1 Ricci signatures of 3-dimensional Riemannian Lie groups

Here we give an account of the methods of *Milnor* in [68] for unimodular Lie groups and *Chebarykov* in [18] for non-unimodular Lie groups.

2.1.1 Milnor's approach

To study the Ricci curvature of left invariant Riemannian metric on 3-dimensional unimodular Lie groups, Milnor used the Euclidean cross product as the main tool to build some orthonormal basis known after as Milnor's frame.

Let G be a connected 3-dimensional unimodular Lie group with a left-invariant Riemannian metric, choose an orientation of the Lie algebra $\mathfrak{g}$ such that the Euclidean cross product($u \times v$ with $u, v \in \mathfrak{g}$) is definite. The following lemma is proved in [68].

Lemma 2.1.1 (see [68]). *There exists a unique self-adjoint endomorphism*

$L : \mathfrak{g} \longrightarrow \mathfrak{g}$ *such that,*

$$[u, v]_{\mathfrak{g}} = L(u \times v), \quad \textit{for all} \quad u, v \in \mathfrak{g}. \tag{2.1}$$

By choosing an orthonormal basis of eigenvectors of L, we get the following result which asserts the existence of what will be called later Milnor's frame.

Theorem 2.1.2 (Milnor, [68])**.** *Let $\mathfrak{g}$ be an unimodular Lie algebra. For any $\langle\ ,\ \rangle$ inner product on $\mathfrak{g}$, there exists an oriented $\langle\ ,\ \rangle$-orthonormal basis (e_1, e_2, e_3) such that*

$$L(e_i) = \lambda_i e_i, \quad \textit{with} \quad \lambda_1 \leq \lambda_2 \leq \lambda_3. \textit{ Thus,}$$

$$[e_1, e_2]_{\mathfrak{g}} = \lambda_3 e_3, \quad [e_2, e_3]_{\mathfrak{g}} = \lambda_1 e_1 \quad \textit{and} \quad [e_3, e_1]_{\mathfrak{g}} = \lambda_2 e_2.$$

The three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are well defined up to order. However, the construction is based on a choice of orientation, thus if we reverse the orientation $\lambda_1, \lambda_2, \lambda_3$ changed the signs. Let us give now a precise description of the possible Lie groups according to the signs of $\lambda_1, \lambda_2, \lambda_3$. By changing signs if necessary, we can assume that at most one of the constants structure $\lambda_1, \lambda_2, \lambda_3$ is nonpositive. There are six distinct cases tabulated as follows:

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group	Lie algebra
+,+,+	$SU(2)$ or $SO(3)$	$\mathfrak{su}(2)-$compact, simple
+,+,-	$SL(2,\mathbb{R})$ or $O(1,2)$	$\mathfrak{sl}(2,\mathbb{R})-$noncompact, simple
+,+,0	$E(2)$	$\mathfrak{e}(2)-$solvable
+,-,0	$E(1,1)$	$\mathfrak{e}$(1,1)$-$solvable
+,0,0	Heisenberg group	$\mathfrak{h}-$nilpotent
0,0,0	$\mathbb{R}^3$	$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}-$commutative

Table.3

With Milnor's frame at hand, we have:

Theorem 2.1.3 (see [68])**.** *Let $\mathfrak{g}$ be an unimodular 3-dimensional Lie algebra. Then the possible signatures of the Ricci operators of all inner products on $\mathfrak{g}$ are given in the following table.*

Lie algebra $\mathfrak{g}$	*Realizable Ricci signatures*
$\mathfrak{su}(2)$	$(0,0,3),\ (0,2,1),\ (2,0,1)$
$\mathfrak{sl}(2,\mathbb{R})$	$(2,0,1),\ (1,2,0)$
$\mathfrak{e}(2)$	$(0,3,0),\ (2,0,1)$
$\mathfrak{e}(1,1)$	$(2,0,1),\ (1,2,0)$
$\mathfrak{h}-$*nilpotent*	$(2,0,1)$
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}-$*commutative*	$(0,3,0)$

Table.4

Proof. In Milnor' frame (e_1, e_2, e_3) given in Theorem 2.1.2, the Ricci operator is diagonal, namely,

$$(\mathrm{ric}(e_i, e_j)) = 2\begin{pmatrix} \mu_2\mu_3 & 0 & 0 \\ 0 & \mu_3\mu_1 & 0 \\ 0 & 0 & \mu_1\mu_2 \end{pmatrix}$$

with $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$. Then, the result follows from a case-by-case analysis. For example, in the case $\mathfrak{g} = \mathfrak{su}(2)$, depending on the values of μ_3, the only realizable Ricci signatures are $(0, 0, 3)$, $(0, 2, 1)$ and $(2, 0, 1)$. $\square$

2.1.2 Chebarykov's approach

To study the Ricci curvature of left invariant Riemannian metric on 3-dimensional nonunimodular Lie groups, Chebarykov used the Gram-Schmidt orthogonalization process as the main tool to recover G.M. Mubarakzyanov's classification of real nonunimodular 3-dimensional metric Lie algebras.

The following table gives the classification of real nonunimodular 3-dimensional Lie algebras by G.M. Mubarakzyanov (see [81]).

Lie algebras	Nonzero cummutation relations
$\mathbb{A}_2 \oplus \mathbb{A}_1$	$[e_1, e_2] = e_2$
$\mathbb{A}_{3,2}$	$[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$
$\mathbb{A}_{3,3}$	$[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$
$\mathbb{A}^p_{3,5}$, $0 <\mid p \mid< 1$	$[e_1, e_3] = e_1$, $[e_2, e_3] = pe_2$
$\mathbb{A}^p_{3,7}$, $0 < p$	$[e_1, e_3] = pe_1 - e_2$, $[e_2, e_3] = e_1 + pe_2$

Table.5

Using the Gram-Schmidt orthogonalization process, we have the following result:

Theorem 2.1.4 (Chebarykov,[18]). *Let $\mathfrak{g}$ be a nonunimodular 3-dimensional real Lie algebra. For any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exists an $\langle\ ,\ \rangle$-orthonormal basis $\{f_1, f_2, f_3\}$ such that the non vanishing constants structure $(c^k_{i,j})$ of $\mathfrak{g}$ in $\{f_1, f_2, f_3\}$ are given by*

Lie algebras $\mathfrak{g}$	*Nonzero constants structure*	*Conditions*
$\mathbb{A}_2 \oplus \mathbb{A}_1$	$c^2_{1,2} = \alpha$, $c^3_{1,2} = \gamma$	$\alpha > 0$
$\mathbb{A}_{3,2}$	$c^1_{1,3} = c^2_{2,3} = \gamma$, $c^1_{2,3} = \delta$	$\gamma > 0$, $\delta > 0$
$\mathbb{A}_{3,3}$	$c^1_{1,3} = c^2_{2,3} = \gamma$	$\gamma > 0$
$\mathbb{A}^p_{3,5}$, $0 <\mid p \mid< 1$	$c^1_{1,3} = \gamma$, $c^2_{2,3} = \gamma p$, $c^1_{2,3} = (1-p)\delta$	$\gamma > 0$
$\mathbb{A}^p_{3,7}$, $0 < p$	$c^1_{1,3} = \gamma(p+\beta)$, $c^2_{1,3} = -\gamma\delta$, $c^2_{2,3} = \gamma(p-\beta)$, $c^1_{2,3} = \frac{\beta^2\gamma}{\gamma}$	$\gamma, \delta > 0$

Table.6

By the mean of this basis, we get the following result.

Theorem 2.1.5 (see [18]). *Let $\mathfrak{g}$ be a nonunimodular 3-dimensional Lie algebra. The possible signatures of the Ricci operators of all inner products on $\mathfrak{g}$ are given in the following table.*

Lie algebras $\mathfrak{g}$	*Realizable Ricci signatures*
$\mathbb{A}_2 \oplus \mathbb{A}_1$	$(2,1,0),\ (2,0,1)$
$\mathbb{A}_{3,2}$	$(3,0,0),\ (2,1,0),\ (2,0,1)$
$\mathbb{A}_{3,3}$	$(3,0,0)$
$\mathbb{A}^p_{3,5},\ 0<p<1$	$(3,0,0),\ (2,1,0),\ (2,0,1)$
$\mathbb{A}^p_{3,5},\ -1<p<0$	$(2,0,1)$
$\mathbb{A}^p_{3,7},\ 0<p$	$(3,0,0),\ (2,1,0),\ (2,0,1)$

Table.7

Proof. The method here is a case-by-case analysis. For example, if $\mathfrak{g} = \mathbb{A}^p_{3,5}$ and $\langle\ ,\ \rangle$ be any inner product on $\mathfrak{g}$, then from Theorem 2.1.4 and a direct computation one gets that the matrix of Ric is:

$$\begin{pmatrix} \frac{1}{2}(p-1)^2\delta^2-\gamma^2(1+p) & \delta\gamma(p-1) & 0 \\ \delta\gamma(p-1) & -\frac{1}{2}(p-1)^2\delta^2-\gamma^2(1+p)p & 0 \\ 0 & 0 & -\frac{1}{2}(1-p)^2\delta^2-\gamma^2(1+p^2) \end{pmatrix}.$$

The coefficient $-\frac{1}{2}(1-p)^2\delta^2-\gamma^2(1+p^2)$ is always negative, thus Ric has at least one negative eigenvalue. Let denoted by Ric_3 , the $2\times 2-$matrix obtained by deleting the third row and the third column. We have:

$$\begin{aligned} \mathrm{tr}(\mathrm{Ric}_3) &= -\gamma^2(1+p)^2, \\ \det(\mathrm{Ric}_3) &= \frac{1}{4}\left(4(1+p)^2p\gamma^4 - 2(1-p)^2(1+p)^2\gamma^2\delta^2 - (1-p)^4\gamma^4\right). \end{aligned}$$

For $p>0$, the only realizable Ricci signatures are $(3,0,0)$, $(2,1,0)$ and $(2,0,1)$. For $p<0$, the only realizable Ricci signature is $(2,0,1)$. $\square$

2.2 Ricci signatures of 4-dimensional Riemannian Lie groups

We give here the description of Kremlev-Nikonorov's approach, which use Milnor's frames and Gram-Schmidt as tools to give the G.M. Mubarakzyanov's classification of real 4-dimensional metric Lie algebras in [55] , [56] and [81].

2.2.1 Four dimensional unimodular Lie algebras

The following table gives the classification of real unimodular 4-dimensional Lie algebras by G.M. Mubarakzyanov (see [55] and [81]).

Lie algebras	Nonzero Lie brackets
$4\mathbb{A}_1$	
$\mathbb{A}_{3,1} \oplus \mathbb{A}_1$	$[e_2, e_3] = e_1$
$\mathbb{A}_{3,4} \oplus \mathbb{A}_1$	$[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$
$\mathbb{A}_{3,6} \oplus \mathbb{A}_1$	$[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$
$\mathbb{A}_{3,8} \oplus \mathbb{A}_1$	$[e_1, e_2] = -e_3$, $[e_3, e_1] = e_2$, $[e_2, e_3] = e_1$
$\mathbb{A}_{3,9} \oplus \mathbb{A}_1$	$[e_1, e_2] = e_3$, $[e_3, e_1] = e_2$, $[e_2, e_3] = e_1$
$\mathbb{A}_{4,1}$	$[e_2, e_4] = e_1$, $[e_3, e_4] = e_2$
$\mathbb{A}_{4,2}^{-2}$,	$[e_1, e_4] = -2e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = e_2 + e_3$
$\mathbb{A}_{4,5}^{\alpha,-1-\alpha}$, $\alpha \in (-1, -1/2]$	$[e_1, e_4] = e_1$, $[e_2, e_4] = \alpha e_2$, $[e_3, e_4] = -(1+\alpha)e_3$
$\mathbb{A}_{4,6}^{-2\beta,\beta}$, $\beta \in (0, \infty)$	$[e_1, e_4] = -2\beta e_1$, $[e_2, e_4] = \beta e_2 - e_3$, $[e_3, e_4] = e_2 + \beta e_3$
$\mathbb{A}_{4,8}$	$[e_2, e_3] = e_1$, $[e_2, e_4] = e_2$, $[e_3, e_4] = -e_2$
$\mathbb{A}_{4,10}$	$[e_2, e_3] = e_1$, $[e_2, e_4] = -e_3$, $[e_3, e_4] = e_2$

Table .8

2.2.1.1 Four dimensional decomposable unimodular Lie algebras

Using the Milnor's frames, we have:

Lemma 2.2.1 (see [55]). *For an arbitrary inner product $\langle\ ,\ \rangle$ on a 4-dimensional decomposable real unimodular Lie algebra $\mathfrak{g}$, there exists an $\langle\ ,\ \rangle$-orthonormal basis in which the nonzero constants structure of $\mathfrak{g}$ have the form:*

$$c_{1,2}^3 = a, \quad c_{1,2}^4 = -am, \quad c_{2,3}^1 = b, \quad c_{2,3}^4 = -bk, \quad c_{1,3}^2 = -c, \quad c_{1,3}^4 = cl,$$

where $k, l, m \in \mathbb{R}$ are arbitrary, $a, b, c \in \mathbb{R}$, and $a \leq b \leq c$.

In dependence on the signs of the numbers a, b and c, one can construct different Lie algebras. All of them are listed in Table.8 which is based on Milnor's results on three-dimensional unimodular Lie algebras.

Lie algebra	$Sign(a), Sign(b), Sign(c)$
$4\mathbb{A}_1$	$0, 0, 0$
$\mathbb{A}_{3,1} \oplus \mathbb{A}_1$	$0, 0, +$
$\mathbb{A}_{3,4} \oplus \mathbb{A}_1$	$-, 0, +$
$\mathbb{A}_{3,6} \oplus \mathbb{A}_1$	$0, +, +$
$\mathbb{A}_{3,8} \oplus \mathbb{A}_1$	$-, +, +$
$\mathbb{A}_{3,9} \oplus \mathbb{A}_1$	$+, +, +$

Table .9

Thus follows:

Theorem 2.2.2 (see [59]). *Let $\mathfrak{g}$ be an unimodular 4-dimensional decomposable Lie algebra. The possible signatures of the Ricci operators of all inner products on $\mathfrak{g}$ are:*

Lie algebras $\mathfrak{g}$	*Realizable Ricci signatures*
$4\mathbb{A}_1$	$(2,1,0)$, $(0,4,0)$
$\mathbb{A}_{3,1}\oplus\mathbb{A}_1$	$(2,1,1)$
$\mathbb{A}_{3,4}\oplus\mathbb{A}_1$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$, $(1,3,0)$
$\mathbb{A}_{3,6}\oplus\mathbb{A}_1$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$, $(0,4,0)$
$\mathbb{A}_{3,8}\oplus\mathbb{A}_1$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$, $(1,3,0)$
$\mathbb{A}_{3,9}\oplus\mathbb{A}_1$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$,$(1,2,1)$,
	$(1,1,2)$, $(1,0,3)$, $(0,3,1)$, $(0,1,3)$

Table .10

Proof. Let $\langle\ ,\ \rangle$ be any inner product on $\mathfrak{g}$, from Lemma 2.2.1 and a direct computations, we get

$$\mathrm{Ric}=\frac{1}{2}\begin{pmatrix} A & clbk & ambk & -b^2k\\ clbk & B & amcl & -c^2l\\ ambk & amcl & C & -a^2m\\ -b^2k & -c^2l & -a^2m & D\end{pmatrix},$$

where

$$A=-a^2m^2-c^2l^2+b^2-(a-c)^2,\quad B=-a^2m^2-b^2k^2+c^2-(a-b)^2,$$
$$C=-c^2l^2-b^2k^2+a^2-(b-c)^2,\quad D=a^2m^2+c^2l^2+b^2k^2.$$

The result follows from a case-by-case analysis. For example if $\mathfrak{g}=\mathbb{A}_{3,6}\oplus\mathbb{A}_1$, here $a=0,b>0$, and $c>0$. We give the values of parameters k,l,m,b,and c for which the Ricci signatures are realized.

Ricci signature	$(b,c,k,l.m)$
$(3,0,1)$	$(1,1,1,0,0)$
$(2,1,1)$	$(1,\sqrt{2},1,0,0)$
$(2,0,2)$	$(1,2,1,0,0)$
$(0,4,0)$	$(1,1,0,0,0)$

Table .11

We prove now that these are the only realizable Ricci signatures. The characteristic polynomial of the Ricci operator matrix is:

$$P(x)=(x-x_1)H(x)=(x-x_1)(x^3+\widetilde{B}x+\widetilde{C}),$$

where $x_1=-\frac{1}{2}\left((b-c)^2+c^2l^2+b^2k^2\right)$,

$$\widetilde{B}=-\frac{1}{4}\left((b^2-c^2-c^2l^2+b^2k^2)^2+b^2c^2(k^2+4k^2l^2+l^2)\right)\leq 0,$$

and $\widetilde{C}$ is an expression depending on the parameters b,c,k,l, and m.

We conclude that $x_1\leq 0$ since $b>0$ and $c>0$. It is obvious that the only realizable Ricci signature is $(0,4,0)$, when $x_1=0$.

For $x_1<0$, the sum of roots of the polynomial $H(x)=x^3+\widetilde{B}x+\widetilde{C}$ is zero. It is clear that, the only realizable Ricci signature is $(1,3,0)$ when $\widetilde{B}=\widetilde{C}=0$. □

2.2.1.2 Four dimensional indecomposable unimodular Lie algebras

From Gram-Schmidt orthogonalization process, we have:

Lemma 2.2.3 (see [55]). *For an arbitrary inner product $\langle\ ,\ \rangle$ on a 4-dimensional indecomposable real unimodular Lie algebra $\mathfrak{g}$, there exists a $\langle\ ,\ \rangle$-orthonormal basis in which the nonzero constants structure of Lie algebra $\mathfrak{g}$ are given in the following table:*

Lie algebra $\mathfrak{g}$	*Constants structure*	*Restrictions*
$\mathbb{A}_{4,1}$	$c_{2,4}^1 = a,\ c_{3,4}^1 = b,\ c_{3,4}^2 = c$	$a > 0, c > 0$
$\mathbb{A}_{4,2}^{-2}$,	$c_{1,4}^1 = -2a,\ c_{2,4}^1 = b,\ c_{2,4}^2 = a,$ $c_{3,4}^1 = c,\ c_{3,4}^2 = d,\ c_{3,4}^3 = a.$	$a > 0, d > 0$
$\mathbb{A}_{4,5}^{\alpha,-1-\alpha}$, $\alpha \in (-1, -1/2]$	$c_{1,4}^1 = a,\ c_{2,4}^1 = b,\ c_{2,4}^2 = c,$ $c_{3,4}^1 = d,\ c_{3,4}^2 = f,\ c_{3,4}^3 = -a - c.$	$a > 0, c < 0$
$\mathbb{A}_{4,6}^{-2\beta,\beta}$, $\beta \in (0, \infty)$	$c_{1,4}^1 = -2a,\ c_{2,4}^1 = b,\ c_{2,4}^2 = a + c,$ $c_{2,4}^3 = d,\ c_{3,4}^1 = f,\ c_{3,4}^2 = g,\ c_{3,4}^3 = -a - c.$	$a > 0, d < 0, g > 0$
$\mathbb{A}_{4,8}$	$c_{2,3}^1 = a,\ c_{2,4}^1 = b,\ c_{2,4}^2 = c,$ $c_{3,4}^1 = d,\ c_{3,4}^2 = f,\ c_{3,4}^3 = -c.$	$a > 0, c > 0$
$\mathbb{A}_{4,10}$	$c_{2,3}^1 = a,\ c_{2,4}^1 = b,\ c_{2,4}^3 = c,$ $c_{3,4}^1 = d,\ c_{3,4}^2 = g.$	$a > 0, c < 0$

Table .12

Thus follows:

Theorem 2.2.4 (see [59]). *Let $\mathfrak{g}$ be an unimodular 4-dimensional indecomposable Lie algebra. Then the possible signatures of the Ricci operators of all inner products on $\mathfrak{g}$ are:*

Lie algebras $\mathfrak{g}$	*Realizable Ricci signatures*
$\mathbb{A}_{4,1}$	$(3,0,1),\ (2,1,1),\ (2,0,2)$
$\mathbb{A}_{4,2}^{-2}$	$(3,0,1),\ (2,1,1),\ (2,0,2)$
$\mathbb{A}_{4,5}^{\alpha,-1-\alpha}$, $\alpha \in (-1, -1/2)$	$(3,0,1),\ (2,1,1),\ (2,0,2),\ (1,3,0)$
$\mathbb{A}_{4,5}^{-\frac{1}{2},-\frac{1}{2}}$	$(2,1,1),\ (1,3,0)$
$\mathbb{A}_{4,6}^{-2\beta,\beta}$, $\beta \in (0, \infty)$	$(3,0,1),\ (2,1,1),\ (2,0,2),\ (1,3,0)$
$\mathbb{A}_{4,8}$	$(3,0,1),\ (2,1,1),\ (2,0,2)$
$\mathbb{A}_{4,10}$	$(3,0,1),\ (2,1,1),\ (2,0,2)$

Table .13

Proof. The result follows from a case-by-case analysis. For example if $\mathfrak{g} = \mathbb{A}_{4,10}$, let $\langle\ ,\ \rangle$ be any inner product on $\mathfrak{g}$, from Lemma 2.2.3 and a direct computation the Ricci operator matrix Ric has the form

$$\frac{1}{2}\begin{pmatrix} a^2+b^2+d^2 & dg & bc & 0 \\ dg & -a^2-b^2-c^2+g^2 & -bd & ad \\ bc & -bd & -a^2-b^2+c^2-g^2 & -ab \\ 0 & ad & -ab & -(g+c)^2-b^2-d^2 \end{pmatrix}.$$

We give the values of parameters $a, b, c, d,$and g for which the Ricci signatures are realized.

Ricci signature	(a, b, c, d, g)
$(3, 0, 1)$	$(1, 2, -2, 0, 1)$
$(2, 1, 1)$	$(1, 0, -1, 0, 1)$
$(2, 0, 2)$	$(1, 0, -2, 0, 1)$

Table .13

We prove now that these are the only realizable Ricci signatures. The operator Ric has at least one positive eigenvalue since $a^2 + b^2 + d^2 > 0$.

Let $|g| = |c|$, consider the $(3\times 3)-$matrix Ric_1 resulting from Ric by deleting the first row and the first column. The characteristic polynomial of the matrix Ric_1 has the form $H(x) = x\left(x + (a^2 + b^2 + d^2)\right)^2$. Since $a > 0$, the matrix Ric_1 has one zero and two negative eigenvalues, then the matrix Ric has at least two negative eigenvalues and one nonpositive eigenvalue (see [45, Theorem 4.3.8]). Thus, the only realizable Ricci signatures in this case are $(3, 0, 1)$ and $(2, 1, 1)$.

Let $|g| < |c|$, consider the $(2 \times 2)-$matrix $\mathrm{Ric}_{1,3}$ obtained from the matrix Ric by deleting the rows and columns with the numbers 1 and 3, and have the form

$$\mathrm{Ric}_{1,3} = \frac{1}{2}\begin{pmatrix} -a^2-b^2-c^2+g^2 & ad \\ ad & -(g+c)^2-b^2-d^2 \end{pmatrix}.$$

This matrix is negative definite. Then matrix Ric has at least one positive and two negative eigenvalues(see [45, Theorem 4.3.8]). Thus, in this case there is not others realizable Ricci signatures a part from those given in table .

Let $|g| > |c|$, consider the $(2 \times 2)-$matrix $\mathrm{Ric}_{1,2}$ obtained from the matrix Ric by deleting the rows and columns with the numbers 1 and 2, and have the form

$$\mathrm{Ric}_{1,2} = \frac{1}{2}\begin{pmatrix} -a^2-b^2+c^2-g^2 & -ab \\ -ab & -(g+c)^2-b^2-d^2 \end{pmatrix}$$

This matrix is negative definite. Then matrix Ric has at least one positive and two negative eigenvalues(see [45, Theorem 4.3.8]). Thus, in this case there is not others realizable Ricci signatures a part from those given in table. $\square$

2.2.2 Four dimensional nonunimodular Lie algebras

The following table gives the classification of real nonunimodular 4-dimensional Lie algebras by G.M. Mubarakzyanov(see [56] and [81]).

Lie algebras	Nonzero cummutation relations
$\mathbb{A}_2 \oplus 2\mathbb{A}_1$	$[e_1, e_2] = e_2$
$2\mathbb{A}_2$	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$
$\mathbb{A}_{3,2} \oplus \mathbb{A}_1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_1 + e_2$
$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2$
$\mathbb{A}^{\alpha}_{3,5} \oplus \mathbb{A}_1,\ 0 < \|\alpha\| < 1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = \alpha e_2$
$\mathbb{A}^{\alpha}_{3,7} \oplus \mathbb{A}_1,\ \alpha > 0$	$[e_1, e_3] = \alpha e_1 - e_2,\ [e_2, e_3] = e_1 + \alpha e_2$
$\mathbb{A}^{\alpha}_{4,2},\ \alpha \neq 0,\ \alpha \neq -2$	$[e_1, e_4] = \alpha e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
$\mathbb{A}_{4,3}$	$[e_1, e_4] = e_1,\ [e_3, e_4] = e_2$
$\mathbb{A}_{4,4}$	$[e_1, e_4] = e_1,\ [e_2, e_4] = e_1 + e_2,\ [e_3, e_4] = e_2 + e_3$
$\mathbb{A}^{\alpha,\beta}_{4,5},\ \alpha\beta \neq 0$ $-1 \leq \alpha \leq \beta \leq 1,\ \alpha + \beta \neq 1$	$[e_1, e_4] = e_1,\ [e_2, e_4] = \alpha e_2,\ [e_3, e_4] = \beta e_3$
$\mathbb{A}^{\alpha,\beta}_{4,6},\ \alpha \neq 0$ $\beta \geq 0,\ \alpha \neq -2\beta$	$[e_1, e_4] = \alpha e_1,\ [e_2, e_4] = \beta e_2 - e_3,\ [e_3, e_4] = e_2 \beta e_3$
$\mathbb{A}_{4,7}$	$[e_2, e_3] = e_1,\ [e_1, e_4] = 2e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$
$\mathbb{A}^{\beta}_{4,9},\ -1 < \beta \leq 1$	$[e_2, e_3] = e_1,\ [e_1, e_4] = (1 + \beta)e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = \beta e_3$
$\mathbb{A}^{\alpha}_{4,11},\ \alpha > 0$	$[e_2, e_3] = e_1,\ [e_1, e_4] = 2\alpha e_1,\ [e_2, e_4] = \alpha e_2 - e_3,\ [e_3, e_4] = e_2 + \alpha e_3$
$\mathbb{A}_{4,12}$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2,\ [e_1, e_4] = -e_2,\ [e_2, e_4] = e_1$

Table .15

Kremlev and Nikonorov used the set of representative of matrices or the Gram-Schmidt orthogonalization method to obtain a convenient parametrization of all inner products on all nonunimodular Lie algebras.Thus follows:

Lemma 2.2.5 (see [56]). *For an arbitrary inner product $\langle\ ,\ \rangle$ on a 4-dimensional real nonunimodular Lie algebra $\mathfrak{g}$, there exists an $\langle\ ,\ \rangle$-orthonormal basis in which the nonzero constants structure of the Lie algebra $\mathfrak{g}$ are:*

Lie algebra $\mathfrak{g}$	*constants structure*	*Restrictions*
$\mathbb{A}_2 \oplus 2\mathbb{A}_1$	$c_{1,2}^2 = a,\ c_{1,2}^3 = b$	$a > 0, b \geq 0$
$2\mathbb{A}_2$	$c_{1,2}^2 = a,\ c_{1,3}^2 = b,\ c_{1,3}^4 = c,\ c_{1,4}^2 = f(a-d)$	$a, g > 0$
	$c_{1,4}^4 = d,\ c_{3,4}^2 = -fg,\ c_{3,4}^4 = g$	
$\mathbb{A}_{3,2} \oplus \mathbb{A}_1$	$c_{1,3}^1 = c_{2,3}^2 = a,\ c_{1,3}^4 = b,\ c_{2,3}^1 = c,$	$a, c > 0$
	$c_{2,3}^4 = d$	
$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$	$c_{1,3}^1 = c_{2,3}^2 = a,\ c_{2,3}^4 = b$	$a > 0, b \geq 0$
$\mathbb{A}_{3,5}^{\alpha} \oplus \mathbb{A}_1,\ 0 < \lvert\alpha\rvert < 1$	$c_{1,3}^1 = a,\ c_{1,3}^4 = b,\ c_{2,3}^1 = c,\ c_{2,3}^2 = a\alpha,$	$a > 0$
	$c_{2,3}^4 = d$	
$\mathbb{A}_{3,7}^{\alpha} \oplus \mathbb{A}_1,\ \alpha > 0$	$c_{1,3}^1 = \alpha l,\ c_{1,3}^2 = -al,\ c_{1,3}^4 = bl,\ c_{2,3}^1 = \frac{l}{a},$	$a, l > 0$
	$c_{2,3}^2 = \alpha l, c_{2,3}^4 = cl$	
$\mathbb{A}_{4,2}^{\alpha},\ \alpha \neq 0,\ \alpha \neq -2$	$c_{1,4}^1 = \alpha l,\ c_{2,4}^1 = a(\alpha-1)l,\ c_{2,4}^2 = l$	$c, l > 0$
	$c_{3,4}^1 = (b(\alpha-1) - ac)l,\ c_{3,4}^2 = cl,\ c_{3,4}^3 = l$	
$\mathbb{A}_{4,3}$	$c_{1,4}^1 = l,\ c_{2,4}^1 = al,\ c_{3,4}^1 = bl,\ c_{3,4}^2 = cl$	$c, l > 0$
$\mathbb{A}_{4,4}$	$c_{1,4}^1 = c_{2,4}^2 = c_{3,4}^3 = l,\ c_{2,4}^1 = al,$	$a, c > 0$
	$c_{3,4}^1 = bl,\ c_{3,4}^2 = cl$	$l > 0$
$\mathbb{A}_{4,5}^{\alpha,\beta},\ \alpha\beta \neq 0$	$c_{1,4}^1 = l,\ c_{2,4}^1 = a(\alpha-1)l,\ c_{2,4}^2 = \alpha l,\ c_{3,4}^2 = \beta l$	$l > 0$
	$c_{3,4}^1 = (ac(\alpha-1) + b(\beta-1))l,\ c_{3,4}^2 = c(\alpha-\beta)l.$	
$\mathbb{A}_{4,6}^{\alpha,\beta},\ \alpha \neq 0$	$c_{1,4}^1 = \alpha l,\ c_{2,4}^1 = al,\ c_{2,4}^2 = c_{3,4}^3 = \beta l,$	$c, l > 0$
	$c_{2,4}^3 = -\frac{l}{c},\ c_{3,4}^1 = bl,\ c_{3,4}^2 = cl$	
$\mathbb{A}_{4,7}$	$c_{1,4}^1 = 2a,\ c_{2,3}^1 = b,\ c_{2,4}^1 = c,\ c_{2,4}^2 = a,$	$a, b > 0$
	$c_{3,4}^1 = d,\ c_{3,4}^2 = f,\ c_{3,4}^3 = a$	$f > 0$
$\mathbb{A}_{4,9}^{\beta},\ -1 < \beta \leq 1$	$c_{1,4}^1 = a(\beta+1),\ c_{2,3}^1 = b,\ c_{2,4}^1 = c,\ c_{2,4}^2 = a,$	$a, b > 0$
	$c_{3,4}^1 = d,\ c_{3,4}^2 = f(1-be),\ c_{3,4}^3 = \alpha\beta$	
$\mathbb{A}_{4,11}^{\alpha},\ \alpha > 0$	$c_{1,4}^1 = 2a\alpha,\ c_{2,3}^1 = b,\ c_{2,4}^1 = c,\ c_{2,4}^2 = a\alpha,$	$a, b > 0$
	$c_{2,4}^3 = -ad,\ c_{3,4}^1 = f,\ c_{3,4}^2 = \frac{a}{d},\ c_{3,4}^3 = a\alpha$	$d > 0$
$\mathbb{A}_{4,12}$	$c_{1,3}^1 = c_{2,3}^2 = a,\ c_{1,4}^1 = c_{2,4}^2 = b,$	$a, d > 0$
	$c_{1,4}^2 = c,\ c_{2,4}^1 = d,\ c_{3,4}^1 = f,\ c_{3,4}^2 = g$	$c < 0$

Table.16

Thus follows:

Theorem 2.2.6 (see [56])**.** *Let $\mathfrak{g}$ be an nonunimodular 4-dimensional indecomposable Lie algebra. Then the possible signatures of the Ricci operators of all inner products on $\mathfrak{g}$ are:*

Lie algebras $\mathfrak{g}$	*Realizable Ricci signatures*
$\mathbb{A}_2 \oplus 2\mathbb{A}_1$	$(2,2,0)$, $(2,1,1)$
$2\mathbb{A}_2$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{3,2} \oplus \mathbb{A}_1$	$(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{3,3} \oplus \mathbb{A}_1$	$(3,1,0)$, $(3,0,1)$
$\mathbb{A}_{3,5}^{\alpha} \oplus \mathbb{A}_1$, $\alpha \in (-1,0)$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{3,5}^{\alpha} \oplus \mathbb{A}_1$, $\alpha \in (0,1)$	$(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{3,7}^{\alpha} \oplus \mathbb{A}_1$, $\alpha > 0$	$(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,2}^{\alpha}$, $\alpha < 0$, $\alpha \neq -2$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,2}^{\alpha}$, $\alpha > 0$, $\alpha \neq 1$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,2}^{1}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$
$\mathbb{A}_{4,3}$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,4}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,5}^{\alpha,\alpha}$, $\alpha \in [-1,-\frac{1}{2})$	$(3,0,1)$
$\mathbb{A}_{4,5}^{\alpha,\alpha}$, $\alpha \in (-\frac{1}{2},0)$	$(2,0,2)$
$\mathbb{A}_{4,5}^{\alpha,\alpha}$, $\alpha \in [-1,0)$	$(3,0,1)$
$\mathbb{A}_{4,5}^{\alpha,\alpha}$, $\alpha \in (0,1)$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$
$\mathbb{A}_{4,5}^{\alpha,1}$, $\alpha \in (0,1)$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$
$\mathbb{A}_{4,5}^{1,1}$	$(0,4,0)$
$\mathbb{A}_{4,5}^{\alpha,\beta}$, $\alpha \in [-1,0)$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,5}^{\alpha,\beta}$, $\alpha \in (0,1)$, $\alpha \neq \beta$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,6}^{\alpha,\beta}$, $\alpha < 0$, $\beta > 0$, $\alpha \neq -2\beta$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,6}^{\alpha,\beta}$, $\alpha > 0$, $\beta > 0$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,6}^{\alpha,0}$	$(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,7}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,9}^{\beta}$, $\beta \in (-1,-\frac{1}{2})$	$(3,0,1)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,9}^{-\frac{1}{2}}$	$(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,9}^{\beta}$, $-1 < \beta \in (-\frac{1}{2},1)$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,9}^{1}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$
$\mathbb{A}_{4,11}^{\alpha}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$
$\mathbb{A}_{4,12}$	$(0,4,0)$, $(3,1,0)$, $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, $(2,0,2)$

Table .17

Proof. The result follows from a case-by-case analysis. For example if $\mathfrak{g} = \mathcal{A}_{4,9}^{\beta}$, let $\langle\ ,\ \rangle$ be any inner product on $\mathfrak{g}$, from Lemma 2.2.5 and a direct computation the Ricci operator matrix 2Ric has the form

$$\begin{pmatrix} b^2+c^2+d^2-4a^2(1+\beta)^2 & -ac\beta+df(1-\beta)-cl & -d(a+l) & 0 \\ -ac\beta+df(1-\beta)-cl & -2al-b^2-c^2+f^2(1-\beta)^2 & m & bd \\ -d(a+l) & m & k & -bc \\ 0 & bd & -bc & -r \end{pmatrix},$$

where $l = 2a(1+\beta) > 0, \quad k = -4a^2\beta(1+\beta) - b^2 - d^2 - f^2(1-\beta)^2,$
$m = -cd - af(1-\beta)^2 - fl(1-\beta), \quad r = 4a^2(\beta^2+\beta+1) + c^2 + d^2 + f^2(1-\beta)^2.$
From [56, Theorem 3], Ric has at least two negative eigenvalues.

The case $\beta = 1$

The matrix of Ricci operator Ric has the form

$$\frac{1}{2}\begin{pmatrix} b^2+c^2+d^2-16a^2 & -5ac & -5d & 0 \\ -5ac & -8a^2-b^2-c^2 & -cd & bd \\ -5ad & -cd & -8a^2-b^2-d^2 & -bc \\ 0 & bd & -bc & -12a^2-c^2-d^2 \end{pmatrix}.$$

The submatrix Ric_1, that produced from Ric by deleting of the first row and the first column, is negative definite. Therefore, the matrix Ric has at least three negative eigenvalues (see [45, Theorem 4.3.8]) . We give the values of parameters $a, b, c,$ and d at which the only Ricci signatures are realized in this case.

Ricci signature	(a,b,c,d)
$(4,0,0)$	$(1,1,0,0)$
$(3,1,0)$	$(1,4,0,0)$
$(3,0,1)$	$(1,6,0,0)$

Table .18

The case $\beta \in (-\frac{1}{2}, 1)$

Let $p(t)$ be the characteristic polynomial of the matrix 2Ric with $c = d = 0$.

If $a = 1$ and $b = 2(1+\beta)$, then

$$p(t) = t(t + 4(\beta^2+\beta+1) + f^2(\beta-1)^2) \times (t^2 + 12(1+\beta)^2 t - (1-\beta)^4 f^4 - (5\beta^2+6\beta+5)(1-\beta)^2 f^2 + 16(2+\beta)(1+2\beta)(1+\beta)^2).$$

For suitable values of f, the Ricci signatures are $(3,1,0)$, $(2,1,1)$ and $(2,2,0)$.

If $a = 1$ and $b = 3(1+\beta)$, then

$$p(t) = (t - 5(1+\beta)^2)(t + 4(\beta^2+\beta+1) + f^2(1-\beta)^2) \times (t^2 + 22(1+\beta)^2 t - (1-\beta)^4 f^4 - (5\beta^2+6\beta+5)(1-\beta)^2 f^2 + (13+9\beta)(9+13\beta)(1+\beta)^2).$$

For suitable values of f, the Ricci signatures are $(3,0,1)$ and $(2,0,2)$.

If $a = b = 1$ and $f = 0$, then

$$p(t) = (t + (2\beta+3)(2\beta+1))(t + 4(\beta^2+\beta+1)(t + (2\beta+1)^2)(t + 5 + 4\beta).$$

For suitable value of f, the Ricci signature is $(4,0,0)$.

The case $\beta = -\frac{1}{2}$

From [57, lemma 2], it is proved that the Ricci operator Ric is non-positive defined if the Ricci signature is $(2,2,0)$.

If $c = f = 0$, the matrix of Ricci operator Ric has the form

$$\frac{1}{2}\begin{pmatrix} b^2+d^2-16a^2 & 0 & -2ad & 0 \\ 0 & -2a^2-b^2 & 0 & bd \\ -2ad & 0 & a^2-b^2-d^2 & 0 \\ 0 & bd & 0 & -3a^2-d^2 \end{pmatrix}.$$

Consider two of its submatrices

$$\begin{pmatrix} b^2+d^2-16a^2 & -2ad \\ -2ad & a^2-b^2-d^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2a^2-b^2 & bd \\ bd & -3a^2-d^2 \end{pmatrix}.$$

The first one has zero trace, hence, it has the signature $(0,0)$ for $b = a$ and $d = 0$ or the signature $(-,+)$ otherwise. The second submatrix is negative defined. Therefore, Ric may have have only signature $(3,0,1)$ and $(2,2,0)$.

If $c = d = 0$ and $b = 2a$, then

$$\mathrm{Ric} = \frac{1}{2}\mathrm{diag}\left(3a^2, \begin{pmatrix} -6a^2+\frac{9}{4}f^2 & -\frac{15}{4}af \\ -\frac{15}{4}af & -3a^2-\frac{9}{4}f^2 \end{pmatrix}, -3a^2-\frac{9}{4}f^2\right).$$

For some suitable a and f, the signatures $(3,0,1)$ and $(2,2,0)$ are realized, since the trace of the depicted $(2\times 2)-$submatrix is negative. Then for $\beta = -\frac{1}{2}$, the only realizable Ricci signatures are $(3,0,1)$, $(2,2,0)$, $(2,1,1)$, and $(2,0,2)$.

The case $\beta \in (-1, -\frac{1}{2})$

From [57, lemma 2], it is proved that the Ricci operator Ric can not have any of the signature $(4,0,0)$, $(3,1,0)$ and $(2,2,0)$.

If $c = d = 0$ and $b = 2a$, then

$$\mathrm{Ric} = \frac{1}{2}\mathrm{diag}\left(-4a^2\beta(2+\beta), A, -4a^2(1+\beta+\beta^2) - f^2(1-\beta)^2\right)$$

where

$$A = \begin{pmatrix} -4a^2(2+\beta)+f^2(1-\beta)^2 & -af(3+\beta)(1-\beta) \\ -af(3+\beta)(1-\beta) & -4a^2(1+\beta+\beta^2)-f^2(1-\beta)^2 \end{pmatrix}$$

The trace of $\mathrm{tr}(A)$ is negative and $\det(A)$ can have any sign for suitable a and f. Therefore, for $\beta \in (-1, -\frac{1}{2})$ the only realizable Ricci signatures are $(3,0,1)$, $(2,1,1)$ and $(2,0,2)$. □

2.3 Ricci signatures of nilpotent Lie groups

We describe the results of Boucetta, which give the Ricci signatures of 2-step nilpotent Lie groups regardless their dimension in [9] and Kremlev, which solved the problem for five dimensional nilpotent Lie groups in [57].

2.3.1 Boucetta's approach

Recall that a real Lie algebra $\mathfrak{g}$ is said to be a 2-step nilpotent if its derived ideal $[\mathfrak{g},\mathfrak{g}]$ is nontrivial and contained in the center $Z(\mathfrak{g})$. Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be an Euclidean 2-step nilpotent Lie algebra of dimension n. Then, we have the following orthogonal splitting:

$$\mathfrak{g} = Z(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}] \oplus \left([\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})\right) \oplus \Big([\mathfrak{g},\mathfrak{g}]^{\perp}\cap\left([\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})\right)^{\perp}\Big) \tag{2.2}$$

$$= [\mathfrak{g},\mathfrak{g}] \oplus \left([\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})\right) \oplus \Big([\mathfrak{g},\mathfrak{g}]^{\perp}\cap\left([\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})\right)^{\perp}\Big) \tag{2.3}$$

From [68, Lemma 2.1, Lemma 2.3], the following result holds:

Theorem 2.3.1 (see [9])**.** *Let $(\mathfrak{g},\langle\ ,\ \rangle)$ be an Euclidean 2-step nilpotent Lie algebra of dimension n. Set $p=\dim Z(\mathfrak{g})$ and $r=\dim[\mathfrak{g},\mathfrak{g}]$. Then there exists an $\langle\ ,\ \rangle$-orthonormal basis $(e_1,...,e_r)$ of $[\mathfrak{g},\mathfrak{g}]$, an $\langle\ ,\ \rangle$-orthonormal basis $(h_{r+1},...,h_{p-r})$ of $[\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})$, an $\langle\ ,\ \rangle-$orthonormal basis $(g_{p+1},...,g_n)$ of $[\mathfrak{g},\mathfrak{g}]^{\perp}\cap\left([\mathfrak{g},\mathfrak{g}]^{\perp}\cap Z(\mathfrak{g})\right)^{\perp}$ and two families of real numbers $0<\mu_1\leq...\leq\mu_r$ and $0<\lambda_1\leq...\leq\lambda_r$ such that non vanishing entries in the matrix of Ricci curvature* ric *in the basis $\mathcal{B}=(e_1,...,e_r;h_{r+1},...,h_{p-r};g_{p+1},...,g_n)$ are*

$$\mathrm{ric}(e_i,e_i)=\mu_i \quad \textit{and} \quad \mathrm{ric}(g_j,g_j)=-\lambda_j, \quad i=1,...,r, \quad j=p+1,...,n.$$

Then follows

Corollary 2.3.2. *For any inner product $\langle\ ,\ \rangle$ on a 2-step nilpotent Lie algebra $\mathfrak{g}$ of dimension n. Then its associated Ricci signature is $(n-p,p-r,r)$. Thus, the Ricci signature is independent of the choice of the inner product.*

2.3.2 Kremlev's approach

The following table give the classification of real nonunimodular 5-dimensional nilpotent Lie algebras by G.M. Mubarakzyanov (see [81]).

Lie algebras	Nonzero cummuation relations
$5\mathbb{A}_1$	
$\mathbb{A}_{3,1}\oplus 2\mathbb{A}_1$	$[e_2,e_3]=e_1$
$\mathbb{A}_{4,1}\oplus\mathbb{A}_1$	$[e_2,e_4]=e_1,\ [e_3,e_4]=e_2$
$\mathbb{A}_{5,1}$	$[e_3,e_5]=e_1,\ [e_4,e_5]=e_2$
$\mathbb{A}_{5,2}^{\alpha}$	$[e_2,e_5]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_3$
$\mathbb{A}_{5,3}^{\alpha}$	$[e_3,e_4]=e_2,\ [e_3,e_5]=e_1,\ [e_4,e_5]=e_3$
$\mathbb{A}_{5,4}^{\alpha}$	$[e_2,e_4]=e_1,\ [e_3,e_5]=e_1$
$\mathbb{A}_{5,5}^{\alpha}$	$[e_3,e_4]=e_1,\ [e_2,e_5]=e_1,\ [e_3,e_5]=e_2$
$\mathbb{A}_{5,6}^{\alpha}$	$[e_3,e_4]=e_1,\ [e_2,e_5]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_3$

Table.19

From [71], we have:

Lemma 2.3.3 (see [71]). *The following list contains all (up to isomorphism) noncommutative nilpotent Lie algebras of dimension* 5*, with the nonzero constants structure in a canonical orthonormal basis for each.*

Lie algebra	*Structure constants*	*Restrictions*
$\mathcal{N}_1^5(\varepsilon,\sigma)$	$c_{1,2}^5=\varepsilon,\ c_{3,4}^5=\sigma$	$\varepsilon\geq\sigma>0$
$\mathcal{N}_1^5(\varepsilon,\sigma,\upsilon,\gamma,\rho)$	$c_{1,2}^3=\varepsilon,\ c_{1,3}^2=b,\ c_{1,2}^5=\upsilon,\ c_{1,4}^5=\sigma$ $c_{2,3}^5=\rho$	$\varepsilon>0,\sigma>0,\rho>0$ $\upsilon\geq0,\gamma\geq0$
$\mathcal{N}_1^5(\varepsilon,\delta,\tau,\sigma,\upsilon,\gamma)$	$c_{1,2}^3=\varepsilon,\ c_{1,2}^4=\tau,\ c_{1,2}^5=\upsilon$ $c_{1,3}^4=\delta,\ c_{1,3}^5=\gamma,\ c_{1,4}^5=\sigma$	$\varepsilon>0,\delta>0,\sigma>0$ $\tau\geq0(\tau=0\Rightarrow\gamma\geq0)$
$\mathcal{N}_1^5(\varepsilon,\delta,\tau,\sigma,\upsilon,\gamma,\rho)$	$c_{1,2}^3=\varepsilon,\ c_{1,2}^4=\tau,\ c_{1,2}^5=\upsilon$ $c_{1,3}^4=\delta,\ c_{1,3}^5=\gamma,\ c_{1,4}^5=\sigma,\ c_{2,3}^5=\rho$	$\varepsilon>0,\delta>0,\sigma>0,\rho>0$ $\tau\geq0(\tau=0\Rightarrow\gamma\geq0)$
$\mathcal{N}_2^5(\delta,\sigma)$	$c_{1,2}^4=\delta,\ c_{1,3}^5=\sigma$	$\delta\geq\sigma>0$
$\mathcal{N}_2^5(\varepsilon,\tau,\upsilon,\gamma)$	$c_{1,2}^3=\varepsilon,\ c_{1,2}^4=\tau,\ c_{1,2}^5=\upsilon,$ $c_{1,3}^4=\gamma$	$\varepsilon>0,\gamma>0,$ $\tau\geq0,\upsilon\geq0$
$\mathcal{N}_2^5(\varepsilon,\tau,\gamma)$	$c_{1,2}^3=\varepsilon,\ c_{1,2}^4=\tau,\ c_{1,3}^4=\gamma,$ $c_{2,3}^5=\gamma$	$\varepsilon>0,\gamma>0,$ $\tau\geq0$
$\mathcal{N}_2^5(\varepsilon,\tau,\upsilon,\gamma,\rho)$	$c_{1,2}^3=\varepsilon,\ c_{1,2}^4=\tau,\ c_{1,2}^5=\upsilon,$ $c_{1,3}^4=\gamma,\ c_{2,3}^5=\rho$	$\varepsilon>0,\gamma>\rho>0,$ $\tau\geq0,\upsilon\geq0$
$\mathcal{N}_3^5(\varepsilon)$	$c_{1,2}^3=\varepsilon$	$\varepsilon>0$

Table .20

Thus follows

Theorem 2.3.4. *Let* $\mathfrak{g}$ *be a* 5*-dimensional noncommutative nilpotent Lie algebra. Then the possible signatures of the Ricci operators of all inner products on* $\mathfrak{g}$ *are:*

Lie algebras $\mathfrak{g}$	*Realizable Ricci signatures*
$\mathbb{A}_{3,1} \oplus 2\mathbb{A}_1$	$(2,2,1)$
$\mathbb{A}_{4,1} \oplus \mathbb{A}_1$	$(3,1,1)$, $(3,0,2)$, $(2,2,1)$, $(2,1,2)$
$\mathbb{A}_{5,1}$	$(3,0,2)$
$\mathbb{A}_{5,2}$	$(4,0,1)$, $(3,1,1)$, $(3,0,2)$, $(2,2,1)$, $(2,1,2)$, $(2,0,3)$
$\mathbb{A}_{5,3}$	$(3,0,2)$, $(2,1,2)$, $(2,0,3)$
$\mathbb{A}_{5,4}$	$(4,0,1)$
$\mathbb{A}_{5,5}$	$(4,0,1)$, $(3,1,1)$, $(3,0,2)$
$\mathbb{A}_{5,6}$	$(4,0,1)$, $(3,1,1)$, $(3,0,2)$, $(2,2,1)$, $(2,1,2)$, $(2,0,3)$

Table .21

Proof. The result follows from a case-by-case analysis. For example if $\mathfrak{g} = \mathbb{A}_{4,1} \oplus \mathbb{A}_1$, let $\langle\ ,\ \rangle$ be any inner product on $\mathfrak{g}$, from Lemma 2.3.3 and a direct computation the Ricci operator matrix Ric has the form

$$\frac{1}{2}\begin{pmatrix} -(\varepsilon^2+\tau^2+\upsilon^2+\gamma^2) & 0 & 0 & 0 & 0 \\ 0 & -(\varepsilon^2+\tau^2+\upsilon^2) & -\tau\gamma & 0 & 0 \\ 0 & -\tau\gamma & \varepsilon^2-\gamma^2 & \varepsilon\tau & \varepsilon\upsilon \\ 0 & 0 & \varepsilon\tau & \tau^2+\gamma^2 & \tau\upsilon \\ 0 & 0 & \varepsilon\upsilon & \tau\upsilon & \upsilon^2 \end{pmatrix}.$$

We give the values of parameters $\varepsilon, \tau, \upsilon$,and γ for which the Ricci signatures are realized.

Ricci signature	$(\varepsilon,\tau,\upsilon,\gamma)$
$(3,1,1)$	$(1,0,0,2)$
$(3,0,2)$	$(1,0,1,1)$
$(2,2,1)$	$(1,0,0,1)$
$(2,1,2)$	$(2,0,0,1)$

Table .22

We prove now that these are the only realizable Ricci signatures. The $(2\times 2)-$matrix $\mathrm{Ric}_{3,4,5}$ obtained from the matrix Ric by deleting the rows and columns with the same numbers 3 , 4 and 5 have the form

$$\mathrm{Ric}_{3,4,5} = -\frac{1}{2}\begin{pmatrix} \varepsilon^2+\tau^2+\upsilon^2+\gamma^2 & 0 \\ 0 & \varepsilon^2+\tau^2+\upsilon^2 \end{pmatrix}$$

with $\varepsilon > 0, \gamma > 0, \tau \geq 0, \upsilon \geq 0$. This matrix is negative definite. Then matrix Ric has at least two negative eigenvalues(see [45, Theorem 4.3.8]). The $(2\times 2)-$matrix $\mathrm{Ric}_{1,2,3}$ obtained from the matrix Ric by deleting the rows and columns with the same numbers 1 , 2 and 3 have the form

$$\mathrm{Ric}_{1,2,3} = \frac{1}{2}\begin{pmatrix} \tau^2+\gamma^2 & \tau\upsilon \\ \tau\upsilon & \upsilon^2 \end{pmatrix}$$

with $\gamma > 0, \tau \geq 0, \upsilon \geq 0$. For $\upsilon > 0$,this matrix is positive definite, then Ric has at least two positive eigenvalues(see [45, Theorem 4.3.8]).

For $\upsilon = 0$, the matrix $(2\times 2)-$matrix $\mathrm{Ric}_{1,2,3}$ has the signature $(0,+)$, then Ric has at least one positive eigenvalue and it is also clear that Ric has at least one null eigenvalue.

At last, consider the $(4 \times 4-)$matrix Ric_1 obtained from the matrix Ric by deleting the row and column with the same number 1. Let $p(t)$ be the characteristic polynomial of the matrix Ric_1, then

$$p(t) = t^4 + \widetilde{B}t^2 + \widetilde{C}t + \widetilde{D}.$$

with $\widetilde{D} = \upsilon^4\gamma^4 + \varepsilon^2\gamma^4\upsilon^2 \geq 0$.

Thus, the signatures of the table are the only realizable Ricci signatures on $\mathfrak{g} = \mathbb{A}_{4,1} \oplus \mathbb{A}_1$. □

Remark 2.3.5. *For every left-invariant Riemannian metrics on three-dimensional unimodular Lie groups, Milnor([68]) constructed certain orthonormal bases of the corresponding metric Lie algebras in which the constants structure are described by at most three parameters and investigate Ricci signatures. He was able to identify amount* 10 *potential signatures candidates, the realizable ones. Such bases are nowadays called the Milnor's frames are powerful tools. Some generalizations have been known: Chebarykov([18]) studied three-dimensional non-unimodular Lie groups, the constants structure are described by at most three parameters and investigate Ricci signatures. He was able to identify amount* 10 *potential signatures candidates, the realizable ones. Kremlev and Nikonorov([55, 56]), studied four-dimensional Lie groups, the constants structure are described by at most six parameters and investigate Ricci signatures. He was able to identify amount* 15 *potential signatures candidates, the realizable ones. Nikitenko([71]), studied five-dimensional nilpotent Lie groups, the constants structure are described by at most seven parameters and Kremlev([58]) investigated Ricci signatures and was able to identify amount* 21 *potential signatures candidates, the realizable ones. In [42], a general procedure to construct Milnor's frames theoretically on any Lie group is given, but in most cases the number of parameters that describe the constants structure is too large then investigating Ricci signatures is almost impossible.*

Chapter Three

The signature of the Ricci curvature of left-invariant Riemannian metrics on nilpotent Lie groups

Let (G, h) be a nilpotent Lie group endowed with a left invariant Riemannian metric, $\mathfrak{g}$ its Euclidean Lie algebra and $Z(\mathfrak{g})$ the center of $\mathfrak{g}$. By using an orthonormal basis adapted to the splitting $\mathfrak{g} = (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus O^+ \oplus (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) \oplus O^-$, where O^+ (resp. O^-) is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $[\mathfrak{g}, \mathfrak{g}]$ (resp. is the orthogonal of $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ in $[\mathfrak{g}, \mathfrak{g}]^\perp$), we show that the signature of the Ricci operator of (G, h) is determined by the dimensions of the vector spaces $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$, $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp$ and the signature of a symmetric matrix of order $\dim[\mathfrak{g}, \mathfrak{g}] - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}])$. This permits to associate to G a subset $\mathbf{Sign}(\mathfrak{g})$ of $\mathbb{N}^3$ depending only on the Lie algebra structure, easy to compute and such that, for any left invariant Riemannian metric on G, the signature of its Ricci operator belongs to $\mathbf{Sign}(\mathfrak{g})$. We show also that for any nilpotent Lie group of dimension less or equal to 6, $\mathbf{Sign}(\mathfrak{g})$ is actually the set of signatures of the Ricci operators of all left invariant Riemannian metrics on G. We give also some general results which support the conjecture that the last result is true in any dimension. All these results are obtained in [23].

3.1 Reduction of the Ricci operator of a Riemannian Lie group and Ricci signature underestimate

In this section, Throughout this chapter, we will use the following convention. The signature of a symmetric operator A on an Euclidean vector space V is the sequence (s^-, s^0, s^+) where $s^+ = \sum_{\lambda_i > 0} \dim \ker(J - \lambda_i \mathrm{I}_V)$, $s^- =$

$\sum_{\lambda_i<0} \dim\ker(J-\lambda_i \mathrm{I}_V)$ and $s^0 = \dim\ker J$, where $\lambda_1,\ldots,\lambda_r$ are the eigenvalues of J and I_V the identity operator of V.

Let $\mathfrak{g}$ be a nilpotent n-dimensional Lie algebra, $Z(\mathfrak{g})$ its center and $[\mathfrak{g},\mathfrak{g}]$ its derived ideal. Note first that, another formulation of Corollary 2.3.2, which first appeared in [9] and which solves Problem 1 for 2-step nilpotent Lie groups is:

Corollary 3.1.1. *Let G be a 2-step nilpotent Lie group. Then, for any left-invariant Riemannian metric on G, the signature of its Ricci curvature is given by*

$$(s^-,s^0,s^+) = (\dim\mathfrak{g} - \dim Z(\mathfrak{g}), \dim Z(\mathfrak{g}) - \dim[\mathfrak{g},\mathfrak{g}], \dim[\mathfrak{g},\mathfrak{g}]).$$

Now, we introduce $\mathbf{Sign}(\mathfrak{g})$. Let us put $d=\dim[\mathfrak{g},\mathfrak{g}]$, $k=\dim Z(\mathfrak{g})$ and $\ell = \dim(Z(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}])$. We associate to $\mathfrak{g}$ the subset of $\mathbb{N}^3$

$$\mathbf{Sign}(\mathfrak{g}) = \left\{ (n-d-p+m^-, p+m^0, \ell+m^+) : \begin{cases} \max(k-d,0)\le p\le k-\ell \\ \\ m^-+m^0+m^+ = d-\ell \end{cases} \right\} \tag{3.1}$$

For instance, if $\mathfrak{g}$ is 2-step nilpotent then $[\mathfrak{g},\mathfrak{g}]\subset Z(\mathfrak{g})$ and hence

$$\mathbf{Sign}(\mathfrak{g}) = \{(n-k, k-d, d)\}.$$

If $\mathfrak{g}$ is a filiform nilpotent Lie algebra then $Z(\mathfrak{g})\subset[\mathfrak{g},\mathfrak{g}]$, $\dim Z(\mathfrak{g}) = 1$, $\dim[\mathfrak{g},\mathfrak{g}] = n-2$ and hence

$$\mathbf{Sign}(\mathfrak{g}) = \left\{(2+m^-, m^0, 1+m^+),\ m^-+m^0+m^+ = n-3\right\}.$$

The signature of the Ricci operator of a left invariant Riemannian metric on Lie group of dimension n belongs to $\{(n^-,n^0,n^+) :\ n^-+n^0+n^+ = n\}$ whose cardinal is $\frac{(n+1)(n+2)}{2}$.

We now consider the Lie subalgebra of left invariant Killing vector fields on G given by

$$K(\langle\ ,\ \rangle) = \{u\in\mathfrak{g}, \mathrm{ad}_u + \mathrm{ad}_u^* = 0\}.$$

It contains obviously the center $Z(\mathfrak{g})$ of $\mathfrak{g}$. Put $K^+(\langle\ ,\ \rangle) = K(\langle\ ,\ \rangle)\cap[\mathfrak{g},\mathfrak{g}]$ and $K^-(\langle\ ,\ \rangle) = K(\langle\ ,\ \rangle)\cap[\mathfrak{g},\mathfrak{g}]^\perp$. Denote by O^+ (resp. O^-) the orthogonal of $K^+(\langle\ ,\ \rangle)$ in $[\mathfrak{g},\mathfrak{g}]$ (resp. the orthogonal of $K^-(\langle\ ,\ \rangle)$ in $[\mathfrak{g},\mathfrak{g}]^\perp$). Then

$$\mathfrak{g} = K^+(\langle\ ,\ \rangle)\oplus O^+\oplus K^-(\langle\ ,\ \rangle)\oplus O^-. \tag{3.2}$$

We call this splitting *characteristic splitting* of $(\mathfrak{g},\langle\ ,\ \rangle)$ and any basis of $\mathfrak{g}$ of the form $\mathbb{B}_1\cup\mathbb{B}_2\cup\mathbb{B}_3\cup\mathbb{B}_4$ (where $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$ and $\mathbb{B}_4$ are, respectively, bases of $K^+(\langle\ ,\ \rangle), O^+, K^-(\langle\ ,\ \rangle), O^-$) is called *characteristic basis*.

We now prove a key lemma that will play a crucial role in the proofs of our main results.

Lemma 3.1.2. *With the hypothesis and the notations above, we have:*

(i) $K^-(\langle\ ,\ \rangle) \subset \ker(\mathrm{ric})$ *and if* $K^+(\langle\ ,\ \rangle) \neq \{0\}$ *then the restriction of* ric *to* $K^+(\langle\ ,\ \rangle)$ *is positive definite.*

(ii) *If* $O^- \neq \{0\}$, *then the restriction of* ric *to* O^- *is negative definite and* $\mathrm{ric}(K^+(\langle\ ,\ \rangle), O^-) = 0$.

(iii) *For any characteristic basis* $\mathbb{B}$ *of* $\mathfrak{g}$, *the matrix of the Ricci tensor in* $\mathbb{B}$ *is given by*

$$\mathrm{Mat}(\mathrm{ric}, \mathbb{B}) = \frac{1}{2}\begin{bmatrix} Z & V & 0 & 0 \\ V^t & X & 0 & W \\ 0 & 0 & 0 & 0 \\ 0 & W^t & 0 & Y \end{bmatrix},$$

and the Ricci signature of $(\mathfrak{g}, \langle\ ,\ \rangle)$ *is given by*

$$(s^-, s^0, s^+) = (\dim[\mathfrak{g},\mathfrak{g}]^\perp - \dim K^-(\langle\ ,\ \rangle) + m^-, \dim K^-(\langle\ ,\ \rangle) + m^0, \dim K^+(\langle\ ,\ \rangle) + m^+), \tag{3.3}$$

where (m^-, m^0, m^+) *is the signature of the symmetric matrix*

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = X - V^t Z^{-1} V - W Y^{-1} W^t. \tag{3.4}$$

Proof. First remark that, for any $u \in \mathfrak{g}$, J_u is skew-symmetric and $J_u = 0$ iff $u \in [\mathfrak{g},\mathfrak{g}]^\perp$. With this remark in mind, by using (1.4), we get for any $u \in K^+(\langle\ ,\ \rangle)$, $\mathrm{ric}(u,u) = -\frac{1}{4}\mathrm{tr}(J_u^2) \geq 0$ and $\mathrm{ric}(u,u) = 0$ if and only if $J_u = 0$. This shows that the restriction of ric to $K^+(\langle\ ,\ \rangle)$ is definite positive. On the other hand, for any $u \in O^-$, by using (1.4) we get $\mathrm{ric}(u,u) = -\frac{1}{4}\mathrm{tr}((\mathrm{ad}_u + \mathrm{ad}_u^*)^2) \leq 0$ and $\mathrm{ric}(u,u) = 0$ iff $u \in K(\langle\ ,\ \rangle)$. This shows that the restriction of ric to O^- is negative definite. We have also, for any $u \in K^-(\langle\ ,\ \rangle)$ and any $v \in \mathfrak{g}$, $\mathrm{ric}(u,v) = 0$. Finally, for any $u \in K^+(\langle\ ,\ \rangle)$ and any $v \in O^-$, $\mathrm{ric}(u,v) = 0$ this completes the proof of $(i) - (ii)$.

In any characteristic basis $\mathbb{B}$ of $\mathfrak{g}$, according to the results shown in $(i)-(ii)$, the matrix $\mathrm{R}(\mathrm{ric}, \mathbb{B})$ has the desired form. Put

$$Q = \begin{bmatrix} I_{n_1} & -Z^{-1}V & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & -Y^{-1}W^t & 0 & I_{n_4} \end{bmatrix}.$$

We can check easily that

$$Q^t\ \mathrm{Mat}(\mathrm{ric}, \mathbb{B})\ Q = \frac{1}{2}\begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & \mathrm{R}(\mathrm{ric}, \mathbb{B}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y \end{bmatrix}.$$

This formula combined with the results in $(i)-(ii)$ give the desired formula for the signature of ric. $\square$

Definition 3.1.3. *Let* (G,h) *be a Riemannian Lie group and* $(\mathfrak{g},\langle\ ,\ \rangle)$ *its associated Euclidean Lie algebra.*

- *We call* $(r^-,r^0,r^+)=(\dim[\mathfrak{g},\mathfrak{g}]^\perp-\dim K^-(\langle\ ,\ \rangle),\dim K^-(\langle\ ,\ \rangle),\dim K^+(\langle\ ,\ \rangle))$ *the Ricci signature underestimate of* $(\mathfrak{g},\langle\ ,\ \rangle)$.
- *For any characteristic basis* $\mathbb{B}$ *of* $\mathfrak{g}$*, we call* $\mathrm{R}(\mathrm{ric},\mathbb{B})$ *defined by* (3.4) reduced matrix *of the Ricci curvature in* $\mathbb{B}$*. It is a symmetric* $(s\times s)$*-matrix with* $s=\dim[\mathfrak{g},\mathfrak{g}]-\dim K^+(\langle\ ,\ \rangle)$.

Note that the order of $\mathrm{R}(\mathrm{ric},\mathbb{B})$ is zero iff $[\mathfrak{g},\mathfrak{g}]\subset K(\langle\ ,\ \rangle)$. In this case $K(\langle\ ,\ \rangle)=[\mathfrak{g},\mathfrak{g}]\oplus K^-(\langle\ ,\ \rangle)$ and we get:

Corollary 3.1.4. *Let* (G,h) *be a Riemannian Lie group such that* $[\mathfrak{g},\mathfrak{g}]\subset K(\langle\ ,\ \rangle)$*. Then the signature of the Ricci curvature of* h *is given by*

$$(s^-,s^0,s^+)=(\dim\mathfrak{g}-\dim K(\langle\ ,\ \rangle),\dim K(\langle\ ,\ \rangle)-\dim[\mathfrak{g},\mathfrak{g}],\dim[\mathfrak{g},\mathfrak{g}]).$$

Remark 3.1.5. *The case where the Riemannian metric is bi-invariant* ($\mathfrak{g}=K(\langle\ ,\ \rangle)$) *is a particular case of the situation in Corollary 3.1.4 and in this case* $Z(\mathfrak{g})=[\mathfrak{g},\mathfrak{g}]^\perp$ *and hence the signature is given by*

$$(s^-,s^0,s^+)=(0,\dim Z(\mathfrak{g}),\dim[\mathfrak{g},\mathfrak{g}]).$$

Moreover, since a skew-symmetric nilpotent endomorphism must vanishes then $K(\langle\ ,\ \rangle)=Z(\mathfrak{g})$. This simple fact combined with the result of Lemma 3.1.2 will have surprising consequences.

3.2 Main result 1

Theorem 3.2.1. *Let* (G,h) *be a nilpotent Lie group endowed with a left invariant Riemannian metric and* $\mathfrak{g}$ *its Lie algebra. Then the signature of the Ricci operator of* (G,h) *belongs to* ***Sign***$(\mathfrak{g})$.

Proof. Let (G,h) be a nilpotent Riemannian Lie group. We distinguish two cases.

- $Z(\mathfrak{g})\subset[\mathfrak{g},\mathfrak{g}]$. In this case, it is obvious that the Ricci signature underestimate of $(\mathfrak{g},\langle\ ,\ \rangle)$ is given by

$$(r^-,r^0,r^+)=(\dim\mathfrak{g}-\dim[\mathfrak{g},\mathfrak{g}],0,\dim Z(\mathfrak{g})).$$

On the other hand, by using (3.1), one can see easily that

$$\mathbf{Sign}(\mathfrak{g}) = \left\{(r^- + m^-, r^0 + m^0, r^+ + m^+),\ m^- + m^0 + m^+ = \dim[\mathfrak{g},\mathfrak{g}] - \dim Z(\mathfrak{g})\right\}.$$

According to Lemma 3.1.2, the Ricci signature of h belongs to $\mathbf{Sign}(\mathfrak{g})$ and we obtain the result in this case. Corollary 3.2.2 follows from the fact that $r^- = \dim\mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}] \geq 2$. In a nilpotent Lie algebra the derived ideal is always of codimension greater than 2.

- $Z(\mathfrak{g}) \not\subseteq [\mathfrak{g},\mathfrak{g}]$. Choose a complement I of $Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$ in $Z(\mathfrak{g})$ and a complement U of $[\mathfrak{g},\mathfrak{g}] \oplus I$ in $\mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{g}_1 \oplus I$ where $\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}] \oplus U$ is an ideal of $\mathfrak{g}$ and I is a central ideal. Moreover, $Z(\mathfrak{g}_1) = Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]$ and $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{g}_1,\mathfrak{g}_1]$. By using the same notations as in (3.1), we get that the Ricci signature underestimate of $(\mathfrak{g}, \langle\ ,\ \rangle)$ is given by

$$(r^-, r^0, r^+) = (n - d - p, p, \ell), \quad p = \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp).$$

We have obviously $p \leq \dim I = \dim Z(\mathfrak{g}) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}])$ and

$$p = \dim Z(\mathfrak{g}) + \dim\mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}] - \dim(Z(\mathfrak{g}) + [\mathfrak{g},\mathfrak{g}]^\perp) \geq \dim Z(\mathfrak{g}) - \dim[\mathfrak{g},\mathfrak{g}].$$

According to Lemma 3.1.2, the Ricci signature of h belongs to $\mathbf{Sign}(\mathfrak{g})$ and we obtain the result in this case. Corollary 3.2.2 follows from the fact that

$$\begin{aligned} r^- &= \dim\mathfrak{g} - \dim[\mathfrak{g},\mathfrak{g}] - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp) \\ &= \dim\mathfrak{g}_1 - \dim[\mathfrak{g}_1,\mathfrak{g}_1] + \dim I - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp) \\ &= \dim\mathfrak{g}_1 - \dim[\mathfrak{g}_1,\mathfrak{g}_1] + \dim Z(\mathfrak{g}) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]) - \dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp) \\ &\geq \dim\mathfrak{g}_1 - \dim[\mathfrak{g}_1,\mathfrak{g}_1] \geq 2. \end{aligned}$$

□

As an immediate consequence of this result, if G is 2-step nilpotent then any left invariant Riemannian metric on G has the signature of its Ricci operator equal to $(\dim\mathfrak{g} - \dim Z(\mathfrak{g}), \dim Z(\mathfrak{g}) - \dim[\mathfrak{g},\mathfrak{g}], \dim[\mathfrak{g},\mathfrak{g}])$. On the other hand, Theorem 3.2.1 has the following corollary which gives a new proof in in [10] to a result proved first in [68].

Corollary 3.2.2. *Let (G, h) be a noncommutative nilpotent Lie group endowed with a left invariant Riemannian metric and $\mathfrak{g}$ its Lie algebra. Then the Ricci operator of (G, h) has at least two negative eigenvalues.*

3.3 Main result 2

Recall that a basis $(X_1,\ldots,X_n)$ of a nilpotent Lie algebra $\mathfrak{g}$ is called nice if:

1. For any i,j with $i\neq j$, $[X_i,X_j]=0$ or there exists k such that $[X_i,X_j]=C_{ij}^kX_k$ with $C_{ij}^k\neq 0$,

2. If $[X_i,X_j]=C_{ij}^kX_k$ and $[X_s,X_r]=C_{sr}^kX_k$ with $C_{ij}^k\neq 0$ and $C_{sr}^k\neq 0$ then $\{i,j\}\cap\{s,r\}=\emptyset$.

This notion appeared first in [64]. One of the most important property of a nice basis $\mathbb{B}$ is that any Euclidean inner product on $\mathfrak{g}$ for which $\mathbb{B}$ is orthogonal has its Ricci curvature diagonal in $\mathbb{B}$. The proof of Theorem 3.5.1 is based mainly on the fact that all the nilpotent Lie algebras of dimension less or equal to 6 have a nice basis except one. It is also known (see [72]) that any filiform $\mathbb{N}$-graded Lie algebra(see [69]) has a nice basis.

Theorem 3.3.1. *Let G be a nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a nice basis and $Z(\mathfrak{g})\subset[\mathfrak{g},\mathfrak{g}]$ with $\dim[\mathfrak{g},\mathfrak{g}]-\dim Z(\mathfrak{g})=1$. Then for any $(s^-,s^0,s^+)\in$ **Sign**$(\mathfrak{g})$ there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-,s^0,s^+).*

Proof. We have obviously $\mathbf{Sign}(\mathfrak{g})=\{(n-d+m^-,m^0,d-1+m^+):\ m^-+m^0+m^+=1\}$, where $d=\dim[\mathfrak{g},\mathfrak{g}]$ and $n=\dim\mathfrak{g}$. Note first that we can choose a nice basis $\mathbb{B}=(X_i)_{i=1}^n$ where $Z(\mathfrak{g})=\mathrm{span}\{X_i\}_{i=1}^{d-1}$ and $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{X_i\}_{i=1}^{d}$. Indeed, suppose that $\mathbb{B}=(X_i)_{i=1}^n$ with $[\mathfrak{g},\mathfrak{g}]=\mathrm{span}\{X_i\}_{i=1}^{d}$. Let $z=\sum_{i=1}^d a_iX_i\in Z(\mathfrak{g})$. Suppose that there exists $a_i\neq 0$ and $X_i\notin Z(\mathfrak{g})$. Then there exists $\ell\in\{1,\ldots,n\}$ such that $[X_\ell,X_i]\neq 0$. So we get $\sum_{j=1}^d a_j[X_\ell,X_j]=0$. From the properties of a nice basis we deduce that $\{[X_\ell,X_j],j=1,\ldots,d,\ [X_\ell,X_j]\neq 0\}$ is a linearly independent family and hence $a_i=0$. This shows that $\{X_i,\ X_i\in Z(\mathfrak{g})\}$ is basis of $Z(\mathfrak{g})$.

We consider the Euclidean product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ for which $\mathbb{B}$ is orthogonal and $a_i=\langle X_i,X_i\rangle$. It is obvious that $\mathbb{B}$ is a characteristic basis of $(\mathfrak{g},\langle\ ,\ \rangle)$ and it is also nice of $\mathfrak{g}$ so $\mathrm{R}(\mathrm{ric},\mathbb{B})$ is diagonal and hence $\mathrm{R}(\mathrm{ric},\mathbb{B})$ is also nice. According to Lemma 3.1.2, the reduced matrix has order 1 and is given by $\mathrm{R}(\mathrm{ric},\mathbb{B})=(2\mathrm{ric}(X_d,X_d))$. Moreover, the Ricci signature of $(\mathfrak{g},\langle\ ,\ \rangle)$ is given by $(n-d+m^-,m^0,d-1+m^+)$ where (m^-,m^0,m^+) is the signature of $\mathrm{R}(\mathrm{ric},\mathbb{B})$. To complete the proof, we will show that we can choose suitable a_i so that $\mathrm{ric}(X_d,X_d)$ can be either zero, positive or negative.

Denote by C_{ij}^k the constants structure of the Lie bracket in $\mathbb{B}$. The basis $(e_i)_{i=1}^n=(\frac{1}{\sqrt{a_i}}X_i)_{i=1}^n$ is an orthonormal basis of $\mathfrak{g}$ and from (1.6)

$$2\mathrm{ric}(X_d,X_d)\ =\ \sum_{i<j}\frac{(C_{ij}^d)^2a_d^2}{a_ia_j}-\sum_{i,j}\frac{(C_{id}^j)^2a_j}{a_i}. \tag{3.5}$$

Note that for any (i, j), such that $[X_i, X_j] = C_{ij}^d X_d$ with $C_{ij}^d \neq 0$, $i \neq d$ and $j \neq d$. Indeed, if $i = d$, we have $[X_d, X_j] = C_{dj}^d X_d$ and hence X_d is an eigenvector of ad_{X_j} with the real non zero eigenvalue $-C_{di}^d$ which is impossible since ad_{X_j} is nilpotent. We have also that if $[X_d, X_i] = C_{di}^j X_j$ with $C_{di}^j \neq 0$ then $i \neq d$ and $j \neq d$. So

$$\mathrm{ric}(X_d, X_d) = \alpha a_d^2 - \beta.$$

Now since $X_d \in [\mathfrak{g}, \mathfrak{g}] \setminus Z(\mathfrak{g})$, $\alpha > 0$, $\beta > 0$ and both α and β depend only on a_i with $i \neq d$. So we can choose a_d such that $\mathrm{ric}(X_d, X_d) = 0$, > 0 or < 0. This completes the proof. □

This Theorem 3.3.1 together with Theorem 3.2.1 solve Problem 1 for a large class of nilpotent Lie groups. Indeed, in the list of indecomposable seven-dimensional nilpotent Lie algebras given in [38] there are more than 35 ones satisfying the hypothesis of Theorem 3.3.1. On the other hand, we will point out the difficulty one can face when trying to generalize Theorem 3.3.1 when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. We will also give a method using the inverse function theorem to overcome this difficulty. Although, we have not yet succeeded to show that this method works in the general case, we will use it successfully in the proof of Theorem 3.5.1. We refer to this method as inverse function theorem trick. One can ask naturally if this theorem is still true when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. By looking to the proof given here, one can conjecture that the answer is true, it suffices to solve some systems of polynomial equations. This can be very difficult. To be precise, we will point out the difficulty one can face when trying to generalize Theorem 3.3.1 when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$. We will also give a method to overcome this difficulty.

3.4 Inverse function theorem trick

Suppose that $\mathfrak{g}$ is a nilpotent Lie algebra having a nice basis $\mathbb{B}$ and satisfying $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. Write $\mathbb{B} = (X_i)_{i=1}^n$ where $(X_i)_{i=1}^\ell$ is a basis of $Z(\mathfrak{g})$ and $(X_i)_{i=1}^d$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$. We have obviously

$$\mathbf{Sign}(\mathfrak{g}) = \left\{ (n - d + m^-, m^0, \ell + m^+) : \ m^- + m^0 + m^+ = d - \ell \right\}.$$

We consider the Euclidean product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle X_i, X_i \rangle$. It is clear that $\mathbb{B}$ is a characteristic basis of $(\mathfrak{g}, \langle\ ,\ \rangle)$ and it is also nice so $\mathrm{R}(\mathrm{ric}, \mathbb{B})$ is diagonal. According to Lemma 3.1.2, the reduced matrix has order $d - \ell$ and is given by

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \mathrm{diag}(2\mathrm{ric}(X_{\ell+1}, X_{\ell+1}), \ldots, 2\mathrm{ric}(X_d, X_d)).$$

Moreover, the signature is given by $(n-d+m^-, m^0, \ell+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. According to (3.5), for any $i = \ell+1, \ldots, d$, we can write in a unique way

$$2\mathrm{ric}(X_i, X_i) = \frac{F_{i-\ell}(a_1, \ldots, a_n)}{a_{n_1} \ldots a_{n_i}},$$

where $F_{i-\ell}$ is a homogeneous polynomial on $(a_1, \ldots, a_n)$. So to generalize Theorem 3.3.1 when $d - \ell \geq 2$, it suffices to find suitable values of $(a_1, \ldots, a_n)$ such that $(F_i(a_1, \ldots, a_n))_{i=1}^{d-\ell}$ have all the possible signs. It is very difficult in the general case. We give now a situation where we can conclude.

Suppose that there exists $(\alpha_1, \ldots, \alpha_n)$ such that $F_j(\alpha_1, \ldots, \alpha_n) = 0$ for $j = 1, \ldots, d - \ell$ and define

$$F : \{(x_1, \ldots, x_{d-\ell}) \in \mathbb{R}^{d-\ell}, x_i > 0\} \longrightarrow \mathbb{R}^{d-\ell} \quad \text{by :}$$

$$F(x_1, \ldots, x_{d-\ell}) = (F_j(\alpha_1, \ldots, \alpha_\ell, x_1, \ldots, x_{d-\ell}, \alpha_{d+1}, \ldots, \alpha_n), \quad j = 1, \ldots d - \ell.$$

We have $F(\alpha_{\ell+1}, \ldots, \alpha_d) = 0$ and if the differential $DF(\alpha_{\ell+1}, \ldots, \alpha_d)$ is invertible we can apply the inverse function theorem and hence F realizes a diffeomorphism from an open set centred at $(\alpha_{\ell+1}, \ldots, \alpha_d)$ into an open ball centred in $(0, , \ldots, 0)$. So, for a suitable choice of a_i, $\mathrm{R}(\mathrm{ric}, \mathbb{B})$ can have all the possible signatures.

So far we have shown that Theorem 3.3.1 is true when $\dim[\mathfrak{g}, \mathfrak{g}] - \dim Z(\mathfrak{g}) \geq 2$ if there exists $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$ satisfying $F_j(\alpha_1, \ldots, \alpha_n) = 0$ for $j = 1, \ldots, d - \ell$ and $\det DF(\alpha_{\ell+1}, \ldots, \alpha_d) \neq 0$.

Definition 3.4.1. *We call nice a nilpotent Lie algebra $\mathfrak{g}$ with $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and having a nice basis for which there exists $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$ satisfying $F_j(\alpha_1, \ldots, \alpha_n) = 0$ for $j = 1, \ldots, d-\ell$ and $\det DF(\alpha_{\ell+1}, \ldots, \alpha_d) \neq 0$.*

So, according to our study above, we have the following result.

Theorem 3.4.2. *Let G be a nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ is nice. Then for any $(s^-, s^0, s^+) \in$ **Sign**$(\mathfrak{g})$ there exists a left invariant Riemannian metric on G such that its Ricci signature is (s^-, s^0, s^+).*

Remark 3.4.3. *It seems reasonable to conjecture that any nilpotent Lie algebra $\mathfrak{g}$ with $Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and having a nice basis is actually nice.*

We give now two examples of nice nilpotent Lie algebras.

Example 3.4.4. *1. We consider the 7-dimensional nilpotent Lie algebra labelled* $(12457L1)$ *in [38] given by*

$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = -e_6,\ [e_1, e_6] = e_7,\ [e_2, e_3] = e_5,\ [e_2, e_5] = -e_6,$

$$[e_3, e_5] = -e_7.$$

We have $Z(\mathfrak{g}) = \{e_7\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5, e_6, e_7\}$ *and* $\mathbb{B} = (e_7, e_3, e_4, e_5, e_6, e_1, e_2)$ *is a nice basis. Let compute* $2\mathrm{ric}(e_i, e_i)$ *for* $i = 3, \ldots, 6$ *for the metric for which* $\mathbb{B}$ *is orthogonal with* $\langle e_i, e_i \rangle = a_i$. *By applying* (3.5), *we get*

$$\begin{aligned}
2\mathrm{ric}(e_3, e_3) &= \frac{a_3^2}{a_1 a_2} - \frac{a_4}{a_1} - \frac{a_5}{a_2} - \frac{a_7}{a_5} = \frac{a_3^2 a_5 - a_2 a_4 a_5 - a_1 a_5^2 - a_1 a_2 a_7}{a_1 a_2 a_5} = \frac{F_1(a_1, \ldots, a_7)}{a_1 a_2 a_5},\\
2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2}{a_1 a_3} - \frac{a_6}{a_1} = \frac{a_4^2 - a_3 a_6}{a_1 a_3} = \frac{F_2(a_1, \ldots, a_7)}{a_1 a_3},\\
2\mathrm{ric}(e_5, e_5) &= \frac{a_5^2}{a_2 a_3} - \frac{a_6}{a_2} - \frac{a_7}{a_3} = \frac{a_5^2 - a_3 a_6 - a_2 a_7}{a_2 a_3} = \frac{F_3(a_1, \ldots, a_7)}{a_2 a_3},\\
2\mathrm{ric}(e_6, e_6) &= \frac{a_6^2}{a_2 a_5} + \frac{a_6^2}{a_1 a_4} - \frac{a_7}{a_1} = \frac{(a_1 a_4 + a_2 a_5) a_6^2 - a_2 a_4 a_5 a_7}{a_1 a_2 a_4 a_5} = \frac{F_4(a_1, \ldots, a_7)}{a_1 a_2 a_4 a_5}.
\end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, a_2, a_3, ..., a_7]$ *and the ideal* I *generated by* $\{F_1, F_2, F_3, F_4\}$. *We take a lexicographic order* $>$ *with* $a_1 > a_2 > \ldots a_6 > a_7$ *of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal* I *contains the polynomial* $h(a_1, \ldots, a_7) = -a_4^2 + a_3 a_6$. *Thus, for* $a_3 = a_4 = a_6 = 1$, *the sequence* $\left(\frac{7}{240}, \frac{1127}{1200}, 1, 1, \frac{7}{5}, 1, \frac{1152}{1127}\right)$ *is a solution of the equations* $F_i(\alpha_1, \ldots, \alpha_7) = 0$ *for* $i = 1, \ldots, 4$ *and satisfies* $\det DF(\alpha_3, \alpha_4, \alpha_5, \alpha_6) \neq 0$ *and hence this Lie algebra is nice.*

2. *We consider the* $\mathbb{N}$*-graded filiform* n*-dimensional Lie algebra* $\mathfrak{m}_0(n) = \mathrm{span}\{X_1, \ldots, X_n\}$ *with the non vanishing Lie brackets* $[X_1, X_i] = X_{i+1}$, $i = 2, \ldots, n-1$. *We have*

$$\boldsymbol{Sign}(\mathfrak{m}_0(n)) = \left\{(2 + m^-, m^0, 1 + m^+),\ m^- + m^0 + m^+ = n - 3\right\}.$$

Let $\langle\ ,\ \rangle$ *be the Euclidean inner product on* $\mathfrak{m}_0(n)$ *for which* $(X_i)_{i=1}^n$ *is an orthogonal basis with* $\langle X_i, X_i \rangle = a_i$. *The basis* $\mathbb{B} = (X_n, X_3, \ldots, X_{n-1}, X_1, X_2)$ *is a characteristic basis of* $\langle\ ,\ \rangle$ *and* $\mathrm{R}(\mathrm{ric}, \mathcal{B}) = \mathrm{diag}(2\mathrm{ric}(X_i, X_i))_{i=3}^{n-1}$. *By using* (3.5), *we get for any* $i = 3, \ldots, n-1$

$$2\mathrm{ric}(X_i, X_i) = \frac{a_i^2}{a_1 a_{i-1}} - \frac{a_{i+1}}{a_1} = \frac{a_i^2 - a_{i-1} a_{i+1}}{a_1 a_{i-1}} = \frac{F_{i-2}(a_1, \ldots, a_n)}{a_1 a_{i-1}}.$$

It is obvious that $F_i(1, \ldots, 1) = 0$ *and* $\det DF(1, \ldots, 1) \neq 0$ *and hence* $\mathfrak{m}_0(n)$ *is nice.*

The results above, the tools we will use to establish them and the examples we will give support the following conjecture.

Conjecture 1. *Let* G *be a nilpotent Lie group and* $\mathfrak{g} = T_e G$ *its Lie algebra. Then, for any* $(s^-, s^0, s^+) \in \boldsymbol{Sign}(\mathfrak{g})$, *there exists a left invariant Riemannian metric on* G *for which the Ricci operator has signature* (s^-, s^0, s^+).

3.5 Main result 3

Theorem 3.2.1 gives a candidate to be the set of all the signatures of the Ricci operators of all left invariant Riemannian metrics on a nilpotent Lie group. Indeed, our second main result together with Theorem 3.2.1 solve Problem 1 completely for nilpotent Lie groups up to dimension 6.

Theorem 3.5.1. *Let G be a nilpotent Lie group of dimension ≤ 6 and $\mathfrak{g}$ its Lie algebra. Then, for any $(s^-, s^0, s^+) \in \textbf{\textit{Sign}}(\mathfrak{g})$, there exists a left invariant Riemannian metric on G for which the Ricci operator has signature (s^-, s^0, s^+).*

According to Theorems 3.2.1 and 3.5.1, it reduces to computing $\mathbf{Sign}(\mathfrak{g})$ for any nilpotent Lie algebra of dimension less or equal to 6. We will use the classification of 5-dimensional and 6-dimensional nilpotent Lie algebras given by Willem A. de Graaf in [22].

Proof. The proof goes as follows. There are, up to an isomorphism, 44 non abelian nilpotent Lie algebras of dimension less or equal to 6: 1 of dimension 3, 2 of dimension 4, 8 of dimension 5 and 33 of dimension 6 (see Tables 1 and 2). Among these Lie algebras, 12 are 2-step nilpotent and we can apply Corollary 3.1.1, 10 satisfy the hypothesis of Theorem 3.3.1 and 15 are nice in the sense of Definition 3.4.1 and we can apply Theorem 3.4.2. At the end, we are left with 7 Lie algebras needing each of them a special treatment.

The Lie algebras $L_{3,2}$, $L_{4,2}$, $L_{5,2}$, $L_{5,4}$, $L_{5,8}$, $L_{6,2}$, $L_{6,4}$ $L_{6,8}$, $L_{6,22}(\epsilon)$, $L_{6,26}$ are obviously 2-step nilpotent and we can apply Corollary 3.1.1.

The Lie algebras $L_{4,3}$, $L_{5,5}$, $L_{5,9}$, $L_{6,10}$, $L_{6,19}(0)$, $L_{6,23}$, $L_{6,24}(\epsilon)$ and $L_{6,25}$ satisfy clearly the hypothesis of Theorem 3.3.1.

We will show now that the Lie algebras $L_{5,6}$, $L_{5,7}$, $L_{6,12}$, $L_{6,13}$, $L_{6,14}$, $L_{6,15}$, $L_{6,16}$, $L_{6,17}$, $L_{6,18}$, $L_{6,19}(\epsilon \neq 0)$, $L_{6,20}$, $L_{6,21}(0)$ and $L_{6,21}(\epsilon \neq 0)$ are nice in the sense of Definition 3.4.1 so that we can apply Theorem 3.3.1.

- **The Lie algebra** $L_{5,6}$.

We have $L_{5,6} = \mathrm{span}\{e_1, \dots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_3] = e_5$$

We have $Z(\mathfrak{g}) = \{e_5\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5\}$ and $\mathbb{B} = (e_5, e_3, e_4, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_3, e_3) &= \frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2} = \frac{F_1(a_1, \dots, a_5)}{a_1 a_2}, \\ 2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2 - a_3 a_5}{a_1 a_3} = \frac{F_2(a_1, \dots, a_5)}{a_1 a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_5]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_5$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_1(a_3, a_4, a_5) = -a_4^2 + a_3a_5$. Thus, for $a_3 = a_4 = a_5 = 1$, the sequence $\left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_5) = 0$ for $i = 1, 2$ and satisfies $\det DF(\alpha_3, \alpha_4) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{5,7}$.

We have $L_{5,7} = \text{span}\{e_1, \dots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5$$

We have $Z(\mathfrak{g}) = \{e_5\} \subset [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_3, e_4, e_5\}$ and $\mathbb{B} = (e_5, e_3, e_4, e_1, e_2)$ is a nice basis. Let compute $2\text{ric}(e_i, e_i)$ for $i = 3, 4$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\text{ric}(e_3, e_3) &= \frac{a_3^2 - a_2a_4}{a_1a_2} = \frac{F_1(a_1, \dots, a_5)}{a_1a_2}, \\ 2\text{ric}(e_4, e_4) &= \frac{a_4^2 - a_3a_5}{a_1a_3} = \frac{F_2(a_1, \dots, a_5)}{a_1a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_5]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_5$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_2(a_3, a_4, a_5) = -a_4^2 + a_3a_5$. Thus, for $a_3 = a_4 = a_5 = 1$, the sequence $(1, 1, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_5) = 0$ for $i = 1, 2$ and satisfies $\det DF(\alpha_3, \alpha_4) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,12}$.

We have $L_{6,12} = \text{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6,\ [e_2, e_5] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_3, e_4, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_1, e_2, e_5)$ is a nice basis. Let compute $2\text{ric}(e_i, e_i)$ for $i = 3, 4$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\text{ric}(e_3, e_3) &= \frac{a_3^2 - a_2a_4}{a_1a_2} = \frac{F_1(a_1, \dots, a_6)}{a_1a_2}, \\ 2\text{ric}(e_4, e_4) &= \frac{a_4^2 - a_3a_6}{a_1a_3} = \frac{F_2(a_1, \dots, a_6)}{a_1a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_6]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_3(a_3, a_4, a_6) = -a_4^2 + a_3a_6$. Thus, for $a_3 = a_4 = a_6 = 1$, the sequence $(1, 1, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_6) = 0$ for $i = 1, 2$ and satisfies $\det DF(\alpha_3, \alpha_4) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,13}$.

We have $L_{6,13} = \operatorname{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_5,\ [e_2, e_4] = e_5,\ [e_1, e_5] = e_6,\ [e_3, e_4] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{e_3, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_5, e_1, e_2, e_4)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned}
2\mathrm{ric}(e_3, e_3) &= (\frac{a_4a_3^2 - a_2a_4a_5 - a_1a_2a_6}{a_1a_2a_4} = \frac{F_1(a_1, \dots, a_6)}{a_1a_2a_4},\\
2\mathrm{ric}(e_4, e_4) &= \frac{(a_2a_4 + a_1a_3)a_5^2 - a_2a_3a_4a_6}{a_1a_2a_3a_4} = \frac{F_2(a_1, \dots, a_6)}{a_1a_2a_3a_4}
\end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_6]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > \dots > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_4(a_1, \dots, a_6) = -a_4a_3^2 + a_2a_4a_5 + a_1a_2a_6$. Thus, for $a_1 = a_3 = a_4 = a_5 = 1$, the sequence $(1, 2, 2, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_6) = 0$ for $i = 1, 2$ and satisfies $\det DF(\alpha_3, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,14}$.

We have $L_{6,14} = \operatorname{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_3] = e_5,\ [e_2, e_5] = e_6,\ [e_3, e_4] = -e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \operatorname{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_5, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned}
2\mathrm{ric}(e_3, e_3) &= (\frac{a_4a_3^2 - a_2a_4^2 - a_1a_4a_5 - a_1a_2a_6}{a_1a_2a_4} = \frac{F_1(a_1, \dots, a_6)}{a_1a_2a_4},\\
2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2 - a_3a_5 - a_1a_6}{a_1a_3} = \frac{F_2(a_1, \dots, a_6)}{a_1a_3}\\
2\mathrm{ric}(e_5, e_5) &= \frac{(a_2a_3 + a_1a_4)a_5^2 - a_1a_3a_4a_6}{a_1a_2a_3a_4} = \frac{F_2(a_1, \dots, a_6)}{a_1a_2a_3a_4}
\end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \ldots, a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_5(a_1, a_3, a_4, a_5, a_6) = -a_4^2 + a_3a_5 + a_1a_6$. Thus, for $a_3 = 1, a_4 = 3, a_5 = 5$, the sequence $\left(\frac{27}{200}, \frac{3}{40}, 1, 3, 5, \frac{800}{27}\right)$ is a solution of the equations $F_i(\alpha_1, \ldots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_3, \alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,15}$.

We have $L_{6,15} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_3] = e_5,\ [e_2, e_4] = e_6\ [e_1, e_5] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_5, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned}
2\mathrm{ric}(e_3, e_3) &= \frac{a_3^2 - a_2a_4 - a_1a_5}{a_1a_2} = \frac{F_1(a_1, \ldots, a_6)}{a_1a_2}, \\
2\mathrm{ric}(e_4, e_4) &= \frac{a_2a_4^2 - a_2a_3a_5 - a_1a_3a_6}{a_1a_2a_3} = \frac{F_2(a_1, \ldots, a_6)}{a_1a_2a_3} \\
2\mathrm{ric}(e_5, e_5) &= \frac{(a_2a_3 + a_1a_4)a_5^2 - a_2a_3a_4a_6}{a_1a_2a_3a_4} = \frac{F_2(a_1, \ldots, a_6)}{a_1a_2a_3a_4}
\end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \ldots, a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_6(a_3, \ldots, a_6) = -a_3^2a_4a_5 + a_3^3a_6$. Thus, for $a_4 = 2, a_5 = a_6 = 1$, the sequence $\left(\frac{4}{3}, \frac{4}{3}, 2, 2, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1, \ldots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_3, \alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,16}$.

We have $L_{6,16} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_2, e_5] = e_6,\ [e_3, e_4] = -e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_5, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4, 5$ for the metric for which

$\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_3,e_3) &= \frac{a_4a_3^2 - a_2a_4^2 - a_1a_2a_6}{a_1a_2a_4} = \frac{F_1(a_1,\dots,a_6)}{a_1a_2a_4}, \\ 2\mathrm{ric}(e_4,e_4) &= \frac{a_4^2 - a_3a_5 - a_1a_6}{a_1a_2a_3} = \frac{F_2(a_1,\dots,a_6)}{a_1a_2a_3} \\ 2\mathrm{ric}(e_5,e_5) &= \frac{a_2a_5^2 - a_1a_4a_6}{a_1a_2a_4} = \frac{F_3(a_1,\dots,a_6)}{a_1a_2a_4} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1,\dots,a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_7(a_1,a_3,a_4,a_5,a_6) = -a_4^2 + a_3a_5 + a_1a_6$. Thus, for $a_4 = a_5 = a_6 = 1$, the sequence $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1,\dots,\alpha_6) = 0$ for $i = 1,2,3$ and satisfies $\det DF(\alpha_3,\alpha_4,\alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,17}$.

We have $L_{6,17} = \mathrm{span}\{e_1,\dots,e_6\}$ with the non vanishing Lie brackets

$$[e_1,e_2] = e_3,\ [e_1,e_3] = e_4,\ [e_1,e_4] = e_5, [e_1,\ e_5] = e_6,\ [e_2,e_3] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{e_3,e_4,e_5,e_6\}$ and $\mathbb{B} = (e_6,e_3,e_4,e_5,e_1,e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i,e_i)$ for $i = 3,4,5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_3,e_3) &= \frac{a_3^2 - a_2a_4 - a_1a_6}{a_1a_2} = \frac{F_1(a_1,\dots,a_6)}{a_1a_2}, \\ 2\mathrm{ric}(e_4,e_4) &= \frac{a_4^2 - a_3a_5}{a_1a_3} = \frac{F_2(a_1,\dots,a_6)}{a_1a_3} \\ 2\mathrm{ric}(e_5,e_5) &= \frac{a_5^2 - a_4a_6}{a_1a_4} = \frac{F_2(a_1,\dots,a_6)}{a_1a_4} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1,\dots,a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_8(a_3,a_4,a_5) = -a_4^2 + a_3a_5$. Thus, for $a_3 = a_4 = a_5 = 1$, the sequence $\left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1,\dots,\alpha_6) = 0$ for $i = 1,2,3$ and satisfies $\det DF(\alpha_3,\alpha_4,\alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,18}$.

We have $L_{6,18} = \mathrm{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_5,\ [e_1, e_5] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_5, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_3, e_3) &= \frac{a_3^2 - a_2a_4}{a_1a_2} = \frac{F_1(a_1, \dots, a_6)}{a_1a_2}, \\ 2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2 - a_3a_5}{a_1a_3} = \frac{F_2(a_1, \dots, a_6)}{a_1a_3} \\ 2\mathrm{ric}(e_5, e_5) &= \frac{a_5^2 - a_4a_6}{a_1a_4} = \frac{F_2(a_1, \dots, a_6)}{a_1a_4} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_9(a_4, a_5, a_6) = -a_5^2 + a_4a_6$. Thus, for $a_4 = a_5 = a_6 = 1$, the sequence $(1, 1, 1, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_3, \alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,19}(\epsilon \neq 0)$.

We have $L_{6,19}(\epsilon \neq 0) = \mathrm{span}\{e_1, \dots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_4,\ [e_1, e_3] = e_5,\ [e_2, e_4] = e_6,\ [e_3, e_5] = \epsilon e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_4, e_5, e_1, e_2, e_3)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i\rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2 - a_1a_6}{a_1a_2} = \frac{F_1(a_1, \dots, a_6)}{a_1a_2}, \\ 2\mathrm{ric}(e_5, e_5) &= \frac{a_5^2 - a_1a_6}{a_1a_3} = \frac{F_2(a_1, \dots, a_6)}{a_1a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \dots, a_6]$ and the ideal I generated by $\{F_1, F_2, F_3\}$. We take a lexicographic order $>$ with $a_1 > a_2 > ... > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_{10}(a_4, a_5) = -a_5^2 + a_4^2$. Thus, for $a_4 = a_5 = 1$, the sequence $(1, 1, 1, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \dots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_3, \alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,20}$.

We have $L_{6,20} = \text{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_4, \ [e_1, e_3] = e_5, \ [e_1, e_5] = e_6, \ [e_2, e_4] = e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_4, e_5, e_1, e_2, e_3)$ is a nice basis. Let compute $2\text{ric}(e_i, e_i)$ for $i = 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\text{ric}(e_4, e_4) &= \frac{a_4^2 - a_1 a_6}{a_1 a_2} = \frac{F_1(a_1, \ldots, a_6)}{a_1 a_2}, \\ 2\text{ric}(e_5, e_5) &= \frac{a_5^2 - a_3 a_6}{a_1 a_3} = \frac{F_2(a_1, \ldots, a_6)}{a_1 a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \ldots, a_6]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > \ldots > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_{11}(a_3, a_5, a_6) = -a_5^2 + a_3 a_6$. Thus, for $a_3 = a_5 = a_6 = 1$, the sequence $(1, 1, 1, 1, 1, 1)$ is a solution of the equations $F_i(\alpha_1, \ldots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,21}(0)$.

We have $L_{6,21}(0) = \text{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5, \ [e_1, e_4] = e_6$$

We have $Z(\mathfrak{g}) = \{e_5, e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \text{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_5, e_6, e_3, e_4, e_1, e_2)$ is a nice basis. Let compute $2\text{ric}(e_i, e_i)$ for $i = 3, 4$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\text{ric}(e_3, e_3) &= \frac{a_3^2 - a_2 a_4 - a_1 a_5}{a_1 a_2} = \frac{F_1(a_1, \ldots, a_6)}{a_1 a_2}, \\ 2\text{ric}(e_4, e_4) &= \frac{a_4^2 - a_3 a_6}{a_1 a_3} = \frac{F_2(a_1, \ldots, a_6)}{a_1 a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \ldots, a_6]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > \ldots > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_{11}(a_3, a_4, a_6) = -a_4^2 + a_3 a_6$. Thus, for $a_3 = 2, a_5 = a_6 = 1$, the sequence $\left(2, \sqrt{2}, 2, \sqrt{2}, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1, \ldots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

- **The Lie algebra** $L_{6,21}(\epsilon \neq 0)$.

We have $L_{6,21}(\epsilon \neq 0) = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_4] = e_6,\ [e_2, e_5] = \epsilon e_6$$

We have $Z(\mathfrak{g}) = \{e_6\} \subset [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{e_3, e_4, e_5, e_6\}$ and $\mathbb{B} = (e_6, e_3, e_4, e_5, e_1, e_2)$ is a nice basis. Let compute $2\mathrm{ric}(e_i, e_i)$ for $i = 3, 4, 5$ for the metric for which $\mathbb{B}$ is orthogonal with $\langle e_i, e_i \rangle = a_i$. By applying (3.5), we get

$$\begin{aligned} 2\mathrm{ric}(e_3, e_3) &= \frac{a_3^2 - a_1 a_4 - a_2 a_5}{a_1 a_2} = \frac{F_1(a_1, \ldots, a_6)}{a_1 a_2}, \\ 2\mathrm{ric}(e_4, e_4) &= \frac{a_4^2 - a_3 a_6}{a_1 a_3} = \frac{F_1(a_1, \ldots, a_6)}{a_1 a_3}, \\ 2\mathrm{ric}(e_5, e_5) &= \frac{a_5^2 - a_3 a_6}{a_2 a_3} = \frac{F_2(a_1, \ldots, a_6)}{a_2 a_3} \end{aligned}$$

We consider a polynomial ring $\mathbb{Q}[a_1, \ldots, a_6]$ and the ideal I generated by $\{F_1, F_2\}$. We take a lexicographic order $>$ with $a_1 > a_2 > \ldots > a_6$ of monomial order. Then by the aid of computer, we see that a Gröbner basis for the ideal I contains the polynomial $h_{12}(a_4, a_5) = a_4^2 - a_5^2$. Thus, for $a_3 = a_4 = a_5 = 1$, the sequence $\left(\frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1\right)$ is a solution of the equations $F_i(\alpha_1, \ldots, \alpha_6) = 0$ for $i = 1, 2, 3$ and satisfies $\det DF(\alpha_4, \alpha_5) \neq 0$ and hence this Lie algebra is nice.

To complete the proof, we treat now the seven remaining Lie algebras using a case by case approach.

- **The Lie algebra** $L_{6,11}$.

This is the only Lie algebra in the list which has no nice basis. Its center is contained in its derived ideal. We have $L_{6,11} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_1, e_4] = e_6, [e_2, e_3] = e_6,\ [e_2, e_5] = e_6$$

and $\mathbf{Sign}(\mathfrak{g}) = \{(3 + m^-, m^0, 1 + m^+), m^- + m^0 + m^+ = 2\}$. We consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,11}$ such that $\mathbb{B} = (e_6, e_3, e_4, e_1, e_2, e_5)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$. It is obvious that $\mathbb{B}$ is an orthogonal characteristic basis and, according to Lemma 3.1.2, the signature of $\langle\ ,\ \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of the characteristic matrix $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (1.6) and (3.4) gives

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \begin{pmatrix} \frac{a_3^2 - a_2 a_4}{a_1 a_2} & 0 \\ 0 & \frac{a_4^2 - a_3 a_6}{a_1 a_3} \end{pmatrix}.$$

If we take $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ we get $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = 0$ and we can use the inverse function theorem trick. So, for a suitable choice of a_i, $\mathrm{R}(\mathrm{ric}, \mathbb{B})$ can have all the possible signatures which prove the theorem for $L_{6,11}$.

- **The Lie algebra** $L_{5,3}$.

We have $L_{5,3} = \mathrm{span}\{e_1, \ldots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4,$$

and $\mathbf{Sign}(\mathfrak{g}) = \{(2,1,2), (2,2,1), (3,0,2), (3,1,1), (4,0,1)\}$. In this case, the parameter p in (3.1) has two values $p = 0$ or 1, so to realize the signatures in $\mathbf{Sign}(\mathfrak{g})$, we will consider two types of Euclidean inner products on $L_{5,3}$. The first ones are those satisfying $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^{\perp}) = 1$ and the second ones are those satisfying $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^{\perp}) = 0$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{5,3}$ for which $\mathbb{B} = (e_4, e_3, e_5, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i \rangle$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(2 + m^-, 1 + m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (3.5) gives $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = (2i\mathrm{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}\right)$ and, for suitable values of the a_i, the Ricci signatures of $\langle\ ,\ \rangle$ are $(2,1,2)$, $(2,2,1)$ or $(3,1,1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5) = (e_4, e_3, e_5 + e_3 + e_1, e_1, e_2)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_3] = -f_1,\ [f_2, f_4] = -f_1,\ [f_3, f_4] = -f_1,\ [f_3, f_5] = f_2,\ [f_4, f_5] = f_2.$$

Consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{5,3}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle f_i, f_i \rangle$. We have chosen $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^{\perp} = \{0\}$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, which is by the way a (1×1)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, its value when $a_1 = a_4 = a_5 = 1$ and $a_3 = 2$ will suffice to our purpose. We get

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \left(\frac{12a_2^4 + 6a_2^3 + 9a_2^2 - a_2 - 3}{4(2a_2^2 + a_2 + 2)}\right).$$

It is clear that we can choose a_2 such that the signature of $\langle\ ,\ \rangle$ is $(3,0,2)$ or $(4,0,1)$. This completes the proof for $L_{5,3}$.

• **The Lie algebra** $L_{6,3}$.

The treatment is similar to $L_{5,3}$ with a slight difference, the parameter p takes 1 or 2. We have $L_{6,3} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4,$$

and $\mathbf{Sign}(\mathfrak{g}) = \{(2,2,2), (2,3,1), (3,1,2), (3,2,1), (4,1,1)\}$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,3}$ for which $\mathbb{B} = (e_4, e_3, e_5, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i\rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^{\perp}) = 2$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(2+m^-, 2+m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (3.5) gives $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = (2\mathrm{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2 a_4}{a_1 a_2}\right)$ and, for suitable values of the a_i, the Ricci signatures of $\langle\ ,\ \rangle$ are $(2,2,2)$, $(2,3,1)$ or $(3,2,1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_4, e_3, e_5, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_4] = -f_1,\ [f_2, f_6] = -f_1,\ [f_4, f_5] = f_2,\ [f_4, f_6] = f_1,\ [f_5, f_6] = -f_2.$$

Consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,3}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle f_i, f_i\rangle$. We have chosen $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^{\perp}) = 1$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-, 1+m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, which is by the way a (1×1)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, its value when $a_1 = a_3 = a_4 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \left(\frac{-4a_2^5 + 2a_2^3 + 3a_2 - 2}{1 - a_2 - 2a_2^3}\right).$$

It is clear that we can choose a_2 such that the signature of $\langle\ ,\ \rangle$ is $(3,1,2)$ or $(4,1,1)$. This completes the proof for $L_{6,3}$.

• **The Lie algebra** $L_{6,5}$.

The treatment is similar to $L_{5,3}$. We have $L_{6,5} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5$$

and $\mathbf{Sign}(\mathfrak{g}) = \{(3,1,2), (3,2,1), (4,0,2), (4,1,1), (5,0,1)\}$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,5}$ for which $\mathbb{B} = (e_5, e_3, e_6, e_1, e_2, e_4)$ is orthogonal with $a_i = \langle e_i, e_i\rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-, 1+m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (3.5) gives $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = (2\mathrm{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2a_5}{a_1a_2}\right)$ and, for suitable values of the a_i, the Ricci signatures of $\langle\ ,\ \rangle$ are $(3, 1, 2)$, $(3, 2, 1)$ or $(4, 1, 1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_1, e_2, e_4, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_2, f_3] = -f_1,\ [f_2, f_6] = -f_1,\ [f_3, f_4] = f_2,\ [f_3, f_6] = f_1,\ [f_4, f_5] = f_1,\ [f_4, f_6] = -f_2.$$

Consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,5}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle f_i, f_i\rangle$. We have chosen $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(4 + m^-, m^0, 1 + m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, which is by the way a (1×1)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, its value when $a_1 = a_3 = a_4 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \left(\frac{4a_2^6 + 6a_2^5 + 6a_2^4 - a_2^3 - 3a_2^2 - 3a_2 - 1}{a_2(2a_2^3 + 3a_2^3 + 2a_2 + 1)}\right).$$

It is clear that we can choose a_2 such that the signature of $\langle\ ,\ \rangle$ is $(4, 0, 2)$ or $(5, 0, 1)$. This completes the proof for $L_{6,5}$.

- **The Lie algebra** $L_{6,9}$.

The treatment is similar to $L_{5,3}$ and $L_{6,5}$. We have $L_{6,9} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5,$$

and $\mathbf{Sign}(\mathfrak{g}) = \{(2, 1, 3), (2, 2, 2), (3, 0, 3), (3, 1, 2), (4, 0, 2)\}$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,9}$ for which $\mathbb{B} = (e_5, e_4, e_3, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i\rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(2+m^-, 1+m^0, 2+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (3.5) gives $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = (2\mathrm{ric}(e_3, e_3)) = \left(\frac{a_3^2 - a_2(a_4 + a_5)}{a_1a_2}\right)$ and, for suitable values of the a_i, the Ricci signatures of $\langle\ ,\ \rangle$ are $(2, 1, 3)$, $(2, 2, 2)$ or $(3, 1, 2)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_4, e_3, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing Lie brackets in this basis are

$$[f_3, f_4] = -f_2,\ [f_3, f_5] = -f_1,\ [f_3, f_6] = -f_2,\ [f_4, f_5] = f_3,\ [f_4, f_6] = f_2,$$

$$[f_5, f_6] = f_1 - f_3.$$

Consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,9}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle f_i, f_i\rangle$. We have chosen $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp = \{0\}$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3 + m^-, m^0, 2 + m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, which is by the way a (1×1)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, its value when $a_1 = a_2 = a_3 = a_5 = a_6 = 1$ will suffice to our purpose. We get

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \left(\frac{12 - a_4 - 35a_4^2}{2(8a_4 + 3)a_4}\right).$$

It is clear that we can choose a_4 such that the signature of $\langle\ ,\ \rangle$ is $(3, 0, 3)$ or $(4, 0, 2)$. This completes the proof for $L_{6,9}$.

- **The Lie algebra** $L_{6,6}$.

The situation here is different from the precedent cases. We still have two types of Euclidean products($p \in \{0, 1\}$) but the order of the reduced matrix of the Ricci curvature is 2. We have $\mathfrak{g} = \mathrm{span}\{e_1, \ldots, e_6\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5$$

Sign$(\mathfrak{g}) = \{(2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 0, 3), (3, 1, 2), (3, 2, 1), (4, 0, 2), (4, 1, 1), (5, 0, 1)\}$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,6}$ for which $\mathbb{B} = (e_5, e_3, e_4, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i\rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]^\perp) = 1$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(2+m^-, 1+m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \mathrm{diag}(2\mathrm{ric}(e_3, e_3), 2\mathrm{ric}(e_4, e_4))$. Now a direct computation using (1.6) gives

$$2\mathrm{ric}(e_3, e_3) = \frac{a_3^2 - a_2a_4 - a_1a_5}{a_1a_2} \quad \text{and} \quad 2\mathrm{ric}(e_4, e_4) = \frac{a_4^2 - a_3a_5}{a_1a_3}.$$

If we take $a_1 = 6, a_2 = 5, a_3 = 4, a_4 = 2, a_5 = a_6 = 1$, we get $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = 0$ and we can apply the inversion theorem trick to get that for a suitable choice of the a_i the Ricci signature of $\langle\ ,\ \rangle$ is $(2,1,3)$, $(2,2,2)$,$(2,3,1)$, $(3,1,2)$, $(3,2,1)$ or $(4,1,1)$.

For the second type, we consider the basis $\mathbb{B}=(f_1,f_2,f_3,f_4,f_5,f_6)$ and the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,6}$ for which $\mathbb{B}$ is orthogonal and $a_i=\langle f_i,f_i\rangle$. We choose $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $Z(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}]^\perp=\{0\}$.

- $\mathbb{B}=(f_1,f_2,f_3,f_4,f_5,f_6)=(e_5,e_3,e_4,e_1,e_2,e_6+e_3)$. The non vanishing Lie brackets in this basis are

  $$[f_2,f_4]=-f_3,\ [f_2,f_5]=-f_1,\ [f_3,f_4]=-f_1,\ [f_4,f_5]=f_2,\ [f_4,f_6]=f_3,\ [f_5,f_6]=f_1.$$

  Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and is not nice. Then, according to Lemma 3.1.2, the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-,m^0,1+m^+)$ where (m^-,m^0,m^+) is the signature of $\mathrm{R}(\mathrm{ric},\mathbb{B})$. Now, a direct computation using (1.6) and (3.4) gives

  $$\mathrm{R}(\mathrm{ric},\mathbb{B})=\mathrm{diag}\left(\frac{a_2^2}{a_4a_5},\frac{-a_1a_2a_6+a_3^2\,(a_2+a_6)}{a_2a_4a_6}\right).$$

  Thus, for suitable values of a_i, the signatures $(3,0,3)$ and $(4,0,2)$ are realizable as the Ricci signature of $\langle\ ,\ \rangle$.

- $\mathbb{B}=(f_1,f_2,f_3,f_4,f_5,f_6)=(e_5,e_3,e_4,e_1,e_2,e_6+e_3+e_1)$. The non vanishing Lie brackets are

  $$[f_2,f_4]=-f_3,\ [f_2,f_5]=-f_1,\ [f_2,f_6]=-f_3,\ [f_3,f_4]=-f_1,\ [f_3,f_6]=-f_1,[f_4,f_5]=f_2,$$

  $$[f_4,f_6]=f_3,\ [f_5,f_6]=-f_2+f_1.$$

  Then $\mathbb{B}$ is a characteristic for $\langle\ ,\ \rangle$. According to Lemma 3.1.2, the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-,m^0,1+m^+)$ where (m^-,m^0,m^+) is the signature of $\mathrm{R}(\mathrm{ric},\mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric},\mathbb{B})$, which is by the way a (2×2)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric},\mathbb{B})$, its value when $a_1=3,a_2=a_4=a_5=2=a_6=1$ will suffice to our purpose. We get

  $$\mathrm{R}(\mathrm{ric},\mathbb{B})=\mathrm{diag}\left(-\frac{18+66a_3+121a_3^2+120a_3^3+73a_3^4+24a_3^5}{18+36a_3+34a_3^2+22a_3^3+6a_3^4},\frac{-57+8a_3^2}{8}\right)$$

  It is clear that for suitable values of a_3, the signature $(5,0,1)$ is realizable as the Ricci signature of $\langle\ ,\ \rangle$. This completes the proof for $L_{6,6}$

- **The Lie algebra** $L_{6,7}$.

The treatment is similar to $L_{6,6}$. We have $\mathfrak{g} = \mathrm{span}\{e_1, \dots, e_5\}$ with the non vanishing Lie brackets

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5$$

Sign$(\mathfrak{g}) = \{(2,1,3),(2,2,2),(2,3,1),(3,0,3),(3,1,2),(3,2,1),(4,0,2),(4,1,1),(5,0,1)\}$.

For the first type, consider the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,7}$ for which $\mathbb{B} = (e_5, e_3, e_4, e_6, e_1, e_2)$ is orthogonal with $a_i = \langle e_i, e_i\rangle$ and $\dim(Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp) = 1$. Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and it is also nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(2+m^-, 1+m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \mathrm{diag}(2\mathrm{ric}(e_3, e_3), 2\mathrm{ric}(e_4, e_4))$. Now a direct computation using (1.6) gives

$$2\mathrm{ric}(e_3, e_3) = \frac{a_3^2 - a_2 a_4}{a_1 a_2} \quad \text{and} \quad 2\mathrm{ric}(e_4, e_4) = \frac{a_4^2 - a_3 a_5}{a_1 a_3}.$$

If we take $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$ we get $\mathrm{R}(\mathrm{ric}, \mathbb{B}) = 0$ and we can apply the inversion theorem trick to get that for a suitable choice of the a_i the Ricci signature of $\langle\ ,\ \rangle$ is $(2,1,3)$, $(2,2,2)$,$(2,3,1)$, $(3,1,2)$, $(3,2,1)$ or $(4,1,1)$.

For the second type, we consider the basis $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6)$ and the Euclidean inner product $\langle\ ,\ \rangle$ on $L_{6,7}$ for which $\mathbb{B}$ is orthogonal and $a_i = \langle f_i, f_i\rangle$. We choose $\mathbb{B}$ and $\langle\ ,\ \rangle$ such that $Z(\mathfrak{g}) \cap [\mathfrak{g},\mathfrak{g}]^\perp = \{0\}$.

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3)$. The non vanishing Lie brackets in this basis are

  $$[f_2, f_4] = -f_3,\ [f_3, f_4] = -f_1,\ [f_4, f_5] = f_2,\ [f_4, f_6] = f_3.$$

  Then $\mathbb{B}$ is a characteristic basis for $\langle\ ,\ \rangle$ and is not nice. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-, m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Now a direct computation using (1.6) and (3.4) gives

  $$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \mathrm{diag}\left(\frac{a_2^2}{a_4 a_5}, \frac{a_2 a_3^2 + (-a_1 a_2 + a_3^2)\, a_6}{a_2 a_4 a_6}\right).$$

  Thus for suitable values of a_i, the signatures $(3,0,3)$ and $(4,0,2)$ are realizable as the Ricci signature of $\langle\ ,\ \rangle$.

- $\mathbb{B} = (f_1, f_2, f_3, f_4, f_5, f_6) = (e_5, e_3, e_4, e_1, e_2, e_6 + e_3 + e_1)$. The non vanishing brackets are

  $$[f_2, f_4] = -f_3,\ [f_2, f_6] = -f_3,\ [f_3, f_4] = -f_1,\ [f_3, f_6] = -f_1,\ [f_4, f_5] = f_2,$$

$$[f_4, f_6] = f_3,\ [f_5, f_6] = -f_2.$$

Then $\mathbb{B}$ is a characteristic for $\langle\ ,\ \rangle$. Then according to Lemma 3.1.2 the Ricci signature of $\langle\ ,\ \rangle$ is $(3+m^-, m^0, 1+m^+)$ where (m^-, m^0, m^+) is the signature of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$. Here the situation is more complicated than the first case because $\mathbb{B}$ is not a nice basis and the computation of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, which is by the way a (2×2)-matrix, is complicated according to formula (3.4). We don't need to give the general expression of $\mathrm{R}(\mathrm{ric}, \mathbb{B})$, its value when $a_1 = 2, a_2 = a_3 = a_4 = a_6 = 1$ will suffice to our purpose. We get

$$\mathrm{R}(\mathrm{ric}, \mathbb{B}) = \mathrm{diag}\left(\frac{8 + 17a_5 - 12a_5^2}{4(1 + 3a_5)a_5}, -2\right)$$

It is clear that we can choose a_5 such that the signature of $\langle\ ,\ \rangle$ is $(5, 0, 1)$. This completes the proof for $L_{6,7}$.

$\square$

We end this work by giving all the realizable Ricci signatures on nilpotent Lie groups up to dimension 6.

Lie algebra $\mathfrak{g}$	Realizable Ricci signatures
$L_{3,2}$	$(2,0,1)$
$L_{4,2}$	$(2,1,1)$
$L_{4,3}$	$(2,1,1),(2,0,2),(3,0,1)$
$L_{5,2}$	$(2,2,1)$
$L_{5,3}$	$(2,1,2),(2,2,1),(3,0,2),(3,1,1),(4,0,1)$
$L_{5,4}$	$(4,0,1)$
$L_{5,5}$	$(3,0,2),(3,1,1),(4,0,1)$
$L_{5,6},L_{5,7}$	$(2,0,3),(2,1,2),(2,2,1),(3,0,2),(3,1,1),(4,0,1)$
$L_{5,8}$	$(3,0,2)$
$L_{5,9}$	$(2,0,3),(2,1,2),(3,0,2)$
$L_{6,2}$	$(2,3,1)$
$L_{6,3}$	$(2,2,2),(2,3,1),(3,1,2),(3,2,1),(4,1,1)$
$L_{6,4}$	$(4,1,1)$
$L_{6,5}$	$(3,1,2),(3,2,1),(4,0,2),(4,1,1),(5,0,1)$
$L_{6,6},L_{6,7}$	$(2,1,3),(2,2,2),(2,3,1),(3,0,3),(3,1,2),$ $(3,2,1),(4,0,2),(4,1,1),(5,0,1)$
$L_{6,8}$	$(3,1,2)$
$L_{6,9}$	$(2,1,3),(2,2,2),(3,0,3),(3,1,2),(4,0,2)$
$L_{6,10}$	$(4,0,2),(4,1,1),(5,0,1)$
$L_{6,11},L_{6,12},L_{6,13},L_{6,20},$ $L_{6,19}(\epsilon),\epsilon\in\{-1,1\}$	$(3,0,3),(3,1,2),(3,2,1),$ $(4,0,2),(4,1,1),(5,0,1)$
$L_{6,14},L_{6,15},L_{6,16},L_{6,17},$ $L_{6,18},L_{6,21}(\epsilon),\epsilon\in\{-1,1\}$	$(2,0,4),(2,1,3),(2,2,2),(2,3,1),(3,0,3),$ $(3,1,2),(3,2,1),(4,0,2),(4,1,1),(5,0,1)$
$L_{6,19}(0),L_{6,23},L_{6,25},$ $L_{6,24}(\epsilon),\epsilon\in\{-1,0,1\}$	$(3,0,3),(3,1,2),(4,0,2)$
$L_{6,21}(0)$	$(2,0,4),(2,1,3),(2,2,2),(3,0,3),(3,1,2),(4,0,2)$
$L_{6,22}(\epsilon),\epsilon\in\{-1,0,1\}$	$(4,0,2)$
$L_{6,26}$	$(3,0,3)$

Table.23: Realizable Ricci signatures on nilpotent Lie groups of dimension ≤ 6.

One-dimensional sectional curvature signatures of nilpotent Lie groups

4.1 Introduction

In this chapter, we give a description of the method of construction of Milnor's frames and we apply it to W.De Graaf's list of real nilpotent Lie algebras of dimension ≤ 4 and we use these bases to solve Problem 2 for these Lie algebras except one. Moreover, we use the method of construction of Milnor's frames to simplify drastically Nikitenko'list of five-dimensional Euclidean nilpotent Lie algebras [71] and we solve Problem 2 for some of them.

4.2 Milnor-type theorems

In this section, we describe a procedure to obtain a Milnor-type theorem (see[42]) for an arbitrary Lie algebra $\mathfrak{g}$ which can be obtained from the moduli space $\mathfrak{BM}$ of left-invariant Riemannian metrics. The space $\mathfrak{BM}$ has been introduced and studied in [61, 42].

Let G be a Lie group, and $\mathfrak{g}$ be the Lie algebra of G. The set $\mathfrak{M}^l$ of all left-invariant Riemannian metrics on G, is naturally identified with

$$\mathfrak{M} := \{\langle\ ,\ \rangle \mid \text{an inner product on } \mathfrak{g}\}. \tag{4.1}$$

Let $n := \dim\mathfrak{g}$, and identify $\mathfrak{g} \cong \mathbb{R}^n$ as vector spaces. For $\langle\ ,\ \rangle \in \mathfrak{M}$ and $g \in \mathrm{GL}_n(\mathbb{R})$, we define

$$g.\langle\ ,\ \rangle := \langle g^{-1}(.), g^{-1}(.)\rangle. \tag{4.2}$$

This induces a transitive action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathfrak{M}$. Thus, we have the identification

$$\mathfrak{M} \cong \mathrm{GL}_n(\mathbb{R})/\mathrm{O}(n). \tag{4.3}$$

Note that $\mathfrak{M}$ is a noncompact Riemannian symmetric space, by equipping with a certain $\mathrm{GL}_n(\mathbb{R})-$invariant metric(see [61]). Let us consider the automorphism group and the scalar group:

$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi \in \mathrm{GL}_n(\mathbb{R}) \mid \varphi[.,.] = [\varphi(.), \varphi(.)]\}, \tag{4.4}$$

$$\mathbb{R}^\times := \{c.id : \mathfrak{g} \longrightarrow \mathfrak{g} \mid \mathrm{c} \in \mathbb{R}, \mathrm{c} \neq 0\}. \tag{4.5}$$

The group $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ naturally acts on $\mathfrak{M}$. The action of $\mathbb{R}^\times$ gives rise to a scaling, and the action of $\mathrm{Aut}(\mathfrak{g})$ induces an isometry of the corresponding left-invariant Riemannian metrics.

Definition 4.2.1 (see [61]). *The orbit space of the action of* $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ *on* $\mathfrak{M}$ *is called the moduli space of left-invariant Riemannian metrics, and denoted*

$$\mathfrak{BM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \mathfrak{M}. \tag{4.6}$$

Note that the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{M}$ is isometric with respect to $\mathrm{GL}_n(\mathbb{R})-$invariant metrics.

Let $\langle\ ,\ \rangle_0$ be the canonical inner product on $\mathfrak{g} \cong \mathbb{R}^n$. For simplicity, the orbit of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ through $\langle\ ,\ \rangle$ is denoted by

$$[\langle\ ,\ \rangle] := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle\ ,\ \rangle := \{\varphi.\langle\ ,\ \rangle \mid \varphi \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})\}. \tag{4.7}$$

Definition 4.2.2 (see [42]). *A subset* $U \subset \mathrm{GL}_n(\mathbb{R})$ *is called a* set of representatives *of* $\mathfrak{BM}$ *if it satisfies*

$$\mathfrak{BM} = \{[h.\langle\ ,\ \rangle_0] \mid h \in U\}. \tag{4.8}$$

A set of representatives do not mean it is a complete set of representatives, it is expected that U is chosen to be as small as possible. We give a criteria for U to be a set of representatives. Let $[[g]]$ denote the double coset of $g \in \mathrm{GL}_n(\mathbb{R})$, defined by

$$[[g]] := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) g \mathrm{O}(n) := \{\varphi g k \mid \varphi \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}), k \in \mathrm{O}(n)\}. \tag{4.9}$$

Lemma 4.2.3 (see [42]). *Let* $U \subset \mathrm{GL}_n(\mathbb{R})$. *Then the following are mutually equivalent:*

(1) U *is a set of representatives of* $\mathfrak{BM}$.

(2) For every $g \in \mathrm{GL}_n(\mathbb{R})$, *there exists* $h \in U$ *such that* $[h.\langle\ ,\ \rangle_0] = [g.\langle\ ,\ \rangle]$.

(3) For every $g \in \mathrm{GL}_n(\mathbb{R})$, *there exists* $h \in U$ *such that* $h \in [[g]]$.

Recall that $\langle\ ,\ \rangle_0$ is the canonical inner product on $\mathfrak{g} \cong \mathbb{R}^n$. Denote by $\{e_1, \ldots, e_n\}$ the canonical orthonormal basis.

Theorem 4.2.4 (Milnor-type theorem,[42]). *Let U be a set of representatives of $\mathfrak{BM}$. Then, for every inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exist $h \in U$, $\varphi \in \mathrm{Aut}(\mathfrak{g})$, and $k > 0$ such that $\{x_1 = \varphi h e_1, \ldots, x_n = \varphi h e_n\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to $k\langle\ ,\ \rangle$.*

Remark 4.2.5. *If U contains l parameters, then the constants structure of $\mathfrak{g}$ in the basis $\{x_1, \ldots, x_n\}$ depend of l parameters.*

4.3 One-dimensional sectional curvature tensor

Definition 4.3.1. *Let (G, h) be a Riemannian Lie group and $\mathfrak{g}$ the associated real Lie algebra with $n := \dim\mathfrak{g} \geq 3$. The one-dimensional sectional curvature tensor* A *is*

$$\mathrm{A} := \frac{1}{n-2}\left(\mathrm{ric} - \frac{sh}{2(n-1)}\right) \tag{4.10}$$

where ric *is the Ricci tensor and s the scalar curvature.*

It is known (see [6]) that

$$R = \mathrm{W} + \mathrm{A} \odot h \tag{4.11}$$

W is the Weyl tensor and for $X, Y, Z, V \in \mathfrak{g}$,

$$(\mathrm{A} \odot h)(X, Y, Z, V) := \mathrm{A}(X, Z)h(Y, V) + \mathrm{A}(Y, V)h(X, Z) - \mathrm{A}(X, V)h(Y, Z) - \mathrm{A}(Y, Z)h(X, V) \tag{4.12}$$

is the Kulkarni-Nomizu product.

The one-dimensional sectional curvature operator denoted by $\mathcal{A}$ is defined by

$$\mathcal{A} = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{sI_n}{2(n-1)}\right), \tag{4.13}$$

where Ric is the Ricci endomorphism(Ricci operator), s the scalar curvature and I_n the identity endomorphism of $\mathfrak{g}$. The spectrum of one-dimensional curvature operator of Riemannian Lie groups is studied in [79, 52, 85, 51, 84, 101, 102, 80] ,and the signatures of one-dimensional curvature operator of Riemannian Lie groups are studied in [102, 101, 52, 85, 84, 79].

Let start now the construction of Milnor's frame for the nilpotent Lie algebras in Graff's list.

4.4 Lie algebra $L_{3,2}$

$L_{3,2} = \mathrm{span}\{e_1, e_2, e_3\}$ with

$$[e_1, e_2] = e_3.$$

In the basis $\mathbb{B} = (e_1, e_2, e_3)$, all the derivations of $L_{3,2}$ are:

$$\mathrm{Der}(L_{3,2}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}+a_{22} \end{bmatrix}, a_{ij} \in \mathbb{R} \right\} \tag{4.14}$$

Thus

$$(\mathrm{Aut}(L_{3,2}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} \end{bmatrix}, a_{11} > 0, a_{22} > 0 \right\} \tag{4.15}$$

We have the following Milnor-Type theorem:

Theorem 4.4.1. *Let $\mathfrak{g} = L_{3,2}$. Then, for any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exist $k > 0$, $\lambda > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle\ ,\ \rangle$ such that the bracket relation is given by*

$$[x_1, x_2] = \lambda x_3. \tag{4.16}$$

Proof. Take any $g \in \mathrm{GL}_3(\mathbb{R})$, from linear algebra (see [30]) there exists $k \in \mathrm{O}(3)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, b_{11} > 0, b_{22} > 0, b_{33} > 0. \tag{4.17}$$

It follows from (4.15) that

$$\varphi_1 := \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} \end{bmatrix} \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \tag{4.18}$$

With

$$a_{11} = \frac{1}{b_{11}}, \quad a_{21} = -\frac{b_{21}}{b_{11}b_{22}}, \quad a_{22} = \frac{1}{b_{22}}, \quad a_{31} = \frac{-b_{22}b_{31} + b_{21}b_{32}}{b_{11}^2 b_{22}^2}, \quad a_{32} = -\frac{b_{32}}{b_{11}b_{22}^2}.$$

This gives

$$\varphi_1 gk = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{b_{33}}{b_{11}b_{22}} \end{bmatrix}.$$

Set $\lambda := \frac{b_{11}b_{22}}{b_{33}}$, then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}, \lambda > 0 \right\}.$$

Take any inner product on $\mathfrak{g}$. By Theorem 4.4.1 , there exists $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\{x_1 = \varphi g_\lambda e_1, x_2 = \varphi g_\lambda e_2, x_3 = \varphi g_\lambda e_3\}$ is orthonormal with respect to $k\langle\ ,\ \rangle$. Hence, we have only to check the bracket relations among them. Note that

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = \frac{1}{\lambda} e_3.$$

We thus obtain

$$\begin{aligned}
[g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2] = e_3 = \lambda g_\lambda e_3, \\
[g_\lambda e_1, g_\lambda e_3] &= [e_1, \lambda^{-1} e_3] = 0, \\
[g_\lambda e_2, g_\lambda e_3] &= [e_2, \lambda^{-1} e_3] = 0.
\end{aligned}$$

Since $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we obtain

$$[x_1, x_2] = [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \varphi[e_1, e_2] = \varphi e_3 = \lambda \varphi g_\lambda e_3 = \lambda x_3.$$

□

Given any inner product $\langle\ ,\ \rangle$ on $L_{3,2}$, following Theorem 4.4.1 and direct computation from (1.6), the one- dimensional operator is

$$\mathcal{A}(\langle\ ,\ \rangle) = \frac{1}{2}\mathrm{diag}\left(-\frac{3}{4}\lambda^2, -\frac{3}{4}\lambda^2, \frac{5}{4}\lambda^2\right).$$

Then for any product $\langle\ ,\ \rangle$ on $L_{3,2}$, the signature of $\mathcal{A}$ is $(2, 0, 1)$.

4.5 Lie algebra $L_{4,2}$

$L_{4,2} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3.$$

In the basis $\mathbb{B} = (e_1, e_2, e_4, e_3)$, all the derivations of $L_{4,2}$ are:

$$\mathrm{Der}(L_{4,2}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11} + a_{22} \end{bmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

Thus

$$(\mathrm{Aut}(L_{4,2}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} \end{bmatrix}, a_{11} > 0, a_{22} > 0 \right\}.$$

We have the following Milnor-Type theorem:

Theorem 4.5.1. *Let $\mathfrak{g} = L_{4,2}$. Then, for any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exist $k > 0$, $\lambda > 0$, and an orthonormal basis $\{x_1, x_2, x_3, x_4\}$ with respect to $k\langle\ ,\ \rangle$ such that the bracket relation is given by*

$$[x_1, x_2] = \lambda x_3.$$

Proof. Take any $g \in \mathrm{GL}_4(\mathbb{R})$, from linear algebra (see [30]) there exists $k \in \mathrm{O}(4)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}, b_{11} > 0, b_{22} > 0, b_{33} > 0, b_{44} > 0.$$

It follows from (4.5) that

$$\varphi_2 := \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} \end{bmatrix} \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}).$$

With

$$\begin{aligned} a_{11} &= \frac{1}{b_{11}}, & a_{21} &= -\frac{b_{21}}{b_{11}b_{22}}, \\ a_{22} &= \frac{1}{b_{22}}, & a_{31} &= \frac{-b_{22}b_{31} + b_{21}b_{32}}{b_{11}b_{22}b_{33}}, \\ a_{32} &= -\frac{b_{32}}{b_{22}b_{33}}, & a_{33} &= \frac{1}{b_{33}}, \\ a_{41} &= \frac{-b_{22}b_{33}b_{41} + b_{21}b_{33}b_{42} + b_{22}b_{31}b_{43} - b_{21}b_{32}b_{43}}{b_{11}^2 b_{22}^2 b_{33}}, & a_{42} &= \frac{-b_{33}b_{42} + b_{32}b_{43}}{b_{11}b_{22}^2 b_{33}}, \\ a_{43} &= -\frac{b_{43}}{b_{11}b_{22}b_{33}}. \end{aligned}$$

This gives

$$\varphi_2 gk = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{b_{44}}{b_{11}b_{22}} \end{bmatrix} \tag{4.19}$$

Set $\lambda := \frac{b_{11}b_{22}}{b_{44}}$, then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}, \lambda > 0 \right\} \tag{4.20}$$

Take any inner product on $\mathfrak{g}$. By Theorem 4.5.1, there exists $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\{x_1 = \varphi g_\lambda e_1, x_2 = \varphi g_\lambda e_2, x_3 = \varphi g_\lambda e_3, x_4 = \varphi g_\lambda e_4\}$ is orthonormal with respect to $k\langle\ ,\ \rangle$. Hence, we have only to check the bracket relations among them. Note that

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = e_2, \quad g_\lambda e_3 = \frac{1}{\lambda} e_3, \quad g_\lambda e_4 = e_4$$

We thus obtain

$$\begin{aligned}
[g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2] = e_3 = \lambda g_\lambda e_3,\\
[g_\lambda e_1, g_\lambda e_3] &= [e_1, \lambda^{-1} e_3] = 0,\\
[g_\lambda e_1, g_\lambda e_4] &= [e_1, e_4] = 0,\\
[g_\lambda e_2, g_\lambda e_3] &= [e_2, \lambda^{-1} e_3] = 0,\\
[g_\lambda e_2, g_\lambda e_4] &= [e_2, e_4] = 0.
\end{aligned}$$

Since $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we obtain

$$[x_1, x_2] = [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \varphi[e_1, e_2] = \varphi e_3 = \lambda \varphi g_\lambda e_3 = \lambda x_3.$$

□

Given any inner product $\langle\ ,\ \rangle$ on $L_{4,2}$, following Theorem 4.5.1 and direct computation from (1.6), the one dimensional operator is

$$\mathcal{A}(\langle\ ,\ \rangle) = \frac{1}{4}\mathrm{diag}\left(-\frac{5}{6}\lambda^2, -\frac{5}{6}\lambda^2, \frac{7}{6}\lambda^2, \frac{1}{6}\lambda^2\right).$$

Then for any product $\langle\ ,\ \rangle$ on $L_{4,2}$, the signature of $\mathcal{A}$ is $(2, 0, 2)$.

4.6 Lie algebra $L_{4,3}$

$L_{4,3} = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4.$$

In the basis $\mathbb{B} = (e_1, e_2, e_3, e_4)$, all the derivations of $L_{4,3}$ are:

$$\mathrm{Der}(L_{4,3}) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & 2a_{11} + a_{22} \end{bmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

Thus

$$(\mathrm{Aut}(L_{4,3}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}^2 a_{22} \end{bmatrix}, a_{11} > 0, a_{22} > 0 \right\}.$$

We have the following Milnor-Type theorem:

Theorem 4.6.1. *Let $\mathfrak{g} = L_{4,3}$. Then, for any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exists $k > 0$, $\lambda_1 > 0$, λ_2, and an orthonormal basis $\{x_1, x_2, x_3, x_4\}$ with respect to $k\langle\ ,\ \rangle$ such that the bracket relation is given by*

$$[x_1, x_2] = \lambda_1 x_3 - \lambda_1\lambda_2 x_4, \quad [x_1, x_3] = x_4.$$

Proof. Take any $g \in \mathrm{GL}_4(\mathbb{R})$, from linear algebra (see [30]) there exists $k \in \mathrm{O}(4)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}, b_{11} > 0, b_{22} > 0, b_{33} > 0, b_{44} > 0.$$

It follows from (4.6) that

$$\varphi_3 := \alpha \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}^2 a_{22} \end{bmatrix} \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).$$

With

$$\begin{aligned} a_{11} &= \frac{b_{33}}{b_{44}}, & a_{21} &= -\frac{b_{21}}{b_{33}}, \\ a_{22} &= \frac{b_{11}}{b_{33}}, & a_{31} &= \frac{-b_{22}b_{31} + b_{21}b_{32}}{b_{22}b_{44}}, \\ a_{42} &= \frac{-b_{11}b_{22}b_{33}b_{42} + b_{11}b_{32}^2 b_{44}}{b_{22}^2 b_{44}^2}, & a_{43} &= -\frac{b_{11}b_{32}}{b_{22}b_{44}}, \\ \alpha &= \frac{b_{44}}{b_{11}b_{33}}, & a_{41} &= \frac{-b_{22}^2 b_{33}b_{41} + b_{21}b_{22}b_{33}b_{42} + b_{22}b_{31}b_{32}b_{44} - b_{21}b_{32}^2 b_{44}}{b_{22}^2 b_{44}^2}. \end{aligned}$$

This gives

$$\varphi_3 gk = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \end{bmatrix}$$

with $\lambda_1 := \frac{b_{22}b_{44}}{b_{33}^2}$ and $\lambda_2 := \frac{b_{11}b_{33}(b_{22}b_{43} - b_{32}b_{44})}{b_{22}b_{44}^2}$ then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \end{bmatrix}, \lambda = (\lambda_1, \lambda_2), \lambda_1 > 0 \right\}.$$

Take any inner product on $\mathfrak{g}$. By Theorem 4.6.1 , there exists $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\{x_1 = \varphi g_\lambda e_1, x_2 = \varphi g_\lambda e_2, x_3 = \varphi g_\lambda e_3, x_4 = \varphi g_\lambda e_4\}$ is orthonormal with respect to $k\langle\ ,\ \rangle$. Hence, we have only to check the bracket relations among them. Note that

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = \lambda_1 e_2, \quad g_\lambda e_3 = e_3 + \lambda_2 e_4, \quad g_\lambda e_4 = e_4.$$

We thus obtain

$$\begin{aligned}
[g_\lambda e_1, g_\lambda e_2] &= \lambda_1[e_1, e_2] = \lambda_1 e_3 = \lambda_1 g_\lambda e_3 - \lambda_1\lambda_2 g_\lambda e_4,\\
[g_\lambda e_1, g_\lambda e_3] &= [e_1, e_3 + \lambda_2 e_4] = e_4 = g_\lambda e_4,\\
[g_\lambda e_1, g_\lambda e_4] &= [e_1, e_4] = 0,\\
[g_\lambda e_2, g_\lambda e_3] &= [e_2, e_3 + \lambda_2 e_4] = 0,\\
[g_\lambda e_2, g_\lambda e_4] &= \lambda_1[e_2, e_4] = 0\\
[g_\lambda e_3, g_\lambda e_4] &= [e_3 + \lambda_2 e_4, e_4] = 0.
\end{aligned}$$

Since $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we obtain

$$[x_1, x_2] = [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \lambda_1 \varphi g_\lambda e_3 - \lambda_1\lambda_2\varphi g_\lambda e_4 = \lambda_1 x_3 - \lambda_1\lambda_2 x_4, \quad [x_1, x_3] = x_4.$$

□

Given any inner product $\langle\ ,\ \rangle$ on $L_{4,3}$, following Theorem 4.6.1 and direct computation from (1.6), the one-dimensional operator is

$$\mathrm{A} = \frac{1}{4}\begin{bmatrix} -\frac{5}{6}(a^2+b^2+1) & 0 & 0 & 0\\ 0 & \frac{1-5(a^2+b^2)}{6} & -b & 0\\ 0 & -b & \frac{7a^2+b^2-5}{6} & ab\\ 0 & 0 & ab & \frac{a^2+7b^2+7}{3}\end{bmatrix} \text{with} a = \lambda_1, b = -\lambda_1\lambda_2. \tag{4.21}$$

4.7 Lie algebra $L_{5,2}$

$L_{5,2} = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ with

$$[e_1, e_2] = e_3.$$

In the basis $\mathbb{B} = (e_1, e_2, e_4, e_5, e_3)$, all the derivations of $L_{5,2}$ are:

$$\mathrm{Der}(L_{5,2}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0\\ a_{21} & a_{22} & 0 & 0 & 0\\ a_{31} & a_{32} & a_{33} & a_{34} & 0\\ a_{41} & a_{42} & a_{43} & a_{44} & 0\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}+a_{22}\end{bmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

Thus

$$(\mathrm{Aut}(L_{5,2}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & \\ a_{21} & a_{22} & 0 & 0 & \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}^2 a_{22} \end{bmatrix}, a_{11} > 0, a_{22} > 0, a_{44} > 0. \right\}.$$

We have the following Milnor-Type theorem:

Theorem 4.7.1. *Let $\mathfrak{g} = L_{5,2}$. Then, for any inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, there exist $k > 0$, $\lambda > 0$, and an orthonormal basis $\{x_1, x_2, x_3, x_4, x_5\}$ with respect to $k\langle\ ,\ \rangle$ such that the bracket relation is given by*

$$[x_1, x_2] = \lambda x_3.$$

Proof. Take any $g \in \mathrm{GL}_5(\mathbb{R})$, from linear algebra (see [30]) there exists $k \in \mathrm{O}(5)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, b_{11} > 0, b_{22} > 0, b_{33} > 0, b_{44} > 0, b_{55} > 0.$$

It follows from (4.7) that

$$\varphi_4 := \begin{bmatrix} a_{11} & 0 & 0 & 0 & \\ a_{21} & a_{22} & 0 & 0 & \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}^2 a_{22} \end{bmatrix} \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}),$$

with

$$\begin{aligned}
a_{11} &= \frac{1}{b_{11}}, & a_{21} &= -\frac{b_{21}}{b_{11}b_{22}}, \\
a_{22} &= \frac{1}{b_{22}}, & a_{31} &= \frac{-b_{22}b_{31} + b_{21}b_{32}}{b_{11}b_{22}b_{33}}, \\
a_{32} &= -\frac{b_{32}}{b_{22}b_{33}}, & a_{33} &= \frac{1}{b_{33}}, \\
a_{41} &= \frac{-b_{22}b_{33}b_{41} + b_{21}b_{33}b_{42} + b_{22}b_{31}b_{43} - b_{21}b_{32}b_{43}}{b_{11}b_{22}b_{33}b_{44}}, & a_{44} &= \frac{1}{b_{44}}, \\
a_{42} &= \frac{-b_{33}b_{42} + b_{32}b_{43}}{b_{22}b_{33}b_{44}}, & a_{43} &= -\frac{b_{43}}{b_{33}b_{44}}, \\
a_{52} &= \frac{-b_{33}b_{44}b_{52} + b_{32}b_{44}b_{53} + b_{33}b_{42}b_{54} - b_{32}b_{43}b_{54}}{b_{11}b_{22}^2b_{33}b_{44}}, & a_{53} &= \frac{-b_{44}b_{53} + b_{43}b_{54}}{b_{11}b_{22}b_{33}b_{44}}, \\
a_{54} &= -\frac{b_{54}}{b_{11}b_{22}b_{44}},
\end{aligned}$$

$$a_{51} = \frac{-b_{22}b_{33}b_{44}b_{51} + b_{21}b_{33}b_{44}b_{52} + b_{22}b_{31}b_{44}b_{53} - b_{21}b_{32}b_{44}b_{53} + b_{22}b_{33}b_{41}b_{54} - b_{21}b_{33}b_{42}b_{54}}{b_{11}^2 b_{22}^2 b_{33} b_{44}} - \frac{b_{22}b_{31}b_{43}b_{54} + b_{21}b_{32}b_{43}b_{54}}{b_{11}^2 b_{22}^2 b_{33} b_{44}}$$

This gives

$$\varphi_4 g k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$$

with $\lambda := \frac{b_{11}b_{22}}{b_{55}}$, then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}, \lambda > 0 \right\}$$

Take any inner product on $\mathfrak{g}$. By Theorem 4.7.1 , there exists $g_\lambda \in U$, $k > 0$, and $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\{x_1 = \varphi g_\lambda e_1, x_2 = \varphi g_\lambda e_2, x_3 = \varphi g_\lambda e_3, x_4 = \varphi g_\lambda e_4, x_5 = \varphi g_\lambda e_5\}$ is orthonormal with respect to $k\langle\ ,\ \rangle$. Hence, we have only to check the bracket relations among them. Note that

$$g_\lambda e_1 = e_1, \quad g_\lambda e_2 = \lambda_1 e_2, \quad g_\lambda e_3 = \frac{1}{\lambda} e_3, \quad g_\lambda e_4 = e_4, \quad g_\lambda e_5 = e_5.$$

We thus obtain

$$\begin{aligned}
[g_\lambda e_1, g_\lambda e_2] &= [e_1, e_2] = e_3 = \lambda g_\lambda e_3, \\
[g_\lambda e_1, g_\lambda e_3] &= [e_1, \frac{1}{\lambda} e_] = 0, \\
[g_\lambda e_1, g_\lambda e_4] &= [e_1, e_4] = 0, \\
[g_\lambda e_1, g_\lambda e_5] &= [e_1, e_5] = 0, \\
[g_\lambda e_2, g_\lambda e_3] &= [e_2, \frac{1}{\lambda} e_3] = 0, \\
[g_\lambda e_2, g_\lambda e_4] &= [e_2, e_4] = 0 \\
[g_\lambda e_2, g_\lambda e_5] &= [e_2, e_5] = 0, \\
[g_\lambda e_3, g_\lambda e_4] &= [\frac{1}{\lambda} e_3, e_4] = 0, \\
[g_\lambda e_3, g_\lambda e_5] &= [\frac{1}{\lambda} e_3, e_5] = 0, \\
[g_\lambda e_4, g_\lambda e_5] &= [e_4, e_5] = 0.
\end{aligned}$$

Since $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we obtain

$$[x_1, x_2] = [\varphi g_\lambda e_1, \varphi g_\lambda e_2] = \lambda \varphi g_\lambda e_3 = \lambda x_3.$$

□

Given any inner product $\langle\ ,\ \rangle$ on $L_{5,2}$, following Theorem 4.7.1 and direct computation from (1.6), the one-dimensional operator is

$$\mathcal{A}(\langle\ ,\ \rangle) = \frac{1}{6}\mathrm{diag}\left(-\frac{7}{8}\lambda^2, -\frac{7}{8}\lambda^2, \frac{9}{8}\lambda^2, \frac{1}{8}\lambda^2, \frac{1}{8}\lambda^2\right).$$

Then for any product $\langle\ ,\ \rangle$ on $L_{5,2}$, the signature of $\mathcal{A}$ is $(2,0,3)$.

4.8 Lie algebra $L_{5,4}$

$L_{5,4} = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ with

$$[e_1, e_2] = e_5, \quad [e_3, e_4] = e_5.$$

We note that $\mathbf{Sign}(L_{5,4}) = \mathbf{Sign}(\mathbb{A}_{5,4})$, thus $L_{5,4}$ and $\mathbb{A}_{5,4}$ are isomorphic. According to Nikitenko in [71], for any inner product $\langle\ ,\ \rangle$ on $L_{5,4}$, there exist $\varepsilon > 0$,$\sigma > 0$ such that $(L_{5,4}, \langle\ ,\ \rangle) \cong \mathcal{N}_1^5(\varepsilon, \sigma)$. Given any inner product $\langle\ ,\ \rangle$ on $L_{5,4}$, direct computation from (1.6) , the one-dimensional operator is

$$\mathcal{A}(\langle\ ,\ \rangle) = \frac{1}{6}\mathrm{diag}\left(\frac{\sigma^2 - 7\varepsilon^2}{8}, \frac{\sigma^2 - 7\varepsilon^2}{8}, \frac{\sigma^2 - 7\varepsilon^2}{8}, \frac{\varepsilon^2 - 7\sigma^2}{8}, \frac{9(\varepsilon^2 + \sigma^2)}{8}\right) \tag{4.22}$$

Then for any product $\langle\ ,\ \rangle$ on $L_{5,4}$, the signature of A is $(2,0,3)$ or $(2,2,1)$.

4.9 Lie algebra $L_{5,5}$

$L_{5,5} = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_5, \quad [e_2, e_4] = e_5.$$

We note that $\mathbf{Sign}(L_{5,5}) = \mathbf{Sign}(\mathbb{A}_{5,5})$, thus $L_{5,5}$ and $\mathbb{A}_{5,5}$ are isomorphic. According to Nikitenko in [71], for any inner product $\langle\ ,\ \rangle$ on $L_{5,5}$, there exist $\varepsilon > 0$,$\sigma > 0$,$\upsilon \geq 0, \gamma \geq 0, \rho > 0$ such that $(L_{5,5}, \langle\ ,\ \rangle) \cong \mathcal{N}_1^5(\varepsilon, \sigma, \upsilon, \gamma, \rho)$. That is the constants structure depend on 5 parameters. We have:

Theorem 4.9.1. *For any inner product $\langle\ ,\ \rangle$ on $L_{5,5}$, there exist an $\langle\ ,\ \rangle-$orthonormal basis in which the constants structure depend of at most 4 parameters.*

Proof. In the basis $\mathbb{B} = (e_1, e_2, e_4, e_3, e_5)$, all the derivations of $L_{5,5}$ are:

$$\mathrm{Der}(L_{5,5}) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{41} + a_{53} & 2a_{11} & 0 & 0 \\ a_{41} & a_{42} & -a_{21} & a_{11} + a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 2a_{11} + a_{22} \end{bmatrix}, a_{ij} \in \mathbb{R} \right\} \tag{4.23}$$

Thus

$$(\mathrm{Aut}(L_{5,5}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & \\ a_{21} & a_{22} & 0 & 0 & \\ a_{31} & a_{41}+a_{53} & a_{11}^2 & 0 & \\ a_{41} & a_{42} & -a_{21} & a_{11}a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}^2 a_{22} \end{bmatrix}, a_{11}>0, a_{22}>0 \right\} \tag{4.24}$$

Take any $g \in \mathrm{GL}_5(\mathbb{R})$, from linear algebra(see [30]) there exists $k \in \mathrm{O}(5)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, b_{11}>0, b_{22}>0, b_{33}>0, b_{44}>0, b_{55}>0. \tag{4.25}$$

It follows from (4.24) that

$$\varphi_5 := \alpha \begin{bmatrix} a_{11} & 0 & 0 & 0 & \\ a_{21} & a_{22} & 0 & 0 & \\ a_{31} & a_{41}+a_{53} & a_{11}^2 & 0 & \\ a_{41} & a_{42} & -a_{21} & a_{11}a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{11}^2 a_{22} \end{bmatrix} \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g}) \tag{4.26}$$

With

$$a_{11} = \frac{b_{44}}{b_{55}},\ a_{42} = -\frac{(b_{33}b_{42}-b_{32}b_{43})\,b_{44}}{b_{22}b_{55}^2},\ a_{22} = \frac{b_{33}}{b_{55}},\ a_{54} = -\frac{b_{33}b_{44}b_{54}}{b_{55}^3},$$

$$a_{31} = -\frac{b_{44}}{b_{11}^2 b_{22} b_{55}^3}\left(-b_{11}b_{21}b_{33}b_{44}b_{52} + b_{11}b_{21}b_{32}b_{44}b_{53} + b_{11}b_{21}b_{33}b_{42}b_{54} - b_{11}b_{21}b_{32}b_{43}b_{54}\right)$$
$$-\frac{b_{44}}{b_{11}^2 b_{22} b_{55}^3}\left(-b_{21}b_{22}b_{33}b_{41}b_{55} + b_{21}^2 b_{33}b_{42}b_{55} + b_{21}b_{22}b_{31}b_{43}b_{55} - b_{21}^2 b_{32}b_{43}b_{55} + b_{11}b_{22}b_{31}b_{44}b_{55}\right),$$

$$a_{41} = -\frac{(b_{22}b_{33}b_{41} - b_{21}b_{33}b_{42} - b_{22}b_{31}b_{43} + b_{21}b_{32}b_{43})\,b_{44}}{b_{11}b_{22}b_{55}^2},\ a_{21} = \frac{b_{43}b_{44}}{b_{55}^2},\ \alpha = \frac{b_{55}^2}{b_{33}b_{44}^2},$$

$$a_{51} = -\frac{b_{44}\,(b_{22}b_{33}b_{44}b_{51} - b_{21}b_{33}b_{44}b_{52} - b_{22}b_{31}b_{44}b_{53} + b_{21}b_{32}b_{44}b_{53} - b_{22}b_{33}b_{41}b_{54})}{b_{11}b_{22}b_{55}^3}$$
$$+\frac{b_{44}\,(b_{21}b_{33}b_{42}b_{54} + b_{22}b_{31}b_{43}b_{54} - b_{21}b_{32}b_{43}b_{54})}{b_{11}b_{22}b_{55}^3},$$

$$a_{52} = -\frac{b_{44}\,(b_{33}b_{44}b_{52} - b_{32}b_{44}b_{53} - b_{33}b_{42}b_{54} + b_{32}b_{43}b_{54})}{b_{22}b_{55}^3},\quad a_{53} = -\frac{b_{44}\,(b_{44}b_{53} - b_{43}b_{54})}{b_{55}^3}.$$

This gives

$$\varphi_5 gk = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.27}$$

with λ_i depend on some b_{ij} , then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \lambda_1 > 0, \lambda_3 > 0 \right\} \tag{4.28}$$

Then U depend on at most 4 parameters. Following Theorem 4.2.4, there exist an $\langle\ ,\ \rangle-$orthonormal basis in which the constants structure depend also on 4 parameters. □

4.10 Lie algebra $L_{5,7}$

$L_{5,7} = \operatorname{span}\{e_1, e_2, e_3, e_4, e_5\}$ with

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5.$$

We note that $\mathbf{Sign}(L_{5,7}) = \mathbf{Sign}(\mathbb{A}_{5,2}) = \mathbf{Sign}(\mathbb{A}_{5,6})$, thus $L_{5,7}$ is isomorphic $\mathbb{A}_{5,2})$ or $\mathbb{A}_{5,6})$. According to Nikitenko in [71], for any inner product $\langle\ ,\ \rangle$ on $L_{5,7}$, there exist $\varepsilon > 0, \delta > 0, \tau \geq 0, \sigma > 0, \tau = 0 \Rightarrow \gamma \geq 0$ such that $(L_{5,7}, \langle\ ,\ \rangle) \cong \mathcal{N}_1^5(\varepsilon, \delta, \tau, \sigma, \upsilon, \gamma)$ or there exist $\varepsilon > 0, \delta > 0, \tau \geq 0$, $\sigma > 0$, $\tau = 0 \Rightarrow \gamma \geq 0$ such that $(L_{5,7}, \langle\ ,\ \rangle) \cong \mathcal{N}_1^5(\varepsilon, \delta, \tau, \sigma, \upsilon, \gamma, \rho)$. That is , the constants structure depend on 7 parameters at most. We have

Theorem 4.10.1. *For any inner product $\langle\ ,\ \rangle$ on $L_{5,7}$, there exist an $\langle\ ,\ \rangle$-orthonormal basis in which the constants structure depend of at most* 5 *parameters.*

Proof. In the basis $\mathbb{B} = (e_1, e_2, e_3, e_4, e_5)$, all the derivations of $L_{5,7}$ are:

$$\operatorname{Der}(L_{5,7}) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11} + a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & 2a_{11} + a_{22} & 0 \\ a_{51} & a_{52} & a_{42} & a_{32} & 3a_{11} + a_{22} \end{bmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

Thus

$$(\mathrm{Aut}(L_{5,7}))^0 \supset \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & a_{11}^2 a_{22} & 0 \\ a_{51} & a_{52} & a_{42} & a_{32} & a_{11}^3 a_{22} \end{bmatrix}, a_{11} > 0, a_{22} > 0 \right\}.$$

Take any $g \in \mathrm{GL}_5(\mathbb{R})$, from linear algebra(see [30]) there exists $k \in \mathrm{O}(5)$ such that

$$gk = \begin{bmatrix} b_{11} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, b_{11} > 0, b_{22} > 0, b_{33} > 0, b_{44} > 0, b_{55} > 0.$$

It follows from (4.10) that

$$\varphi_6 := \alpha \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{32} & a_{11}^2 a_{22} & 0 \\ a_{51} & a_{52} & a_{42} & a_{32} & a_{11}^3 a_{22} \end{bmatrix} \in \mathbb{R}^{\times}\mathrm{Aut}(\mathfrak{g})$$

With

$$a_{11} = \frac{b_{33}}{b_{44}}, \quad a_{22} = \frac{b_{11}}{b_{33}}, \quad a_{32} = -\frac{b_{11}b_{33}^2 b_{54}}{b_{44}^4}, \quad a_{42} = -\frac{b_{11}b_{33}\left(b_{44}b_{53} - b_{43}b_{54}\right)}{b_{44}^4},$$

$$a_{52} = -\frac{b_{11}b_{33}\left(b_{33}b_{44}b_{52} - b_{32}b_{44}b_{53} - b_{33}b_{42}b_{54} + b_{32}b_{43}b_{54}\right)}{b_{22}b_{44}^4}, \quad a_{21} = -\frac{b_{21}}{b_{33}}, \quad \alpha = \frac{b_{44}}{b_{11}b_{33}},$$

$$a_{31} = \frac{-b_{31}b_{44}^3 + b_{21}b_{33}^2 b_{54}}{b_{44}^4}, \quad a_{41} = \frac{b_{33}\left(-b_{41}b_{44}^2 + b_{21}b_{44}b_{53} + b_{31}b_{33}b_{54} - b_{21}b_{43}b_{54}\right)}{b_{44}^4},$$

$$\begin{aligned} a_{51} = &-\frac{b_{33}\left(b_{22}b_{33}b_{44}b_{51} - b_{21}b_{33}b_{44}b_{52} - b_{22}b_{31}b_{44}b_{53} + b_{21}b_{32}b_{44}b_{53} - b_{22}b_{33}b_{41}b_{54}\right)}{b_{22}b_{44}^4} \\ &+ \frac{b_{33}\left(b_{21}b_{33}b_{42}b_{54} + b_{22}b_{31}b_{43}b_{54} - b_{21}b_{32}b_{43}b_{54}\right)}{b_{22}b_{44}^4}. \end{aligned}$$

This gives

$$\varphi_6 gk = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$$

with λ_i depend on some b_{ij}, then the set of representatives of $\mathfrak{BM}$ is

$$U = \left\{ g_\lambda := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}, \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5), \lambda_1 > 0, \lambda_5 > 0 \right\}.$$

Then U depend on at most 5 parameters. Following Theorem 4.2.4, there exists an $\langle\ ,\ \rangle$-orthonormal basis in which the constants structure depend also on 5 parameters. $\square$

4.11 Lie algebra $L_{5,8}$

$L_{5,4} = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ with

$$[e_1, e_2] = e_4, \quad [e_1, e_3] = e_5$$

We note that $\mathbf{Sign}(L_{5,8}) = \mathbf{Sign}(\mathbb{A}_{5,1})$, thus $L_{5,8}$ and $\mathbb{A}_{5,1}$ are isomorphic. According to Nikitenko in [71], for any inner product $\langle\ ,\ \rangle$ on $L_{5,8}$, there exist $\delta > 0$, $\sigma > 0$ such that $(L_{5,8}, \langle\ ,\ \rangle) \cong \mathcal{N}_2^5(\delta, \sigma)$. Given any inner product $\langle\ ,\ \rangle$ on $L_{5,8}$, direct computation from (1.6) , the one-dimensional operator is

$$\mathcal{A}(\langle\ ,\ \rangle) = \frac{1}{6}\text{diag}\left(-\frac{\sigma^2+\delta^2}{8}, \frac{\sigma^2-7\delta^2}{8}, \frac{\delta^2-7\sigma^2}{8}, \frac{\sigma^2+9\delta^2}{8}, \frac{\delta^2+9\sigma^2)}{8}\right).$$

Then for any product $\langle\ ,\ \rangle$ on $L_{5,8}$, the signature of $\mathcal{A}$ is $(2,1,2)$ or $(2,0,2)$ or $(2,0,3)$.

We end this work by giving all the realizable one dimensional curvature tensors signatures on nilpotent Lie groups up to dimension 5.

Lie algebra $\mathfrak{g}$	Realizable one dimensional curvature tensors signatures
$L_{3,2}$	$(2,0,1)$,
$L_{4,2}$	$(2,0,2)$,
$L_{5,2}$	$(2,0,3)$,
$L_{5,4}$	$(2,0,3),(2,2,1),(4,0,1)$
$L_{5,4}$	$(4,0,1)$
$L_{5,8}$	$(2,1,2),(3,0,2),(2,0,3)$.

Table.24: Realizable one dimensional curvature tensors signatures on some nilpotent Lie groups of dimension ≤ 5.

Conjecture 1 in the class of completely solvable Lie groups

In [40], the authors established the classification up to automorphism of the left-invariant Riemannian metrics on all simply connected three-dimensional Lie groups. In [76], the authors determined a particular subclass of these left-invariant Riemannian metrics called locally symmetric left-invariant Riemannian metrics on 3-dimensional Lie groups. Most simply connected three-dimensional Lie groups are completely solvable and admit these particular Riemannian metrics, among these Lie groups are $\widetilde{E_0}(2)$, the universal covering of the connected component of the Euclidean group and G_I one of the nonunimodular Lie groups. For more details on $\widetilde{E_0}(2)$, G_I and their Lie algebras see [40]. We study the Conjecture 1 when the Lie group is $\widetilde{E_0}(2)$ or G_I.

Simply connected Lie group $\widetilde{E_0}(2)$

The brackets on a canonical basis (e_1, e_2, e_3) of its associated Lie algbera $\mathfrak{g}$ are:

$$[e_1, e_2] = 0 \quad [e_3, e_1] = -e_2, \quad [e_3, e_2] = e_1.$$

Thus, $\mathbf{Sign}(\mathfrak{g}) = \{(2,1,0),(2,0,1),(1,1,1),(3,0,0),(1,2,0),(1,0,2)\}$.

In [76] , it is proved that locally symetric left-invariant Riemannian metrics on $\widetilde{E_0}(2)$ are equivalent up to automorphism to the metric whose associated matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix}, \ \nu > 0.$$

Let $\langle\ ,\ \rangle$ be the associated inner product on $\mathfrak{g}$. We see that,

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1,\ \langle e_3, e_3\rangle = \nu,\ \langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0.$$

Taking

$$X = e_1, \quad Y = e_2, \quad Z = \frac{1}{\sqrt{\nu}} e_3,$$

they form an orthonormal basis and satisfy

$$[X,Y]=0,\quad [Z,X]=-\frac{1}{\nu}Y,\quad [Z,Y]=\frac{1}{\nu}X.$$

The direct computation using (1.4), gives

$$\mathrm{ric}(X,X)=\mathrm{ric}(Y,Y)=\mathrm{ric}(Z,Z)=0.$$

For any such metrics, the Ricci signature is $(0,3,0)\notin \mathbf{Sign}(\mathfrak{g})$.

Simply connected Lie group G_I

The brackets on a canonical basis (e_1,e_2,e_3) of its associated Lie algbera $\mathfrak{g}_I$ are:

$$[e_1,e_2]=0\quad [e_3,e_1]=e_1,\quad [e_3,e_2]=e_2.$$

Thus, $\mathbf{Sign}(\mathfrak{g}_I)=\{(2,1,0),(2,0,1),(1,1,1),(3,0,0),(1,2,0),(1,0,2)\}$.

In [76], it is proved that all left-invariant Riemannianmetrics on G_I are locally symetric left-invariant Riemannian metrics and are equivalent up to automorphism to the metric whose associated matrix is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{bmatrix},\ \nu>0.$$

Let $\langle\ ,\ \rangle$ be the associated inner product on $\mathfrak{g}_I$. We see that,

$$\langle e_1,e_1\rangle=\langle e_2,e_2\rangle=1,\ \langle e_3,e_3\rangle=\nu,\ \langle e_1,e_2\rangle=\langle e_1,e_3\rangle=\langle e_2,e_3\rangle=0.$$

Taking

$$X=\frac{1}{\sqrt{\nu}}e_3,\quad Y=e_1,\quad Z=e_2,$$

they form an orthonormal basis and satisfy

$$[X,Y]=\frac{1}{\nu}Y,\quad [X,Z]=\frac{1}{\nu}Z,\quad [Y,Z]=0.$$

The direct computation using (1.4), gives

$$\mathrm{ric}(X,X)=\mathrm{ric}(Y,Y)=\mathrm{ric}(Z,Z)=-\frac{2}{\nu}.$$

For any such metrics, the Ricci signature is $(3,0,0)\notin \mathbf{Sign}(\mathfrak{g}_I)$.

Remark .0.1. *From, these examples it is obvious that the Conjecture 1 cannot be extended to the class of completely solvable Lie groups wchich are the generalization of nilpotent Lie groups.*

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Published papers

During this research, our contributions have been published to the journals as listed below:

- **M.B. Djiadeu Ngaha**, M. Boucetta, and J. Wouafo Kamga, The signature of the Ricci curvature of left-invariant Riemannian metrics on nilpotent Lie groups, *Differential Geometry and its Applications*, **47** (2016), 26-42.
- R. Nimpa Pefoukeu, **M.B. Djiadeu Ngaha**, J. Wouafo Kamga, Locally symmetric left-invariant Riemannian metrics on 3-dimensional Lie groups, *Mathematische Nachrichten*, **accepted**.