Doctoral Thesis

Lie racks and Leibniz algebras

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Abstract

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The tangent space at the identity of a pointed Lie rack carries a structure of left Leibniz algebra [34]. Conversely, given a (left) Leibniz algebra $\mathfrak{L}$ there exists a (local) pointed Lie rack $(X, 1)$ such that $T_1X = \mathfrak{L}$ [20]. A symmetric Leibniz algebra is an algebra which is both left and right Leibniz algebra. For any symmetric Leibniz algebra $\mathfrak{L}$, there is a straightforward construction of a global pointed Lie rack $(X, 1)$ such that $T_1X = \mathfrak{L}$. The obtained Lie rack is said to be associated to the symmetric Leibniz algebra $(\mathfrak{L}, [\ ,\ ])$. We first classify symmetric Leibniz algebras of dimension 3 and 4 and we determine all the associated Lie racks. Some of these Lie racks provide non-trivial topological quandles. Then we study some algebraic properties of these quandles and we give a necessary and sufficient condition for them to be quasi-trivial.
The second purpose of this thesis deals with the study of analytic linear Lie rack structures on Leibniz algebras. We first introduce these Lie rack structures on a vector space V which are given by a sequence $(A_{n,1})_{n\geq 1}$ of $n+1$-multilinear maps on V. We show that a sequence $(A_{n,1})_{n\geq 1}$ of $n+1$-multilinear maps on a vector space V defines an analytic linear Lie rack structure if and only if $[\ ,\ ] := A_{1,1}$ is a left Leibniz bracket, the $A_{n,1}$ are invariant for $(V, [\ ,\ ])$ and satisfy a sequence of multilinear equations. Some of these equations have a cohomological interpretation and can be solved when the zero and the 1-cohomology of the left Leibniz algebra $(V, [\ ,\ ])$ are trivial. On the other hand, given a left Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$, we show that there is a large class of (analytic) linear Lie rack structures on $(\mathfrak{h}, [\ ,\ ])$ which can be built from the canonical one and invariant multilinear symmetric maps on $\mathfrak{h}$. A left Leibniz algebra on which all the analytic linear Lie rack structures are built in this way will be called rigid. We use our characterizations of analytic linear Lie rack structures to show that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$ are rigid. Then we conjecture that any simple Lie algebra is rigid as a left Leibniz algebra.

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List of Abbreviations

TSPLR	**T**angent **s**pace of **p**ointed **L**ie **R**ack
ALLRS	**A**nalytic **l**inear **L**ie **r**ack **s**tructures

Chapter 1

Introduction

In the 1980's, Joyce [32] and Matveev [38] introduced independently the notion of *quandle*. This notion has been derived from knot theory, in the way that the three axioms of a quandle algebraically encode the three Reidemeister moves (I, II, III) of oriented knot diagrams [23] (see Appendix B). Quandles provide many knot invariants. Joyce and Matveev introduced the fundamental quandle or knot quandle. They showed that it is a complete invariant of a knot (up to a weak equivalence). Racks which are a generalization of quandles were introduced by Conway and Wraith, see the historical account in the work of Fenn and Rourke [24]. Recently (see [16, 17]), there has been investigations on quandles and racks from an algebraic point of view and their relationship with other algebraic structures as Lie algebras, Leibniz algebras, Frobenius algebras, Yang Baxter equation, and Hopf algebras etc.

The pointed Lie rack structure was primarily introduced by M. Kinyon [34] as an algebraic structure compatible with an underling pointed smooth manifold. In [34], M. Kinyon proved that if $(X, 1)$ is a pointed Lie rack (with left self-distributivity), T_1X carries a structure of (left) Leibniz algebra. Moreover, in the case when the Lie rack structure is associated to a Lie group[1] G then the associated (left) Leibniz algebra is the Lie algebra[2] of G. This leads naturally to claim that pointed Lie racks provide an answer to the problem of integrating Leibniz algebras raised by Loday [36] which consists in finding an appropriate generalization of the Lie's third theorem A.3.2. However, pointed Lie racks need additional requirements to be a global answer of integrating Leibniz algebras problem. More precisely, if we denote by LRack the category of simply connected Lie racks, and by Lgroups the category of simply connected Lie groups, RLeibniz the category of right Leibniz algebras and Liealgebra the category of Lie algebras. We have Lgroups $\subset$ LRack and Theorem 2.4.1 and Proposition 2.4.2 ensure that there is a tangent functor $T_{Leibniz} : \text{LRack} \longrightarrow \text{RLeibniz}$ which sends, in addition, Lgroups to Liealgebra. The third Lie theorem A.3.2 asserts that $T_{Lie} : \text{Lgroups} \longrightarrow \text{Liealgebra}$ is invertible and it is natural to ask if $T_{Leibniz}$ is invertible and the restriction of its inverse coincides with the inverse of T_{Lie}. This problem is well known by the *coquecigrue problem for Leibniz algebras*. Until now, there is no natural answer to this problem, however, several partial answers have been published since then as in [34] where a positive answer was given for split Leibniz algebras. In [20], S. Covez gave a local answer to the integration problem. He showed that every Leibniz algebra can be integrated into a local augmented Lie rack. In [14], Bordemann gave a global process of integrating Leibniz algebras. Unfortunately, this process is not functorial.

Among these investigations, there is one that concerns particularly symmetric Leibniz algebras which are both left and right Leibniz algebras. S. Benayadi and M. Bordemann [9] succeeded in giving a functorial process for integrating symmetric Leibniz algebras detailed in Section 3.2. By using this process and a well-known interaction between symmetric Leibniz algebras and Lie algebras (see Proposition 3.1.1) we achieve the first steps in the study of Lie rack structures on Leibniz algebras.

In literature, the study of (pointed) Lie racks and Leibniz algebras was always motivated by either the problem of integrating Leibniz algebras or the study of each notion separately. Up to our knowledge, there have been no study of Lie rack structures on Leibniz algebras on its own interest. In this thesis, we study two large classes of Lie rack structures on Leibniz algebras:

1. Lie racks associated to symmetric Leibniz algebras.

2. Analytic linear Lie rack structures on Leibniz algebras.

[1] Any Lie group G has a pointed Lie rack structure given by the conjugation operation on G, i.e., $g \rhd h = h^{-1}gh,\ \forall g, h \in G$.
[2] Any Lie algebra is a (symmetric) Leibniz algebra.

Overview of basic concepts. A pointed Lie rack is a pointed smooth manifold $(X,1)$ equipped with a binary operation $\rhd$ such that:

- $\rhd : X \times X \longrightarrow X$ is smooth,
- for any $x \in X$, $\mathrm{R}_x : X \longrightarrow X$, $y \longmapsto y \rhd x$ is a diffeomorphism,
- for any $x,y,z \in X$, $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$ (right self-distribution),
- for any $x \in X$, $x \rhd 1 = x$ and $1 \rhd x = 1$.

Any Lie group G has a Lie rack structure given by $h \rhd g = g^{-1}hg$. Distinctive properties and examples of (pointed Lie) racks are presented in Section 2.3. We have mentioned before that Lie racks constitute an interesting topic in the study of Leibniz algebras.

Leibniz algebras were first introduced and investigated in the papers of Blokh [12, 13] under the name of D-algebras. Then they were rediscovered by Loday [36] who called them Leibniz algebras. A right Leibniz algebra is an algebra $(\mathfrak{L},[\ ,\ ])$ over a field $\mathbb{K}$ such that, for every element $u \in \mathfrak{L}$, $\mathrm{ad}_u : \mathfrak{h} \longrightarrow \mathfrak{h}$, $v \mapsto [u,v]$ is a derivation of $\mathfrak{L}$, i.e.,

$$[[v,w],u] = [[v,u],w] + [v,[w,u]], \quad v,w \in \mathfrak{L}.$$

Lie racks and Leibniz algebras have equally the left and the right version, but these two sides are parallel. For instance, we may pass from a right Leibniz algebra $(\mathfrak{L},[\ ,\ ])$ (see below the definition) to a left Leibniz algebra by considering the new multiplication $u.v = [v,u]$ and vice-versa. So, we will choose working with either left or right upon some needed requirements. Here, we give, briefly, the global frame of our specific algebraic structures adopting the right versions. On the other hand, any Lie algebra is a right Leibniz algebra. Conversely a right Leibniz algebra is a Lie algebra if and only if its bracket is skew-symmetric. Many results of the theory of Lie algebras can be extended to Leibniz algebras (see [2, 3, 8, 10, 15, 37]).

Tangent functor: in [34], Kinyon proved that if $(X,1)$ is a pointed Lie rack, then T_1X carries a structure of left Leibniz algebra. Moreover, in the case when the Lie rack structure is associated to a Lie group G then the associated left Leibniz algebra is the Lie algebra of G.

Let $(X,1)$ be a pointed Lie rack with right distributivity. We denote by $\mathrm{Ad}_x : T_1X \longrightarrow T_1X$ the differential of R_x at 1. We have

$$\mathrm{R}_{x\rhd y} = \mathrm{R}_y \circ \mathrm{R}_x \circ \mathrm{R}_y^{-1} \quad \text{and} \quad \mathrm{Ad}_{x\rhd y} = \mathrm{Ad}_y \circ \mathrm{Ad}_x \circ \mathrm{Ad}_y^{-1}.$$

Thus $\mathrm{Ad} : X \longrightarrow \mathrm{GL}(T_1X)$ is a homomorphism of Lie racks. If we put

$$[u,v]_{\rhd} = \frac{d}{dt}_{|t=0} \mathrm{Ad}_{c(t)}(u), \quad u,v \in T_1X, c :]-\varepsilon,\varepsilon[\longrightarrow X,\ c(0) = 1,\ c'(0) = v.$$

$(T_1X,[\ ,\ ]_{\rhd})$ becomes a right Leibniz algebra. When, there is not confusion, the bracket $[\ ,\ ]_{\rhd}$ is simply denoted by $[\ ,\ ]$. For a detailed proof, see Section 2.4.

In our work, we are primarily interested in the following two classes of Lie racks on Leibniz algebras for which we have succeeded to prove many important results:

Part 1: Lie racks associated to symmetric Leibniz algebras

Let $(\mathfrak{L},\cdot)$ be a symmetric Leibniz algebra. According to Proposition 3.1.1, there exists a Lie bracket $[\ ,\ ]$ on $\mathfrak{L}$ and a bilinear symmetric map $\omega : \mathfrak{L} \times \mathfrak{L} \longrightarrow Z(\mathfrak{L})$, where $Z(\mathfrak{L})$ is the center of $(\mathfrak{L},\ [\ ,\ ])$, such that,

$$u \cdot v = [u,v] + \omega(u,v) \text{ for all } u,v \in \mathfrak{L}$$

and ω satisfies

$$\omega([u,v],w) = \omega(\omega(u,v),w) = 0. \tag{1.1}$$

Now, we describe the process of integrating symmetric Leibniz algebras [9].

Denote by $\mathfrak{L} \cdot \mathfrak{L} = \mathrm{span}\{u.v,\ u,\ v \in \mathfrak{L}\}$, $\mathfrak{a} = \mathfrak{L}/\mathfrak{L}\cdot\mathfrak{L}$, $q : \mathfrak{L} \longrightarrow \mathfrak{a}$ and define $\beta : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{L}$ given by

$$\beta(q(u),\ q(v)) = \omega(u,v).$$

By virtue of 1.1, β is well defined. Moreover, since $[u,v] = \frac{1}{2}(u \cdot v - v \cdot u)$, q is a Lie algebra homomorphism when $\mathfrak{a}$ is considered as an abelian Lie algebra. Consider G the connected and simply connected Lie group whose Lie algebra is $(\mathfrak{L}, [\,,\,])$ and $\exp : \mathfrak{L} \longrightarrow G$ its exponential. Then there exists a homomorphism of Lie groups $\kappa : G \longrightarrow \mathfrak{a}$ such that $d_e\kappa = q$. Finally, consider $\chi : G \times G \longrightarrow G$ given by

$$\chi(h, g) = \exp(\beta(\kappa(h), \kappa(g))) \text{ for all } h, g \in G.$$

Define now on G the binary product by putting

$$h \rhd g := g^{-1}hg\chi(h, g) \;\text{ for all } h, g \in G. \tag{1.2}$$

Thus, we have the following main theorem.

Theorem 1.0.1. *Let $(\mathfrak{L}, .)$ be a symmetric Leibniz algebra. Let G be the connected simply connected Lie group associated to the underlying Lie algebra endowed with the binary operation $\rhd$ defined by (3.6). Then $(G, \rhd)$ is a pointed Lie rack whose associated right Leibniz algebra is exactly $(\mathfrak{L}, .)$.*

We call the obtained Lie rack $(G, \rhd)$ the *Lie rack associated to symmetric Leibniz algebra* $(\mathfrak{L}, .)$. Thanks to Proposition 2.3.1, we show that any Lie rack $(G, \rhd)$, the set of its idempotents

$$Q((G, \rhd)) = \{g \in G\,;\, \chi(g, g) = 1_G\}$$

is a topological quandle[3].

It has been said before that the above method gives, surprisingly, a global answer to the problem of integrating symmetric Leibniz [9]. It had been a crucial role in the study of Lie racks associated to symmetric Leibniz algebras. So, keeping this method in mind and by using Proposition 3.1.1, we establish our results concerning this class of Lie rack structures.

Our **main results** fall into three steps:

1. We classify symmetric Leibniz algebras of dimension 3 and 4. We proceed as follows:
   - Take a Lie algebra $\mathfrak{g}$ with non trivial center in the list of [11].
   - Determine, by straightforward computation, symmetric maps ω satisfying (1.1).
   - In the spirit of Proposition 3.1.2, we act by the group of automorphisms of $\mathfrak{g}$ on the obtained ω to reduce the parameters.

   The results of this part are summarized in Table 3.1.

2. For any symmetric Leibniz algebra from Table 3.1, we construct the associated Lie racks. Then we give the associated topological quandles defined in 1. The results of this part are presented in Subsection 3.3.2.

3. We study some algebraic properties of the derived topological quandles, namely quasi-triviality and medial property. The results are detailed in 4.3.

All these results constitute the subject of our published paper:

H. Abchir, F-E. Abid, and M. Boucetta, A class of Lie racks associated to symmetric Leibniz algebras, Journal of Algebra and its Applications (2021).

The second class of Lie rack structures on Leibniz algebras is differently treated. We adopt the left version of Lie racks and Leibniz algebras.

Part 2: Analytic linear Lie rack structures on Leibniz algebras.

A *linear* Lie rack structure on a finite dimensional vector space V is a Lie rack operation $(x, y) \mapsto x \rhd y$ pointed at 0 and such that for any x, the map $\mathrm{L}_x : y \mapsto x \rhd y$ is linear. A linear Lie rack operation $\rhd$ is called *analytic* if for any $x, y \in V$,

$$x \rhd y = y + \sum_{n=1}^{\infty} A_{n,1}(x, \ldots, x, y), \tag{1.3}$$

[3] A topological quandle is a topological rack X with the idempotency condition, i.e., for any $x \in X$, $x \rhd x = x$

where for each n, $A_{n,1} : V \times \ldots \times V \longrightarrow V$ is an $n+1$-multilinear map which is symmetric in the n first arguments. In this case, $A_{1,1}$ is the left Leibniz bracket associated to $\rhd$.

If $(\mathfrak{h}, [\ ,\ ])$ is a left Leibniz algebra then the operation $\overset{c}{\rhd}: \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ given by

$$u \overset{c}{\rhd} v = \exp(\mathrm{ad}_u)(v)$$

defines an analytic linear Lie rack structure on $\mathfrak{h}$ such that the associated left Leibniz bracket on $T_0\mathfrak{h} = \mathfrak{h}$ is the initial bracket $[\ ,\ ]$. We call $\overset{c}{\rhd}$ the *canonical* linear Lie rack structure associated to $(\mathfrak{h}, [\ ,\ ])$.

During our study of linear Lie rack structures with an emphasis on analytic linear Lie rack structures, we came to prove many aim results. Here, we summarize all such results. At first, we show that there is a large class of linear Lie rack structures on $(\mathfrak{h}, [\ ,\ ])$ containing the canonical one as stated in the following proposition.

Proposition 1.0.2. *Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra, $F : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function and $P : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathbb{R}$ a symmetric multilinear p-form such that, for any $y, x_1 \ldots, x_p \in \mathfrak{h}$,*

$$\sum_{i=1}^{p} P(x_1, \ldots, [y, x_i], \ldots, x_p) = 0.$$

Then the operation $\rhd$ given by

$$x \rhd y = \exp(F(P(x, \ldots, x))\mathrm{ad}_x)(y)$$

is a linear Lie rack structure on $\mathfrak{h}$ and its associated left Leibniz bracket is $[\ ,\ ]_\rhd = F(0)[\ ,\ ]$. Moreover, if F is analytic then $\rhd$ is analytic.

This proposition shows that a left Leibniz algebra might be associated to many non equivalent pointed Lie rack structures. It motivates, in addition, the study of linear Lie rack structures and gives a meaning to the following definition.

Definition 1.0.1. *A left Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ is called rigid if any analytic linear Lie rack structure $\rhd$ on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd = [\ ,\ ]$ is given by*

$$x \rhd y = \exp(F(P(x, \ldots, x))\mathrm{ad}_x)(y),$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is analytic with $F(0) = 1$ and $P : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathbb{R}$ is a symmetric multilinear p-form such that, for any $y, x_1 \ldots, x_p \in \mathfrak{h}$,

$$\sum_{i=1}^{p} P(x_1, \ldots, [y, x_i], \ldots, x_p) = 0.$$

Then we introduce the notion of (left) Leibniz algebras rigidity which was suggested to us by the one used in the study of linearization of Poisson structures (see [18]). The introduction of the notion of rigidity is motivated by the construction of a large family of non rigid Leibniz algebras (see Proposition 4.1.5, and Corollary 4.1.6). Now, we list our main theorems.

Theorem 1.0.3. *Let V be a real finite dimensional vector space and $(A_{n,1})_{n\geq 1}$ a sequence of $n+1$-multilinear maps symmetric in the n first arguments. We suppose that the operation $\rhd$ given by*

$$x \rhd y = y + \sum_{n=1}^{\infty} A_{n,1}(x, \ldots, x, y)$$

converges. Then $\rhd$ is a Lie rack structure on V if and only if for any $p, q \in \mathbb{N}^*$ and $x, y, z \in V$,

$$A_{p,1}(x, A_{q,1}(y,z)) = \sum_{s_1+\ldots+s_q+k=p} A_{q,1}(A_{s_1,1}(x,y), \ldots, A_{s_q,1}(x,y), A_{k,1}(x,z)), \quad (1.4)$$

where for sake of simplicity $A_{p,1}(x,y) := A_{p,1}(x, \ldots, x, y)$.

In particular, if $p = q = 1$ we get that $[\ ,\ ] := A_{1,1}$ is a left Leibniz bracket which is actually the left Leibniz bracket associated to $(V, \rhd)$.

Theorem 1.0.4. *Let* $(\mathfrak{h}, [\ ,\ ])$ *be a left Leibniz algebra such that* $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$. *Let* $(A_{n,1})_{n\geq 0}$ *be a sequence where* $A_{0,1}(x,y) = y$ *and* $A_{1,1}(x,y) = [x,y]$ *and, for any* $n \geq 2$, $A_{n,1} : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathfrak{h}$ *is multilinear invariant and symmetric in the n first arguments. We suppose that the* $A_{n,1}$ *satisfy* 4.6. *Then there exists a unique sequence* $(B_n)_{n\geq 2}$ *of invariant symmetric multilinear maps* $B_n : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathfrak{h}$ *such that, for any* $x, y \in \mathfrak{h}$,

$$A_{n,1}(x,y) = A^0_{n,1}(x,y) + \sum_{\substack{1 \leq k \leq [\frac{n}{2}] \\ s = l_1 + \ldots + l_k \leq n}} A^0_{k,1}(B_{l_1}(x), \ldots, B_{l_k}(x), A^0_{n-s,1}(x,y)), \quad (1.5)$$

where $A_{p,1}(x,y) = A_{p,1}(x, \ldots, x, y)$ *and* $B_l(x) = B_l(x, \ldots, x)$.

Theorem 1.0.5. *Let* $\mathfrak{h}$ *be either* $\mathfrak{sl}_2(\mathbb{R})$ *or* $\mathfrak{so}(3)$ *and* $\rhd$ *an analytic linear Lie rack structure on* $\mathfrak{h}$ *such that* $[\ ,\ ]_\rhd$ *is the Lie algebra bracket of* $\mathfrak{h}$. *Then there exists an analytic function* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *given by*

$$F(u) = 1 + \sum_{k=1}^{\infty} a_k u^k$$

such that, for any $x, y \in \mathfrak{h}$,

$$x \rhd y = \exp(F(\langle x, x \rangle)\mathrm{ad}_x)(y),$$

where $\langle x, x \rangle = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_x)$. *So* $\mathfrak{h}$ *is rigid.*

These results are the subject of our published paper:

H. Abchir, F-E. Abid, and M. Boucetta, Analytic linear Lie rack structures on Leibniz algebras, Journal of Communications in Algebra, Vol. 48, No. 8, pp. 3249-3267 March 2020.

This report is organized as follows. In Chapter 2, we review some standard facts on Lie racks and Leibniz algebras and we describe the link between these two concepts. In Chapter 3, we state in details the main results about Lie racks associated to symmetric Leibniz algebras. The chapter 4 is devoted to a detailed exposition of Theorem 4.1.1, 4.2.1, and 4.3.1 with its corollaries. At the end of this report, we give two appendices: the first Appendix A contains an overview of Lie theory intended to help the readers in understanding the results stated in this report. In the Appendix B, we recall some aspects of the interaction between racks and knot theory.

Chapter 2

Lie racks and Leibniz algebras

2.1 Leibniz algebras

In this section, we provide some basic definitions and algebraic properties in Leibniz algebras theory. Leibniz algebras were introduced by Loday [36] as a generalization of Lie algebras. Therefore, many results and properties are extended from Lie theory. Although, the theory of Leibniz algebras is a large subject on its own (see [2, 3, 15, 8, 10, 37]). Recently, there are fundamental and significant results restricted to Leibniz algebras and their relationship with other algebraic structures, namely Lie racks which are treated further in Section 2.3 (see for instance [20, 14, 34]).

2.1.1 Definitions and immediate properties

An algebra $\mathfrak{L}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a bilinear map $\cdot : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ called "multiplication".

Let $(\mathfrak{L}, \cdot)$ be an algebra. For any $u \in \mathfrak{L}$, we denote $\mathrm{L}_u,\ \mathrm{R}_u : \mathfrak{L} \longrightarrow \mathfrak{L}$ the endomorphisms given by $\mathrm{L}_u(v) = u \cdot v$ and $\mathrm{R}_u(v) = v \cdot u$, $\forall\ v \in \mathfrak{L}$. The map L_u (resp. R_u) is called the left multiplication by u (resp. right multiplication by u). In the study of Leibniz algebras the multiplication is often written (by analogy with the Lie algebras) in bracket form $[\ ,\ ]$.

Definition 1. A left Leibniz algebra is an algebra $(\mathfrak{L}, [\ ,\ ])$ such that for any $u \in \mathfrak{L}$, the left multiplication L_u is a derivation, i.e.,

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]] \tag{2.1}$$

for any $v, w \in \mathfrak{L}$. We call the identity 2.1, the left Leibniz identity.

It is clear that 2.1 is equivalent to each one of the following relations:

$$\begin{aligned} \mathrm{L}_{[u,v]} &= [\mathrm{L}_u, \mathrm{L}_v], & (2.2)\\ \mathrm{R}_{[u,v]} &= \mathrm{R}_v \circ \mathrm{R}_u + \mathrm{L}_u \circ \mathrm{R}_v, & (2.3)\\ \mathrm{R}_{[u,v]} &= [\mathrm{L}_u, \mathrm{R}_v]. & (2.4) \end{aligned}$$

Similarly, we define a right Leibniz algebra:

Definition 2. A right Leibniz algebra is an algebra $(\mathfrak{L}, [\ ,\ ])$ such that for any $u \in \mathfrak{L}$, the right multiplication R_u is a derivation, i.e.,

$$[[v,w],u] = [[v,u],w] + [v,[w,u]] \tag{2.5}$$

for any $v, w \in \mathfrak{L}$.

The identity (2.5) is called the right Leibniz identity which is equivalent to the following relations.

$$\begin{aligned} [\mathrm{R}_u, \mathrm{R}_v] &= \mathrm{R}_{[v,u]}, & (2.6)\\ [\mathrm{R}_u, \mathrm{L}_v] &= \mathrm{L}_{[v,u]}. & (2.7) \end{aligned}$$

Proposition 2.1.1. *Let $(\mathfrak{L}, [\ ,\ ])$ be an algebra. Then*

1. *If* $(\mathfrak{L}, [\ ,\ ])$ *is a left Leibniz algebra, we have* $\mathrm{L}_{[u,v]} = -\mathrm{L}_{[v,u]}$, $\forall u, v \in \mathfrak{L}$.

2. *If* $(\mathfrak{L}, [\ ,\ ])$ *is a right Leibniz algebra, we have* $\mathrm{R}_{[u,v]} = -\mathrm{R}_{[v,u]}$, $\forall u, v \in \mathfrak{L}$.

Proof. These two equalities come from Leibniz identities. Let $u, v, w \in \mathfrak{L}$.

1. We have

$$\begin{aligned}[v,[u,w]] &= [[v,u],w] + [u,[v,w]]\\ [v,[u,w]] &= [u,[v,w]] - [[u,v],w],\end{aligned}$$

Thus, $[[u,v],w] = -[[v,u],w]$.

2. Similarly, we have

$$\begin{aligned}[[w,u],v] &= [[w,v],u] - [w,[v,u]]\\ [[w,u],v] &= [[w,v],u] + [w,[u,v]]\end{aligned}$$

So, $[w,[v,u]] = -[w,[u,v]]$.

□

We remark that a left Leibniz algebra is not necessarily a right Leibniz algebra as shown in the following example.

Example 1. Let $\mathfrak{L}$ be 2-dimensional algebra with a suitable basis $\{x,y\}$. The following product on $\mathfrak{L}$

$$[x,x] = 0,\ [x,y] = 0,\ [y,x] = x,\quad [y,y] = x$$

makes it into a left Leibniz algebra, but $(\mathfrak{L}, [\ ,\ ])$ is not a right Leibniz algebra, since $[[y,y],y] \neq [y,[y,y]] + [[y,y],y]$.

Proposition 2.1.2 ([10])**.** *Let* $(\mathfrak{L}, [\ ,\ ])$ *be a left Leibniz algebra. Then* $(\mathfrak{L}, [\ ,\ ])$ *is a right Leibniz algebra if and only if*

$$[x,[y,z]] = -[[y,z],x],\ \forall x,y,z \in \mathfrak{L}.$$

Although, the concepts of left and right Leibniz algebras are parallel. For example, we may pass from a left algebra $(\mathfrak{L}, [\ ,\ ])$ to a right Leibniz algebra by considering a new multiplication $u.v = [v,u]$. Hence, it will be convenient to use Leibniz algebras referring unspecified left or right Leibniz algebras. Throughout of this chapter, we will deal mostly with right Leibniz algebras. Furthermore, if the Leibniz product satisfies, in addition, the skew-symmetry condition

$$[x,x] = 0$$

then the Leibniz identity can be easily reduced to the Jacobi identity. Therefore Lie algebras are a particular class of Leibniz algebras. However the class of Leibniz algebras is far more large as the following example shows.

Example 2. Let $(A,\cdot)$ be an associative algebra and T an endomorphism of A such that $T^2 = T$. The bracket on A given by

$$[a,b] = a\cdot Tb - b\cdot Ta$$

is a Leibniz bracket and it is a Lie bracket if and only if $T = \mathrm{Id}_A$.

Here, we cite some simple examples of real Leibniz algebras $\mathfrak{L}$ of dimension two [10, 25].

Examples 1. If $(\mathfrak{L} :=< a,b >, [\ ,\ ])$ be two dimensional Leibniz algebra. Then $\mathfrak{L}$ is isomorphic to one of the following algebras:

1. The two dimensional Lie algebras.

2. The left Leibniz algebras:

   (i) $\mathfrak{L}_1$, endowed with the bracket given by $[b,b] = a$

   (ii) $\mathfrak{L}_2$, endowed with the bracket given by $[b,a] =$, and $[b,b] = a$

3. The right Leibniz algebra $\mathfrak{L}_3$, endowed with the bracket given by $[a,b]=a$

For more examples, see [27]. We can easily check that the Leibniz algebra (i) is both a left and right Leibniz algebra. We then have the following definition.

Definition 3. An algebra $(\mathfrak{L},[\ ,\ ])$ which is both left and right Leibniz algebra is called symmetric Leibniz algebra.

Using Equations 2.2 and 2.6, we obtain the following proposition.

Proposition 2.1.3. *Let $(\mathfrak{L},\cdot)$ be an algebra. The following assertions are equivalent:*

1. $(\mathfrak{L},\cdot)$ *is a symmetric Leibniz algebra.*
2. *For any $u,v\in\mathfrak{L}$, $[\mathrm{L}_u,\mathrm{L}_v]=\mathrm{L}_{[u,v]}=-\mathrm{R}_{[u,v]}$.*
3. *For any $u,v\in\mathfrak{L}$, $[\mathrm{R}_u,\mathrm{R}_v]=\mathrm{R}_{[v,u]}=-\mathrm{L}_{[v,u]}$.*

As an example, any Lie algebra is symmetric Leibniz algebra. However, the class of symmetric Leibniz algebras is far more bigger than the class of Lie algebras as we will see later.

Ideals and subalgebras

Definition 4. Let $(\mathfrak{L},\ [\ ,\ ])$ be a Leibniz algebra.

1. A Leibniz subalgebra of $\mathfrak{L}$ is a vector subspace $\mathfrak{g}$ of $\mathfrak{L}$ such that $[\mathfrak{g},\mathfrak{g}]\subseteq\mathfrak{g}$.
2. A left (resp. right) ideal of $\mathfrak{L}$ is a vector subspace I such that $[\mathfrak{L},I]\subseteq I$ (resp. $[I,\mathfrak{L}]\subseteq I$).
3. An ideal of $\mathfrak{L}$ is a left and right ideal of I.

We denote

$$Z^l(\mathfrak{L})=\{u\in\mathfrak{L},\ [u,\mathfrak{L}]=0\},\ Z^r(\mathfrak{L})=\{u\in\mathfrak{L},\ [\mathfrak{L},u]=0\}\quad\text{and}\quad Z(\mathfrak{L})=Z^l(\mathfrak{L})\cap Z^r(\mathfrak{L})$$

respectively, the left center, the right center and the center of $\mathfrak{L}$. Note that the center $Z(\mathfrak{L})$ of $\mathfrak{L}$ contains elements of the form

$$[u,v]+[v,u],\ [u,u],\ [[u,v],[v,u]]$$

We denote

$$\mathrm{Leib}(\mathfrak{L})=\mathrm{span}\left\{[u,v]+[v,u],\ u,v\in\mathfrak{L}\right\}.$$

Proposition 2.1.4. *Let $(\mathfrak{L},\ [\ ,\ ])$ be a Leibniz algebra.*

1. $\mathrm{Leib}(\mathfrak{L})$ *is an ideal of $\mathfrak{L}$.*
2. $\mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ *is a Lie algebra denoted by $\mathfrak{L}_{lie}$, and $\mathfrak{L}$ is a Lie algebra if and only if* $\mathrm{Leib}(\mathfrak{L})=\{0\}$.

Notice that $\mathrm{Leib}(\mathfrak{L})$ can be viewed as the space generated by the elements of the form $[u,u]$ for $u\in\mathfrak{L}$. Furthermore, if we consider $\mathfrak{L}$ a left (resp. right) Leibniz algebra, then we have $\mathrm{Leib}(\mathfrak{L})\subset Z^l(\mathfrak{L})$ (resp. $\mathrm{Leib}(\mathfrak{L})\subset Z^l(\mathfrak{L})$).

Definition 5. Let $\mathfrak{L}$ and $\mathfrak{h}$ two Leibniz algebras. A morphism of Leibniz algebras $f:\mathfrak{L}\longrightarrow\mathfrak{h}$ is a linear map which respects the product, that is

$$f([u,v])=[f(u),f(v)],\ \forall u,v\in\mathfrak{L}.$$

Example 3. Let $(\mathfrak{L},\ [\ ,\ ])$ be a left Leibniz algebra. The linear map $\mathrm{ad}:\mathfrak{L}\longrightarrow\mathrm{End}(\mathfrak{L})$ which maps u to $\mathrm{ad}_u(v)=[u,v]$, $\forall v\in\mathfrak{L}$ is a morphism of Leibniz algebras. Using Leibniz identity, we can easily check that $\mathrm{ad}_{[u,v]}=[\mathrm{ad}_u,\mathrm{ad}_v]$, $\forall u,v\in\mathfrak{L}$.

Proposition 2.1.5. *Let $f:\mathfrak{L}\longrightarrow\mathfrak{h}$ be a morphism of Leibniz algebras, then* $\mathrm{Ker}(f)$ *is an ideal in $\mathfrak{L}$.*

Proof. Let $u \in \mathrm{Ker}(f)$, and let $v \in \mathfrak{L}$. We have

$$f([u,v]) = [f(u), f(v)] = 0$$

and

$$f([v,u]) = [f(v), f(u)] = 0,$$

hence $\mathrm{Ker}(f)$ is an ideal in $\mathfrak{L}$. □

2.1.2 Some classes of Leibniz algebras

As Lie algebras, we introduce some important classes of Leibniz algebras, namely, nilpotent, solvable and simple Leibniz algebras. For more details, see [2, 4, 15].

For a given right Leibniz algebra $(\mathfrak{L}, [\ ,\ ])$ the sequences of two-sided ideals defined recursively as follows:

$$\mathfrak{L}^1 = \mathfrak{L},\ \mathfrak{L}^{k+1} = [\mathfrak{L}^k, \mathfrak{L}],\ k \geq 1, \quad \mathfrak{L}^{[1]} = \mathfrak{L},\ \mathfrak{L}^{[s+1]} = [\mathfrak{L}^{[s]}, \mathfrak{L}^{[s]}],\ s \geq 1.$$

are said to be the *lower central* and the *derived series* of $\mathfrak{L}$, respectively.

Definition 6. 1. A Leibniz algebra $\mathfrak{L}$ is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $\mathfrak{L}^n = 0$.

2. A Leibniz algebra $\mathfrak{L}$ is said to be solvable, if there exists $m \in \mathbb{N}$ such that $\mathfrak{L}^{[m]} = 0$.

The minimal number n (respectively m) with such property is said to be the index of nilpotency (respectively the index of solvability) of the Leibniz algebra $\mathfrak{L}$. It is easy to see that the index of nilpotency of an arbitrary n-dimensional Leibniz algebra is at most $n+1$.

Example 4 ([15])**.** Let $\mathfrak{L}$ be 2-dimensional nilpotent right Leibniz algebra. Then $\mathfrak{L}$ is an abelian algebra or it is isomorphic to
$\mathfrak{L}_1 : [e_1, e_1] = e_2$.

For more examples of nilpotent and solvable right Leibniz algebras, see for instance [2, 4, 15]. Similarly to the Lie algebras case, the following analogue of Engel's theorem is valid for Leibniz algebras.

Theorem 2.1.6 ([2])**.** *A finite dimensional right Leibniz algebra $\mathfrak{L}$ is nilpotent if and only if the operator R_x is nilpotent for every $x \in \mathfrak{L}$.*

Theorem 2.1.7 ([2])**.** *A finite dimensional right Leibniz algebra $\mathfrak{L}$ is solvable if and only if $[\mathfrak{L}, \mathfrak{L}]$ is nilpotent.*

Simple Lie algebras are very important in the theory of Lie algebras[1]. But Leibniz algebras which are not Lie algebras cannot be simple in the classical sense, because any Leibniz algebra $\mathfrak{L}$ contains the nonzero proper ideal $\mathrm{Leib}(\mathfrak{L})$.

Definition 7. A Leibniz algebra $\mathfrak{L}$ is called simple if it contains no ideals other than $\{0\}$, $\mathrm{Leib}(\mathfrak{L})$, and $\mathfrak{L}$.

Notice that a Leibniz algebra $\mathfrak{L}$ is simple if and only if the Lie algebra $\mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ is simple. It follows that the above definition agrees with the definition of simple Lie algebras, since $\mathrm{Leib}(\mathfrak{L}) = \{0\}$ in this case.

Theorem 2.1.8 ([2])**.** *Let $\mathfrak{L}$ be a finite dimensional simple (right) Leibniz algebra, which is non-Lie, and $\mathfrak{L}/\mathrm{Leib}(\mathfrak{L})$ is abelian. Then $\mathfrak{L}$ is two-dimensional and therefore isomorphic to one of the following algebras with non-vanishing products:*

(i) $[e_1, e_2] = e_1, \quad [e_2, e_2] = e_1$

(ii) $[e_2, e_2] = e_1$

where e_1, e_2 is the basis of $\mathfrak{L}$.

[1] See Definition 35

2.2 Cohomology of Leibniz algebras

In this section, we briefly introduce the notions of representations and the cohomology of right Leibniz algebras, see [37, 39]. For the cohomology of left Leibniz algebras, see for instance [20, 19].

2.2.1 Representations

Definition 8. Let $(\mathfrak{L}, [\ ,\ ])$ be a right Leibniz algebra. A $\mathfrak{L}$-representation is a vector space M equipped with two bilinear maps

$$[\ ,\ ]_L : \mathfrak{L} \times M \longrightarrow M \quad \text{and} \quad [\ ,\ ]_R : M \times \mathfrak{L} \longrightarrow M$$

satisfying:

$$\begin{array}{lll} (MLL) & [m,[x,y]]_R & = [[m,x]_R,y]_R - [[m,y]_R,x]_R, \\ (LML) & [x,[m,y]_R]_L & = [[x,m]_L,y]_R - [[x,y],m]_L, \\ (LLM) & [x,[y,m]_L]_L & = [[x,y],m]_L - [[x,m]_L,y]_R, \end{array}$$

for any $m \in M$ and $x, y \in \mathfrak{L}$.

In Leibniz theory some authors call a $\mathfrak{L}$-representation by $\mathfrak{L}$-**module**. The relations (MLL) and (LLM) implies that

$$[x,[m,y]_R]_L + [x,[y,m]_L]_L = 0.$$

By definition, any Leibniz algebra $(\mathfrak{L}, [\ ,\ ])$ is a $\mathfrak{L}$-representation by taking $[\ ,\ ]_L = [\ ,\ ]_R = [\ ,\ ]$.

Definition 9. A morphism of Leibniz $\mathfrak{L}$-representations is a linear map which which respects the two brackets $[\ ,\ ]_L$ and $[\ ,\ ]_R$.

Definition 10. Let $(\mathfrak{L}, [\ ,\ ])$ be a Leibniz algebra and let M be a $\mathfrak{L}$-representation.

1. If for any $x \in \mathfrak{L}$, and $m \in M$, $[x,m]_L = -[m,x]_R$, then M is called symmetric.
2. If for any $x \in \mathfrak{L}$, and $m \in M$, $[x,m]_L = 0$, then M is called anti-symmetric.
3. If for any $x \in \mathfrak{L}$, and $m \in M$, $[x,m]_L = -[m,x]_R = 0$, then M is called trivial.

The most important examples of anti-symmetric Leibniz $\mathfrak{L}$-modules are the ideal $\mathrm{Leib}(\mathfrak{L})$ which also can be generated by the elements of the form $[u,u]$ for $u \in \mathfrak{L}$, and the right center $Z^r(\mathfrak{L})$. Moreover, we have a short exact sequence of Leibniz algebras:

$$0 \longrightarrow \mathrm{Leib}(\mathfrak{L}) \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{L}_{lie} \longrightarrow 0$$

which is an abelian extension of Leibniz algebras (see for instance [20, 36, 37]).

In general, any Leibniz $\mathfrak{L}$-module M gives rise to a symmetric $\mathfrak{L}$-module and an anti-symmetric $\mathfrak{L}$-module as follows: the quotient of the Leibniz $\mathfrak{L}$-module M by the submodule generated by the relations $[x,m] + [m,x]$ for all $x \in \mathfrak{L}$ and all $m \in M$, defines a symmetric Leibniz $\mathfrak{L}$-module denoted M^s. The kernel of the projection map $M \longrightarrow M^s$ is an anti-symmetric representation, which is denoted M^a. Therefore, for each Leibniz $\mathfrak{L}$-module M, there is a short exact sequence of Leibniz $\mathfrak{L}$-modules

$$0 \longrightarrow M^a \longrightarrow M \longrightarrow M^s \longrightarrow 0.$$

Remark 1. Let $\mathfrak{L}$ be a Lie algebra and let M be a $\mathfrak{L}$-module in the Lie sense A.1.4. If we take

$$[x,m]_L = -[m,x]_R = [x,m],\ \forall x \in \mathfrak{L},\ m \in M,$$

then M is a symmetric $\mathfrak{L}$-representation (in the Leibniz sense) M^s. Otherwise, if we put

$$[x,m]_L = 0 \quad \text{and} \quad [m,x]_R = [x,m],\ \forall x \in \mathfrak{L},\ m \in M,$$

hence M is an anti-symmetric $\mathfrak{L}$-representation (in the Leibniz sense) M^a.

2.2.2 Cohomology of Leibniz algebras

The cochain complex

Let $\mathfrak{L}$ be a right Leibniz algebra and let M be a $\mathfrak{L}$-representation. For any non-negative integer n, set $C^n(\mathfrak{L}, M) := \mathrm{Hom}(\otimes^n \mathfrak{L}, M)$ and consider the a coboundary operator

$$\delta^n : \mathrm{Hom}(\otimes^n \mathfrak{L}, M) \longrightarrow \mathrm{Hom}(\otimes^{n+1} \mathfrak{L}, M)$$

given by

$$\begin{aligned} \delta^n(\omega)(x_0, \ldots, x_n) &= [x_0, \omega(x_1, \ldots, x_n)]_L + \sum_{i=1}^{n} (-1)^i [\omega(x_1, \ldots, \hat{x}_i, \ldots, x_n), x_i]_R \\ &+ \sum_{1 \leq i < j \leq n} (-1)^{j+1} \omega(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_n), \end{aligned}$$

where $\omega \in C^n(\mathfrak{L}, M)$ and $(x_0, \ldots, x_n) \in \mathfrak{L}^{n+1}$. In [37], Loday proved that $\{C^n(\mathfrak{L}, M), \delta^n\}_{n \geq 0}$ is a cochain complex, i.e, $\delta^{n+1} \circ \delta^n = 0$ for every non-negative integer n.

Definition 11. Let $\mathfrak{L}$ be a right Leibniz algebra and let M be a $\mathfrak{L}$-representation. The cohomology of $\mathfrak{L}$ with coefficients in M is the cohomology of the cochain complex $\{C^n(\mathfrak{L}, M), \delta^n\}_{n \geq 0}$. Then

$$H^n(\mathfrak{L}, M) := \mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1}).$$

$H^0(\mathfrak{L}, M)$ and $H^1(\mathfrak{L}, M)$

Let $\mathfrak{L}$ be a Leibniz algebra and let M be a $\mathfrak{L}$-representation. By definition $C^0(\mathfrak{L}, M) = M$ and $\delta^0(m)(x) = [x, m]_L$. Thus

$$H^0(\mathfrak{L}, M) = \{m \in M | [x, m]_L = 0 \ \forall x \in \mathfrak{L}\}.$$

It is called the *submodule left invariants of* M . Note that if the $\mathfrak{L}$-representation M is anti-symmetric, then $H^0(\mathfrak{L}, M) = M$.

Then we obtain from the defining coboundary operator that

$$\delta^1(\omega)(x, y) = [x, \omega(y)]_L + [\omega(x), y]_R - \omega([x, y]),$$

and so

$$Z^1(\mathfrak{L}, M) = \{\omega \in \mathrm{Hom}(\mathfrak{L}, M) | [x, \omega(y)]_L + [\omega(x), y]_R = \omega([x, y])\}$$

defines the module of derivations from $\mathfrak{L}$ to M, denoted $\mathrm{Der}(\mathfrak{L}, M)$. Moreover, the module of inner derivations from $\mathfrak{L}$ to M, denoted $\mathrm{InnDer}(\mathfrak{L}, M)$, is given by

$$B^1(\mathfrak{L}, M) = \{\omega \in \mathrm{Hom}(\mathfrak{L}, M) | \omega(x) = [m, x]_R\}.$$

Hence, we have

$$H^1(\mathfrak{L}, M) = \mathrm{Der}(\mathfrak{L}, M)/\mathrm{InnDer}(\mathfrak{L}, M).$$

Link between Lie cohomology and Leibniz cohomology

The concept of Lie cohomology of Lie algebra $\mathfrak{L}$ is developed in [28], and is defined further in (A.7). Let $\mathfrak{L}$ be a Lie algebra which is effectively a Leibniz algebra, and let M be a $\mathfrak{L}$-module. Denote

$$C^n_{Lie}(\mathfrak{L}, M) := \mathrm{Hom}(\wedge^n \mathfrak{L}, M),$$

the n-cochain of $\mathfrak{L}$ as a Lie algebra. The canonical inclusion of $\mathrm{Hom}(\wedge^n \mathfrak{L}, M)$ into $\mathrm{Hom}(\otimes^n \mathfrak{L}, M)$ makes sense to the following proposition.

Proposition 2.2.1. *[37, 39] Let $\mathfrak{L}$ be a Lie algebra and let M be a $\mathfrak{L}$-module . We have an epimorphism of cochain complexes*

$$C^n_{Lie}(\mathfrak{L}, M) \overset{i^n}{\longrightarrow} C^n(\mathfrak{L}, M^s)$$

given by the canonical inclusion of $Hom(\wedge^n \mathfrak{L}, M)$ *into* $Hom(\otimes^n \mathfrak{L}, M)$.

Remark 2. Let $(\mathfrak{L}, [\ ,\ ])$ be a Leibniz algebra which is a Lie algebra. The cohomology of $\mathfrak{L}$ as left (or right) Leibniz algebra is different from its cohomology as a Lie algebra, however H^0 and H^1 are the same for both cohomologies.

2.3 Lie racks

2.3.1 Racks and quandles: general properties

In this section, we introduce the notion of racks and quandles for which we give some immediate properties.

Definition 12. 1. A rack is a non-empty set X together with a map $\rhd : X \times X \longrightarrow X$, $(x, y) \mapsto x \rhd y$ such that

(i) for any fixed element $x \in X$, the map $\mathrm{R}_x : X \longrightarrow X$, $y \mapsto y \rhd x$ is a bijection,

(ii) for any $x, y, z \in X$, we have $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$.

2. A rack X is called pointed, if there exists a distinguished element $1 \in X$ such that

$$x \rhd 1 = \mathrm{R}_1(x) = \mathrm{id}_X(x) = x \text{ and } 1 \rhd x = \mathrm{R}_x(1) = 1 \quad \forall x \in X.$$

3. A rack X is called a quandle if the idempotency condition holds, i.e., for any $x \in X,\ x \rhd x = x$.

4. A quandle X is called a Kei if for any $x, y \in X,\ (y \rhd x) \rhd x = y$, i.e., R_x is an involution.

When X is a rack, we write $x \rhd^{-1} y$ for $\mathrm{R}_y^{-1}(x)$.
With these definitions, we can see that kei is a type of quandle and a quandle is a rack.

$$\{kei\} \subset \{quandle\} \subset \{rack\}.$$

The requirement that $(x \rhd y) \rhd z = (x \rhd z) \rhd (y \rhd z)$ says that the binary operation $\rhd$ is **right self-distributive**, i.e., $\rhd$ distributes over $\rhd$ on the *right* the same way that multiplication distributes over addition. In particular, the rack operation is in general *nonassociative*, i.e.,

$$(x \rhd y) \rhd z \neq x \rhd (y \rhd z).$$

In fact, there is the notion of a rack where the binary operation $\rhd$ satisfies **left self-distributivity**:

$$x \rhd (y \rhd z) = (x \rhd y) \rhd (x \rhd z)$$

for all x, y, and $z \in X$. One can always transform a *left self-distributive* rack into a *right self-distributive* rack (and vice-versa) by sending the bijective map $y \longmapsto x \rhd y$ to its inverse which is then denoted $z \longmapsto z \rhd x$. Thus, it is very important to keep track of the order of the elements as well as the parentheses when doing rack computation.

There are obvious definitions of **subrack**, **subquandle** or **subkei**.

Definition 13. Let $(X, \rhd)$ be a rack. A subset $S \subseteq X$ is a subrack of X if $(S, \rhd)$ is itself a rack.

In particular, if we take $(X, \rhd)$ a kei, we can easily show that the closure under $\rhd$ is necessary and sufficient for a subset $S \subseteq X$ to be a subkei.

A map $\phi : X \longrightarrow Y$ between racks $(X, \rhd_1)$ and $(Y, \rhd_2)$ is a homomorphism of racks if ϕ preserves the rack operations, i.e., $\forall x, y \in X,\ \phi(x \rhd_1 y) = \phi(x) \rhd_2 \phi(y)$. If furthermore ϕ is a bijection, it is called an isomorphism of racks. In particular, a bijective rack homomorphism $\phi : X \longrightarrow X$ is called a rack automorphism. The set of all rack automorphisms of X forms a group denoted $\mathrm{Aut}(X)$. The *self-distributivity* axiom means that for any $x \in X$, R_x is an automorphism. Furthermore

$$\mathrm{R}_{x \rhd y} = \mathrm{R}_y \circ \mathrm{R}_x \circ \mathrm{R}_y^{-1}, \quad \forall x, y \in X.$$

The group of inner automorphisms $\mathrm{Inn}(X)$ generated by all bijections R_x is a normal subgroup of $\mathrm{Aut}(X)$. One can define homomorphisms, automorphisms and inner automorphisms of keis or quandles in exactly the same way.

For an arbitrary rack X, one can associate a quandle as follows.

Proposition 2.3.1. *Let $(X, \rhd)$ be a rack and $Q(X)$ be the set of its idempotents,*

$$Q(X) = \{x \in X\ ,\ x \rhd x = x\},$$

then $(Q(X), \rhd)$ is a quandle.

Proof. We note first that $Q(X)$ is closed by the binary operation $\rhd$. Indeed, if $x, y \in Q(X)$, then

$$(x \rhd y) \rhd (x \rhd y) = (x \rhd x) \rhd y = x \rhd y.$$

The right distributivity is obviously satisfied.

For any $x, y \in Q(X)$, the restriction of the right translation R_y to $Q(X)$ is injective by assumption. Actually, the map R_y is bijective when considered as a map defined on X. Let $y, z \in Q(X)$. There exists a unique $x \in X$ such that $\mathrm{R}_y(x) = z$ and then, $x = \mathrm{R}_y^{-1}(z) = z \rhd^{-1} y$. Since R_y^{-1} is a quandle morphism, we have

$$x \rhd x = \mathrm{R}_y^{-1}(z) \rhd \mathrm{R}_y^{-1}(z) = \mathrm{R}_y^{-1}(z \rhd z) = \mathrm{R}_y^{-1}(z) = x$$

Then the unique $x \in X$ such that $x = \mathrm{R}_y^{-1}(z)$ belongs to $Q(X)$. This ends the proof. □

Related to the notion of subrack, if $f : X \longrightarrow Y$ is a rack (or quandle) homomorphism, then the set

$$\mathrm{Im}(f) := \{y \in Y,\ y = f(x) \text{ for some } \ x \in X\}$$

is a subrack (or subquandle) if f is onto. However, if we suppose that f is only a kei homomorphism, then $\mathrm{Im}(f)$ is obviously a subkei of Y, which is referred as the *image subkei of f*.

The following result shows that for any rack can endowed by another rack structure under some assumptions.

Proposition 2.3.2. *Let $(X, \rhd)$ be a rack and $J : X \longrightarrow X$ a map such that, for any $x, y \in X$, $J(y \rhd x) = J(y) \rhd x$, that is $J \circ \mathrm{R}_x = \mathrm{R}_x \circ J$ for any x. Then the operation*

$$x \rhd_J y = x \rhd J(y)$$

defines a rack structure on X. The rack $(X, \rhd_J)$ is called the gauged rack of $(X, \rhd)$ by the map J.

Proof. We have, for any $x, y, z \in X$,

$$\begin{aligned}
(x \rhd_J y) \rhd_J z &= (x \rhd J(y)) \rhd J(z) \\
&= (x \rhd J(z)) \rhd (J(y) \rhd J(z)) \\
&= (x \rhd J(z)) \rhd (J(y \rhd J(z)) \\
&= (x \rhd_J z) \rhd_J (y \rhd_J z). \quad \square
\end{aligned}$$

Examples

Here, we present some known examples of racks and quandles [24].

1. **Trivial rack.** Perhaps the simplest example of rack operation is so-called the *trivial kei, the trivial quandle or the trivial rack: let X be non empty set with the trivial binary operation $x \rhd y := x$, for any $x, y \in X$.*

2. ***Takasaki kei.*** *Let $X = \mathbb{Z}_n$ be the integers mod n with kei operation*

$$x \rhd y := 2y - x \quad mod\ n.$$

This is known as a Takasaki kei, also sometimes called a cyclic kei, or dihedral quandle, and is denoted by R_n. The automorphisms and inner automorphisms group of the Takasaki kei are *given by*

$$\begin{aligned} \mathrm{Aut}(X) &= \mathit{Aff}(X) = \{f_{a,b} : X \longrightarrow X,\ f_{a,b}(x) = ax + b, \quad a \in \mathbb{Z}_n^\times, b \in \mathbb{Z}\}. \\ \mathrm{Inn}(X) &= D_{\frac{m}{2}}, \quad m = l.c.m(2, n). \end{aligned}$$

For more details about these results see [6, 7].

3. ***Conjugation quandles.*** *Let G be a group. G is a quandle by taking the conjugation as a quandle operation.*

$$x \rhd y := y^{-1}xy \quad \forall x,\ y \in G.$$

We denote this quandle by $\mathrm{Conj}(G)$. *This operation can be generalized for any integer* $n \geq 1$ *as follows:*

$$x \rhd y := y^{-n}xy^n \quad \forall x,\ y \in G.$$

In [7], the authors studied the automorphisms group of quandles arising from groups, as examples, the conjugation quandles. They stated the following results.

$$\begin{aligned} \mathrm{Inn}(\mathrm{Conj}(G)) &= G/Z(G). \\ \mathrm{Aut}(\mathrm{Conj}(G)) &= \mathrm{Aut}(G) \ \mathit{iff}\ Z(G) = 1_G. \end{aligned}$$

4. ***Core kei.*** *The rule*

$$x \rhd y := yx^{-1}y$$

also defines a kei operation in a group G called Core kei and denoted by $core(G)$.

5. ***Coxeter keis.*** *Let V be a* $\mathbb{F}$*-vector filed with symmetric bilinear form* $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{F}$. *Let X be the subset of V consisting of vectors u such that* $< u, u > \neq 0$. *The formula*

$$u \rhd v := \frac{2\langle u, v\rangle}{\langle v, v\rangle} v - u$$

defines a quandle structure on X, more precisely a kei structure called Coxeter kei. Geometrically, $u \rhd v$ *presents the reflection of u across the line containing v.*

6. ***Alexander quandle.*** *Let A be an additive abelian group and* $t \in \mathrm{Aut}(A)$. *Then the set A equipped with the binary operation*

$$x \rhd y := tx + (\mathrm{id}_A - t)y$$

is a quandle called the Alexander quandle of A with respect to t.
For instance, let $\mathbb{Z}[t,\ t^{-1}]$ *be the ring of Laurent polynomials in the variable t. Let M be a* $\mathbb{Z}[t,\ t^{-1}]$*-module. The operation*

$$x \rhd y := tx + (1 - t)y$$

for any $x,\ y \in M$, *makes M into a quandle.*

7. ***Generalized Alexander quandles.*** *Let G be a group and* $\varphi \in \mathrm{Aut}(G)$, *then the set G with binary operation*

$$x \rhd y := \varphi(xy^{-1})y$$

gives a quandle structure on G, which we denote by $Alex(G, \varphi)$. *These quandles are referred in the literature as generalized Alexander quandles. In particular, if A is an additive abelian group and* $\varphi = -\mathrm{id}_A$, *then* $x \rhd y = 2y - x$. *Thus we have* $Alex(A, \varphi) = T(A)$, *the Takasaki kei of A. On the other hand, for arbitrary* φ, *we have* $x \rhd y = \varphi(x) + (\mathrm{id}_A - \varphi)(y)$. *Therefore, in this case,* $Alex(A, \varphi)$ *is the usual Alexander quandle of A with respect to* φ.

8. *The 2-sphere* $S^2 = \{x \in \mathbb{R}^3,\ \|x\| = 1\}$ *with the following product*

$$x \rhd y := 2(x.y)y - x,$$

where $x.y$ *is the scalar product in* $\mathbb{R}^3$ *of the vectors x and y, is a quandle.*

9. ***Canonical rack structures on Leibniz algebras.*** *Let* $(\mathfrak{L}, [\ ,\ ])$ *is a right Leibniz algebra, the operation* $\overset{c}{\rhd}: \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ *given by*

$$u \overset{c}{\rhd} v = \exp(\mathrm{ad}_v)(u) \tag{2.8}$$

defines rack structure on $\mathfrak{L}$ *which is called the canonical rack structure on* $\mathfrak{L}$*.*

In what follows, we introduce briefly some particular classes of racks, quandles.

1. **Augmented racks.** This class of racks comprises a set X with a right action by a group G, which is written as

$$(x, g) \longmapsto x.g, \quad \text{where } x,\ x.g \in X \quad \text{and} \quad g \in G,$$

and a function $\varepsilon\ : X \longrightarrow G$ satisfying the **augmented identity**:

$$\varepsilon(x.g) = g^{-1}\varepsilon(x)g \quad \text{for all } \ x \in X, g \in G,$$

which is precisely the same as saying that ε is a G-map when the action on G on itself is taken to be conjugation. We can define a rack operation on X by

$$x \rhd y = x.\varepsilon(y)$$

with inverse rack operation

$$x \rhd^{-1} y = x.\varepsilon(y)^{-1}.$$

2. **Medial quandles.** A quandle Q is called medial, if for any $x, y, z, w \in Q$, we have

$$(x \rhd y) \rhd (z \rhd w) = (x \rhd z) \rhd (y \rhd w).$$

It is easily to see that Alexander quandles presented in Example (6) are medial quandles. For more details and examples see for instance [31].

3. **Latin quandles.** Consider Q an arbitrary quandle, and for any $x \in Q$, denote $\mathrm{L}_x : Q \longrightarrow Q$, the left multiplication which maps y into $x \rhd y$.

   - A quandle Q is called latin if for any $x \in Q$, $\mathrm{L}_x \in S_Q$.
   - A quandle Q is called semi-latin if for any $x \in Q$, L_x is one-to-one.

   It is worth to point out that lain quandle is connected (see further 2.3.1). Here, we give some examples.

   - The quandle $\mathrm{core}(\mathbb{Z})$ is semi-latin but not latin.
   - For an abelian group G, the quandle $\mathrm{core}(G)$ is semi-latin if and only if G has no 2-torsion.
   - Let G be a group. If we assume that for any $a, b, c \in G$, $aN_G(c) = bN_G(c)$ iff $a = b$. Then the (left distributive) conjugation quandle $\mathrm{Conj}(G)$ is latin.

   For more details see [42].

Orbits and stabilizers

Given an arbitrary rack $(X, \rhd)$. Then, the map

$$\begin{array}{rccc} \mathrm{R}: & X & \longrightarrow & \mathrm{Inn}(X) \\ & x & \longmapsto & R_x \end{array}$$

induces a right action of the group $\mathrm{Inn}(X)$ on the set X. The orbit of an element $x \in X$, denoted $\Omega(x)$, is given

$$\Omega(x) := \{\varphi(x),\ \varphi \in \mathrm{Inn}(X)\}.$$

The stabilizer of an element $x \in X$, written $\mathrm{Stab}(x)$, is defined by

$$\mathrm{Stab}(x) := \{y \in X,\ x \rhd y = x\}.$$

We say that a rack $(X, \triangleright)$ is *algebraically connected* or simply *connected*, when there will be no confusion with topological connectivity, if the inner automorphism group $\mathrm{Inn}(X)$ acts transitively on X. In other words, X is connected iff for each pair $x,\ y$ in X there are $x_1, \ldots, .., x_n$ in X and $\varepsilon_1, \ldots, \varepsilon_n$ in $\{-1, 1\}$ such that

$$x \triangleright^{\varepsilon_1} x_1 \triangleright^{\varepsilon_2} \ldots \triangleright^{\varepsilon_n} x_n = y.$$

As example, latin quandles are connected [42]. Here, we present an important algebraic class of quandles related to the notion of orbits.

Quasi-trivial quandles. The notion of *quasi-trivial quandles* is always related to link-homotopy notion [23, 29, 30]. Here, we restrict our attention to introduce this notion and give some examples.

Definition 14. A quandle $(Q, \triangleright)$ is called quasi-trivial if $x \triangleright \varphi(x) = x$ for each $x \in Q$ and $\varphi \in \mathrm{Inn}(Q)$.

This is equivalent to $x \triangleright y = x$ for all x and y in the same orbit.

Example 5 (Reduced quandle RQ)**.** Let Q be a quandle. We define the **reduced quandle** RQ by the quandle obtained from Q by adjoining the relation $x \triangleright y = x$ for all x and y belonging to the same orbit. From the definition, RQ is obviously a quasi-trivial quandle. In particular, if $Q(L)$ is the fundamental quandle of the link L, then its *reduced fundamental quandle* $RQ(L)$ is quasi-trivial and it is shown to be a link homotopy invariant [29, 30, 32].

Example 6 (Dihedral quandle R_4)**.** The quandle operation table of R_4 is given by

$\triangleright$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

It follows that the orbit decomposition is $\mathrm{R}_4 = \{0, 2\} \cup \{1, 3\}$. Now, let x and y be elements of R_4 belonging to the same orbit, we take $\{0, 2\}$ as example, we have

$$\begin{aligned} \mathrm{R}_0(0) = 0 \triangleright 0 &= 0, \\ \mathrm{R}_2(0) = 0 \triangleright 2 &= 0, \\ \mathrm{R}_0(2) = 2 \triangleright 0 &= 2, \\ \mathrm{R}_2(2) = 2 \triangleright 2 &= 2. \end{aligned}$$

Similarly, for all $x,\ y \in \{1, 3\}$, we have $\mathrm{R}_x(y) = x$. Therefore, R_4 is quasi-trivial quandle. However, the dihedral quandle R_{2n} is not quasi trivial for $n \geq 3$. Indeed, as an example, we take the dihedral quandle R_6: in the same manner we can see that $\mathrm{R}_6 = \{0, 2, 4\} \cup \{1, 3, 5\}$, and we have $1 \triangleright 3 = 5$ that contradicts the definition of quasi-trivial quandle.

2-Engel groups and quasi-triviality of conjugation quandles

Definition 15. A group G is a 2-Engel group if it satisfies the condition that x commutes with $g^{-1}xg$ for all $x,\ g \in G$.

Recall that the conjugation quandle $\mathrm{Conj(G)}$ is the pair $(G, \triangleright)$ where G is a group and $\triangleright$ its conjugation operation, i.e., $x \triangleright y = y^{-1}xy$ for all $x,\ y \in G$.

Proposition 2.3.3. *The conjugation quandle* $\mathrm{Conj(G)}$ *of a group G is quasi-trivial if and only if G is* 2*-Engel group.*

Proof. We know that the quandle $\mathrm{Conj(G)}$ is quasi-trivial, then $x \triangleright y = x$ for any for any x and y belong to the same orbit. Let Ω be one of the orbits of G, and let x and y be in Ω, then

$$\begin{aligned} x \triangleright y = x \quad &\Longrightarrow \quad y^{-1}xy = x \\ &\Longleftrightarrow \quad xy^{-1}xy = xx = y^{-1}xyx \end{aligned}$$

Thus, $xy^{-1}xy = y^{-1}xyx$.

Similarly, we obtain the same result for each orbit of G, therefore, $xy^{-1}xy = y^{-1}xyx$ for all x and y in G.

Conversely, if we suppose that G is 2-Engel, then for all $x, z \in G$, we have

$$\begin{aligned} xz^{-1}xz = z^{-1}xzx &\Longrightarrow (z^{-1}xz)^{-1}x(z^{-1}xz) = x \\ &\Longrightarrow x \rhd (x \rhd z) = x \\ &\Longrightarrow x \rhd y = x \end{aligned}$$

where $y = x \rhd z$, we thus get x and y are in the same orbit, and furthermore $x \rhd y = x$, then G is quasi-trivial. □

Quasi-triviality of Alexander quandles $A = \mathbb{Z}_n[t^{\pm}]/(1-t)^2$

Proposition 2.3.4. *In an Alexander quandle*[6] *A, two elements x and y are in the same orbit if and only if there exists z such that $x - y = (1-t)z$.*

Proof. If x and y belong to the same orbit, then $x \rhd w = y$ for some $w \in A$. Then

$$\begin{aligned} x - y &= x - (x \rhd w), \\ &= x - xt - (1-t)w, \\ &= (1-t)(x-w). \end{aligned}$$

Conversely, if $x - y = (1-t)z$ for some $z \in A$, we get

$$\begin{aligned} y &= x - (1-t)z = x - xt + xt - (1-t)z \\ &= xt + (1-t)(x-z). \end{aligned}$$

Taking $w = x - z$ we can write $y = x \rhd w$, and thus x and y belong to the same orbit. □

Corollary 2.3.5. *The Alexander quandle $A = \mathbb{Z}_n[t^{\pm}]/(1-t)^2$ is quasi-trivial quandle for any positive integer n.*

Proof. Let x and y be in the same orbit, then $x - y = (1-t)z$ for some $z \in A$, and therefore $x = y + (1-t)z$. Since $t - t^2 = 1 - t$, we obtain

$$\begin{aligned} x \rhd y &= tx + (1-t)y = t(x-y) + y \\ &= t(1-t)z + y \\ &= (t - t^2)z + y \\ &= (1-t)z + y \\ &= x. \end{aligned}$$

□

In this thesis, we will be primarily interested in (pointed) racks in which the algebraic structure is compatible with an underlying smooth manifold structure, referred to as (pointed) Lie racks [34].

Definition 16.
1. A topological rack is a topological space X with a rack structure such that the product $\rhd : X \times X \longrightarrow X$ is continuous and, for all $x \in X$, $\mathrm{R}_x : X \longrightarrow X$, $y \longmapsto y \rhd x$ is a homeomorphism.
2. A Lie rack is a smooth manifold X with a rack structure such that the product $\rhd : X \times X \longrightarrow X$ is smooth and, for all $x \in X$, $\mathrm{R}_x : X \longrightarrow X$, $y \longmapsto y \rhd x$ is a diffeomorphism.

Any Lie group G has a pointed Lie rack structure given by the conjugation operation on G. Notice that every Leibniz algebra is a pointed Lie rack by considering the canonical rack structure 2.8 which is pointed at 0.

Examples. Note that Kinyon and Bordemann showed in their articles [14, 34] the left version of the following examples of pointed Lie racks. Here, we suggest the right version of such pointed Lie racks.

1. **Linear pointed Lie rack.** Let H be a Lie group, and V is a (right) H-module. We have $Q = H \times V$ endowed with rack-operation $\rhd$ given by

$$(A,\, u) \rhd (B,\, v) = (B^{-1}AB,\, uB),\ \forall A, B \in H \quad \text{and} \quad u, v \in V$$

is a Lie rack. Setting $\mathbf{1} = (1,0)$, then $(Q, \rhd, \mathbf{1})$ is a pointed Lie rack which is called *linear (pointed) Lie rack*. Indeed, it is clear that the binary operation $\rhd : Q \times Q \longrightarrow Q$ is a smooth map. Then for any $(B,\, v) \in Q$, the map $\mathrm{R}_{(B,\,v)} : Q \longrightarrow Q$ which sends $(A,\, u)$ to $(A,\, u) \rhd (B,\, v)$ is invertible with smooth inverse $\mathrm{R}^{-1}_{(B,\,v)} = \mathrm{R}_{(B^{-1},\,v)}$.
Let us show that the right self-distributivity condition is satisfied. Let $(A,\, u), (B,\, v), (C,\, w) \in Q$. On the one hand, we have

$$\begin{aligned} ((A,\, u) \rhd (B,\, v)) \rhd (C, w) &= (B^{-1}AB,\, uB) \rhd (C, w) \\ &= ((BC)^{-1}A(BC), u(BC)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} ((A,\, u) \rhd (C,\, w)) \rhd ((B,\, v) \rhd (C,\, w)) &= (C^{-1}AC,\, uC) \rhd (C^{-1}BC,\, vC) \\ &= (C^{-1}B^{-1}C(C^{-1}AC)C^{-1}BC, (uC)C^{-1}BC) \\ &= ((BC)^{-1}A(BC), u(BC)). \end{aligned}$$

2. **Augmented pointed Lie rack.** Recall that a pointed differentiable manifold is a pair (M, e) where M is a differentiable manifold and e is a fixed element of M.

Definition 17. An augmented Lie rack $(M,\, \varepsilon,\, G,\, \mathrm{r})$ consists of the following data:

- (M, e) is a pointed differentiable manifold,
- G is a Lie group, with a smooth right G-action $\mathrm{r} : M \times G \longrightarrow M$ written $(x,\, g) \longmapsto \mathrm{r}(g)(x) = xg$
- $\varepsilon : M \longrightarrow G$ is a smooth map of pointed manifolds such that for all $g \in G$ and $x \in M$

$$\begin{aligned} ge &= e \\ \varepsilon(xg) &= g^{-1}\varepsilon(x)g \end{aligned}$$

It is obvious to require that M endowed with the multiplication $\rhd$ defined for all $x,\, y \in M$ by

$$x \rhd y \stackrel{1}{=} \mathrm{r}(\varepsilon(y))(x)$$

is a pointed Lie rack.

2.4 From pointed Lie racks to Leibniz algebras

Let $(X, 1)$ be a pointed Lie rack with left distributivity. Kinyon showed in [34] that the tangent space T_1X carries a structure of left Leibniz algebra. In what follows, we show that, in the same way, one can get a structure of a right Leibniz algebra on the tangent space T_1X if the pointed Lie rack $(X, 1)$ is considered with right distributivity.
For each $x \in X$, $\mathrm{R}_x(1) = 1$. We consider the linear map

$$\mathrm{Ad}_x = T_1\mathrm{R}_x : T_1X \longrightarrow T_1X.$$

We have

$$\mathrm{Ad}_{x \rhd y} = \mathrm{Ad}_y \circ \mathrm{Ad}_x \circ \mathrm{Ad}_y^{-1}.$$

Thus $\mathrm{Ad}\, : X \longrightarrow \mathrm{GL}(T_1X)$ is an homomorphism of Lie racks. If we put

$$[u,v] := \mathrm{ad}_u(v) = \frac{d}{dt}_{|t=0}\mathrm{Ad}_{(c(t))}(u),\ \ \forall\, u,v \in T_1X, \tag{2.9}$$

where $c\ :]-\varepsilon,\, \varepsilon[\longrightarrow X$ is a smooth path in X such that $c(0) = 1$ and $c'(0) = v$. We have the following theorem.

Theorem 2.4.1 ([34]). *Let $(X,1)$ be a pointed Lie rack. Then the tangent space T_1X endowed with the bracket*

$$[u,v] := \mathrm{ad}_u(v) = \frac{d}{dt}_{|t=0}\mathrm{Ad}_{(c(t))}(u),\ \ \forall\, u,\, v \in T_1X,$$

is a right Leibniz algebra. Moreover, if $X = \mathrm{Conj}(G)$ where G is a Lie group then $(T_1X,\, [\ ,\ ])$ is the Lie algebra of G.

Proof. We shall claim that the product 2.9 satisfies the right Leibniz identity, that is

$$[[u,v],w] \ = \ [[u,w],v] \,+\, [u,[v,w]] \quad \forall\, u,\, v,\, w \in T_1X. \tag{2.10}$$

To show that, we use the rack identity

$$(a \rhd b) \rhd c \ = (a\ \rhd c) \rhd (b \rhd c) \tag{2.11}$$

Let $u,\, v,\, w \in T_1X$, and let γ_u, γ_v and γ_w be smooth paths in X such that

$$\gamma_u(0) = \gamma_v(0) = \gamma_w(0) = 1,$$

and

$$\frac{\partial}{\partial s}_{|s=0}\gamma_u(s) = u,\ \frac{\partial}{\partial r}_{|r=0}\gamma_v(r) = v,\ \frac{\partial}{\partial t}_{|t=0}\gamma_w(t) = w.$$

On the one hand, we have

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}(\gamma_u(s) \rhd \gamma_v(r)) \rhd \gamma_w(t) \ &= \ \frac{\partial}{\partial s}_{|s=0}(\mathrm{R}_{\gamma_w(t)} \circ \mathrm{R}_{\gamma_v(r)})(\gamma_u(s)),\\ &= \ (\mathrm{Ad}_{\gamma_w(t)} \circ \mathrm{Ad}_{\gamma_v(r)})(u),\end{aligned}$$

on the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial s}_{|s=0}(\gamma_u(s) \rhd \gamma_w(t)) \rhd (\gamma_v(r) \rhd \gamma_w(t)) \ &= \ \frac{\partial}{\partial s}_{|t=0}(\mathrm{R}_{\gamma_v(r)\rhd\gamma_w(t)} \circ \mathrm{R}_{\gamma_w(t)})(\gamma_u(s)),\\ &= \ (\mathrm{Ad}_{\gamma_v(r)\rhd\gamma_w(t)} \circ \mathrm{Ad}_{\gamma_w(t)})(u),\\ &= \ (\mathrm{Ad}_{R_{\gamma_w(t)}(\gamma_v(r))} \circ \mathrm{Ad}_{\gamma_w(t)})(u).\end{aligned}$$

Next, we differentiate with respect to the variable r, so we get

$$\frac{\partial}{\partial r}_{|r=0}((\mathrm{Ad}_{\gamma_w(t)} \circ \mathrm{Ad}_{\gamma_v(r)})(u)) = (\mathrm{Ad}_{\gamma_w(t)} \circ \mathrm{ad}_v)(u),$$

and

$$\frac{\partial}{\partial r}_{|r=0}(\mathrm{Ad}_{\mathrm{R}_{\gamma_w(t)}(\gamma_v(r))} \circ \mathrm{Ad}_{\gamma_w(t)})(u) = (\mathrm{ad}_{\mathrm{Ad}_{\gamma_w(t)}(v)} \circ \mathrm{Ad}_{\gamma_w(t)})(u).$$

Moreover, if we differentiate with respect to the variable t, we obtain

$$\frac{\partial}{\partial t}_{|t=0}(\mathrm{Ad}_{\gamma_w(t)} \circ \mathrm{ad}_v)(u) = \mathrm{ad}_w(\mathrm{ad}_v(u))$$

and

$$\frac{\partial}{\partial t}_{|t=0}(\mathrm{ad}_{\mathrm{Ad}_{\gamma_w(t)}(v)} \circ \mathrm{Ad}_{\gamma_w(t)})(u) = \mathrm{ad}_{ad_w(v)}(u) + \mathrm{ad}_v(ad_w(u)).$$

Hence

$$\mathrm{ad}_w(\mathrm{ad}_v(u)) = \mathrm{ad}_{ad_w(v)}(u) + \mathrm{ad}_v(ad_w(u)),$$

that is equivalent to
$$[[u,v],w] \;=\; [u,[v,w]] + [[u,w],v],$$
this shows that the (right) Leibniz identity 2.10 is satisfied. □

Examples

1. *Canonical pointed Lie rack structures on Leibniz algebras.* Consider $(\mathfrak{L},[\;,\;])$ a right Leibniz algebra, the associated (right) Leibniz bracket on $T_0\mathfrak{L} = \mathfrak{L}$ of the canonical pointed Lie rack operation 2.8, is the initial bracket $[\;,\;]$.

2. *Tangent Leibniz algebra of linear pointed Lie rack 2.9.* Let $\mathfrak{h}$ be the Lie algebra of the Lie group H, and V is a (right) H-module. We identify the tangent space $T_{(1,0)}Q$ with $\mathfrak{L} := \mathfrak{h} \times V$. Therefore, $\mathfrak{L}$ endowed with the following bracket
$$[X+u, Y+v] := [X,Y] + uY$$
is a Leibniz algebra. $(\mathfrak{L} := \mathfrak{h} \times V, [\;,\;])$ is so called *split Leibniz algebra*.

3. *Tangent Leibniz algebra structure of augmented (pointed) Lie rack 2.9* has been studied by M.Bordmann and F. Wagemann in [14]. In this article, the authors introduced the notion of **augmented Leibniz algebras** which is exactly the tangent Leibniz structure of augmented pointed Lie racks.

Proposition 2.4.2. *Let $f : X \longrightarrow Y$ be a morphism of pointed Lie racks. Then $T_1 f : T_1 X \longrightarrow T_1 Y$ is a morphism of Leibniz algebras.*

Proof. We have f a morphism of pointed Lie racks, then $f(1) = 1$ and $f(x \rhd y) = f(x) \rhd f(y)$ for any $x, y \in X$. We shall show that $T_1 f([u,v]) = [T_1 f(u), T_1 f(v)]$ for any $u, v \in T_1 X$. Indeed, let $u, v \in T_1 X$, and let γ_u and γ_v be smooth paths in X such that
$$\gamma_u(0) = \gamma_v(0) = 1,\; \frac{\partial}{\partial t}_{|t=0}\gamma_u(t) = u, \quad \text{and} \quad \frac{\partial}{\partial s}_{|s=0}\gamma_v(s) = v.$$
We have
$$\frac{\partial}{\partial t}_{|t=0} f(\gamma_u(s) \rhd \gamma_v(s)) = T_1 f(\mathrm{Ad}_{\gamma_v(s)}(u)), \tag{2.12}$$
and
$$\frac{\partial}{\partial t}_{|t=0} f(\gamma_u(s)) \rhd f(\gamma_v(s)) = \mathrm{Ad}_{f(\gamma_v(s))}(T_1 f(u)). \tag{2.13}$$
We differentiate 2.12 and 2.13 with respect to the variable s, we then have
$$\frac{\partial}{\partial s}_{|s=0} T_1 f(\mathrm{Ad}_{\gamma_v(s)}(u)) = T_1 f([u,v]),$$
and
$$\frac{\partial}{\partial s}_{|s=0} \mathrm{Ad}_{f(\gamma_v(s))}(T_1 f(u)) = [T_1 f(u), T_1 f(v)]$$
Thus, $T_1 f$ is a morphism of Leibniz algebras. □

Chapter 3

A class of Lie racks associated to symmetric Leibniz algebras

We have said that there are only partial answers to the problem of integrating Leibniz algebras (see [20, 14]). However, S. Benayadi and M. Bordemann [9] gave a natural method for integrating symmetric Leibniz algebras, which are both right and left Leibniz algebras. This method is based on the characterization of symmetric Leibniz algebras given in [8]. More precisely, given a symmetric Leibniz algebra $(\mathcal{L},\cdot)$, the product is Lie-admissible and defines a Lie algebra bracket $[\ ,\ ]$ on $\mathcal{L}$. Let G be the connected and simply-connected Lie group associated to $(\mathcal{L},[\ ,\ ])$. Then, naturally one can build on G a Lie rack structure such that the right Leibniz algebra on T_eG is exactly $(\mathcal{L},\cdot)$. The obtained *Lie rack is said to be associated to the symmetric Leibniz algebra* $(\mathcal{L},\cdot)$. In this chapter, we determine all symmetric Leibniz algebras in dimension 3 and 4, up to an isomorphism, and for each of them we build the associated Lie rack. We get a family of Lie racks and some topological quandles. We study some algebraic properties of these quandles.

3.1 Characterization of symmetric Leibniz algebras

We have seen that any Lie algebra is a symmetric Leibniz algebra which is both left and right algebra. However, the class of symmetric Leibniz algebras is far more bigger than the class of Lie algebras as we will see in this section.
Let $\mathfrak{L}$ be a real vector space equipped with a bilinear map $\cdot\, : \mathfrak{L}\times\mathfrak{L}\longrightarrow\mathfrak{L}$. Let $[\ ,\ ]$ and $\circ$ be respectively its antisymmetric and symmetric parts. For all $u,\ v\ \in\mathfrak{L}$, they are defined by:

$$[u,v]=\frac{1}{2}(u\cdot v-v\cdot u)\quad \mathit{and}\quad u\circ v=\frac{1}{2}(u\cdot v+v\cdot u). \tag{3.1}$$

Thus

$$u\cdot v=[u,v]+u\circ v. \tag{3.2}$$

The following proposition gives a useful **characterization of symmetric Leibniz algebras** (see [[8] , Proposition 2.11]).

Proposition 3.1.1. *Let $(\mathfrak{L},\cdot)$ be an algebra. The following assertions are equivalent:*

1 $(\mathfrak{L},\cdot)$ is a symmetric Leibniz algebra.

2 The following conditions hold :

- *(a) $(\mathfrak{L},[\ ,\ ])$ is a Lie algebra.*
- *(b) For any $u,v\in\mathfrak{L}$, $u\circ v$ belongs to the center of $(\mathfrak{L},[\ ,\ ])$.*
- *(c) For any $u,v,w\in\mathfrak{L}$, $([u,v])\circ w=0$ and $(u\circ v)\circ w=0$.*

According to this proposition, any symmetric Leibniz algebra is given by a Lie algebra $(\mathfrak{L},\ [\ ,\ ])$ and a bilinear symmetric map $\omega\ :\mathfrak{L}\times\mathfrak{L}\longrightarrow Z(\mathfrak{L})$ where $Z(\mathfrak{L})$ is the center of the Lie algebra, such that, for any $u,\ v,\ w\in\mathfrak{L}$,

$$\omega([u,v],w)=\omega(\omega(u,v),w)=0. \tag{3.3}$$

Then the product of the symmetric Leibniz algebra is given by

$$u.v = [u,v] + \omega(u,v), \quad u,v \in \mathfrak{L}. \tag{3.4}$$

Note that if $Z(\mathfrak{L}) = 0$ or $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$ then the solutions of 3.3 are trivial.
The following proposition is easy to prove.

Proposition 3.1.2. *Let $(\mathfrak{L}, [\ ,\ ])$ a Lie algebra and ω and μ two solutions of 3.3. Then $(\mathfrak{L}, \bullet_\omega)$ is isomorphic to $(\mathfrak{L}, \bullet_\mu)$ (as symmetric Leibniz algebras) if and only if there exists an automorphism A of $(\mathfrak{L}, [\ ,\ ])$ such that*

$$\mu(u,v) = A^{-1}\omega(Au, Av).$$

3.2 Lie racks and topological quandles associated to symmetric Leibniz algebras

We have said that the problem of integrating a Leibniz algebra into a pointed Lie rack has not a natural answer. For our purpose, there is a functorial process of integrating symmetric Leibniz algebras which was communicated to us privately by S. Benayadi and M. Bordemann and we will give it now in details. Let $(\mathfrak{L}, \cdot)$ be a symmetric Leibniz algebra. According to Proposition 3.1.1, there exists a Lie bracket $[\ ,\ ]$ on $\mathfrak{L}$ and a bilinear symmetric map $\omega : \mathfrak{L} \times \mathfrak{L} \longrightarrow Z(\mathfrak{L})$, where $Z(\mathfrak{L})$ is the center of $(\mathfrak{L}, [\ ,\ ])$, such that,

$$u \cdot v = [u,v] + \omega(u,v) \text{ for all } u,v \in \mathfrak{L}$$

and ω satisfies 3.3.

Method 1. *Denote by $\mathfrak{L} \cdot \mathfrak{L} = span\{u \cdot v,\, u,\, v \in \mathfrak{L}\}$, $\mathfrak{a} = \mathfrak{L}/\mathfrak{L} \cdot \mathfrak{L}$, $q : \mathfrak{L} \longrightarrow \mathfrak{a}$ and define $\beta : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{L}$ given by*

$$\beta(q(u), q(v)) = \omega(u,v).$$

By virtue of 3.3, β is well defined. Moreover, since $[u,v] = \frac{1}{2}(u \cdot v - v \cdot u)$, q is a Lie algebra homomorphism when $\mathfrak{a}$ is considered as an abelian Lie algebra. Consider G the connected and simply connected Lie group whose Lie algebra is $(\mathfrak{L}, [\ ,\ ])$ and $\exp : \mathfrak{L} \longrightarrow G$ its exponential. Hence there exists an homomorphism of Lie groups $\kappa : G \longrightarrow \mathfrak{a}$ such that $d_e\kappa = q$. Finally, consider $\chi : G \times G \longrightarrow G$ given by

$$\chi(h, g) = \exp(\beta(\kappa(h), \kappa(g))) \text{ for all } h, g \in G. \tag{3.5}$$

Define now on G the binary product by putting

$$h \rhd g := g^{-1}hg\chi(h, g) \text{ for all } h, g \in G. \tag{3.6}$$

We show the result obtained by S. Benayadi and M. Boredmann.

Theorem 3.2.1. *Let $(\mathfrak{L}, .)$ be a symmetric Leibniz algebra. Let G be the connected simply connected Lie group associated to the underlying Lie algebra endowed with the binary operation $\rhd$ defined by (3.6). Then $(G, \rhd)$ is a pointed Lie rack whose associated right Leibniz algebra is exactly $(\mathfrak{L}, .)$.*

Proof. At first, we note that the map χ satisfies the following properties for all elements $h, h_1, h_2, h_3 \in G$:

- The map χ is symmetric, since β is symmetric.
- $\chi(h, 1) = 1 = \chi(1, h)$ because $\kappa(1) = 0$ and β is bilinear.
- $$\begin{aligned} \chi(h_1h_2, h_3) &= \exp(\beta(\kappa(h_1h_2), \kappa(h_3))) = \exp(\beta(\kappa(h_1) + \kappa(h_2), \kappa(h_3))) \\ &= \exp(\beta(\kappa(h_1), \kappa(h_3)))\exp(\beta(\kappa(h_2), \kappa(h_3))) \\ &= \chi(h_1, h_3)\chi(h_2, h_3) \end{aligned}$$
- Since the map β takes its values in the centre of $\mathfrak{L}$, it follows that,

$$1 = \chi(h_1^{-1}h_1, h_2) = \chi(h_1^{-1}, h_2)\chi(h_1, h_2)$$

and so $\chi(h_1^{-1}, h_2) = \chi(h_1, h_2)^{-1}$.

- Since κ is a morphism between connected simply connected Lie groups, then for all $x \in \mathfrak{L}$ we have $\kappa(\exp(x)) = \exp_{\mathfrak{a}}(T_1\kappa(x))$. The latter is equal to $q(x)$ because the exponential map of the vector space Lie group $\mathfrak{a}$ is the identity. Then

$$\kappa(\chi(h_1, h_2)) = \kappa(\exp(\beta(\kappa(h_1), \kappa(h_2)))) = q(\beta(\kappa(h_1), \kappa(h_2))) = 0,$$

hence

$$\chi(h_1\chi(h_2, h_3), h_4) = \chi(h_1, h_4).$$

Now we will use those properties to show the theorem.

First, it is easy to see that the binary operation $\rhd : G \times G \longrightarrow G$ is a smooth map. Then for any $g \in G$, the map $\mathrm{R}_g : G \longrightarrow G$ which sends h to $h \rhd g$ is invertible with smooth inverse $\mathrm{R}_g^{-1} = \mathrm{R}_{g^{-1}}$. This follows easily from the identity $\chi(h \rhd g, h_1) = \chi(h, h_1)$ for any $h, g, h_1 \in G$.

Let us show that the self-distributivity condition is satisfied. For that we will use the following identity. If $z \in G$ writes $z = \chi(h_1, h_2)$ for $h_1, h_2 \in G$, then

$$\begin{aligned} h_1 \rhd (zh_2) &= (zh_2)^{-1}h_1(zh_2)\chi(h_1, zh_2) \\ &= h_2^{-1}z^{-1}h_1zh_2\chi(h_1, h_2) \\ &= h_2^{-1}h_1h_2\chi(h_1, h_2) \\ &= h_1 \rhd h_2. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} (h_1 \rhd h_2) \rhd h_3 &= (h_2^{-1}h_1h_2\chi(h_1, h_2)) \rhd h_3 \\ &= h_3^{-1}h_2^{-1}h_1h_2h_3\chi(h_1, h_2)\chi(h_2^{-1}h_1h_2\chi(h_1, h_2), h_3) \\ &= (h_2h_3)^{-1}h_1(h_2h_3)\chi(h_1, h_2)\chi(h_1, h_3) \\ &= (h_2h_3)^{-1}h_1(h_2h_3)\chi(h_1, h_2h_3) \\ &= h_1 \rhd (h_2h_3). \end{aligned}$$

On the other hand

$$\begin{aligned} (h_1 \rhd h_3) \rhd (h_2 \rhd h_3) &= h_1 \rhd (h_3(h_2 \rhd h_3)) \\ &= h_1 \rhd (h_2h_3\chi(h_2, h_3)) \\ &= \chi(h_2, h_3)^{-1}h_3^{-1}h_2^{-1}h_1h_2h_3\chi(h_2, h_3)\chi(h_1, h_2h_3\chi(h_2, h_3)) \\ &= (h_2h_3)^{-1}h_1h_2h_3\chi(h_1, h_3h_2) \\ &= h_1 \rhd (h_2h_3), \end{aligned}$$

This proves the self-distributivity condition. Moreover, we have

$$h_1 \rhd 1 = h_1\chi(1, h_1) = h_1 \quad \text{and} \quad 1 \rhd h_1 = h_1^{-1}h_1\chi(1, h_1) = 1$$

Finally $(G, \rhd)$ is a pointed Lie rack.

For the last claim in the theorem we must show that the corresponding Leibniz product on $\mathfrak{L}$ is exactly the product we started with in $\mathfrak{L}$. Indeed, according to Theorem 2.4.1, we get for all $u, v \in \mathfrak{L}$ and

$x \in G$:

$$\begin{aligned}
\frac{\partial}{\partial t}_{|t=0} \mathrm{R}_x(\exp(tu)) &= \frac{\partial}{\partial t}_{|t=0}(\exp(tu) \rhd x), \\
&= \frac{\partial}{\partial t}_{|t=0}(x^{-1}\exp(tu)x\chi(x, \exp(tu)), \\
&= \frac{\partial}{\partial t}_{|t=0}(\exp(tx^{-1}ux)\exp(\beta(\kappa(x), \kappa(\exp(tu))))), \\
&= \frac{\partial}{\partial t}_{|t=0}(\exp(tAd_{x^{-1}}(u))\exp(\beta(\kappa(x), tq(u)))), \\
&= Ad_{x^{-1}}(u) + \beta(\kappa(x), q(u)).
\end{aligned}$$

Replacing x by the curve $t \longmapsto \exp(tv)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}_{|t=0} T_1 \mathrm{R}_{\exp(tv)}(u)) &= \frac{\partial}{\partial t}_{|t=0}(Ad_{\exp(-tv)}(u) + \beta(\kappa(\exp(tv)), q(u))), \\
&= \frac{\partial}{\partial t}_{|t=0}(Ad_{\exp(-tv)}(u)) + \beta(q(v), q(u)), \\
&= [-v, u] + v \circ u, \\
&= [u, v] + u \circ v = u.v.
\end{aligned}$$

This establishes the formula (3.2) and completes the proof. □

We say that the obtained Lie rack $(G, \rhd)$ is associated to the symmetric Leibniz algebra $(\mathfrak{L}, .)$.

The Lie racks associated to symmetric Leibniz algebras give rise to a class of topological quandles in the following way. Let $(\mathfrak{L}, .)$ be a symmetric Leibniz algebra and $(G, \rhd)$ be the associated Lie rack. Proposition 2.3.1 states that, in particular, if $(X, \rhd)$ is a Lie rack then $(Q(X), \rhd)$ is a topological quandle. Hence,

$$Q((G, \rhd)) = \{g \in G\,;\, \chi(g, g) = 1_G\}$$

is a topological quandle.

3.3 Lie racks associated to symmetric Leibniz algebras in dimensions 3 and 4

In this section, by using Proposition 3.1.1 we determine first all the symmetric Leibniz algebras of dimension 3 and 4, up to an isomorphism and, for each of them, we use Method 1 described in the last section to build the associated Lie racks.

3.3.1 Symmetric Leibniz algebras of dimension 3 and 4

We proceed in the following way:

1. We pick a Lie algebra $\mathfrak{g}$ with non trivial center in the list of [11].
2. By a direct computation, we determine the symmetric maps ω satisfying 3.3.
3. In the spirit of Proposition 3.1.2, we act by the group of automorphisms of $\mathfrak{g}$ on the obtained ω to reduce the parameters.

By doing so, we get for any Lie algebra $\mathfrak{g}$ of dimension 3 or 4 with non trivial center all non equivalent symmetric Leibniz structures for which $\mathfrak{g}$ is the underlying Lie algebra. In the last section, we give an example of detailed computations. The results are summarized in Table 3.1.

Lie algebra	Non-vanishing Lie brackets	Symmetric Leibniz Non-vanishing brackets	Name	Conditions
$\mathfrak{g}_{3,1}$	$[e_2, e_3] = e_1$	$e_2 \cdot e_2 = \mu e_1,\ e_3 \cdot e_3 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$	$\mathfrak{g}_{3,1}^1$	$\mu \in \mathbb{R}$
		$e_2 \cdot e_3 = (\gamma+1)e_1, e_3 \cdot e_2 = (\gamma-1)e_1$	$\mathfrak{g}_{3,1}^2$	$\gamma \neq 0$
$\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$	$[e_1, e_2] = e_1$	$e_2 \cdot e_3 = e_3 \cdot e_2 = e_4,\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_1$	$(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^1$	
		$e_2 \cdot e_2 = e_3 - e_4,\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_1$	$(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^2$	
		$e_2 \cdot e_4 = e_4 \cdot e_2 = e_3,\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_1$	$(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^3$	
		$e_2 \cdot e_2 = \varepsilon e_3,\ e_4 \cdot e_4 = e_3,\ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_1$	$(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^4$	$\varepsilon = 0, 1, -1$
$\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$	$[e_2, e_3] = e_1$	$e_2 \cdot e_2 = \mu e_1,\ e_3 \cdot e_3 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$	$(\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^1$	$\mu \in \mathbb{R}$
		$e_2 \cdot e_3 = (\gamma+1)e_1, e_3 \cdot e_2 = (\gamma-1)e_1$	$(\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^2$	$\gamma \neq 0$
$\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1$	$[e_3, e_1] = e_1$ $[e_2, e_3] = e_1 - e_2$	$e_3 \cdot e_3 = e_4,\ e_3 \cdot e_1 = -e_3 \cdot e_1 = e_1$ $e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1 - e_2$	$(\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1)^1$	
$\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1$	$[e_3, e_1] = e_1$ $[e_2, e_3] = -e_2$	$e_3 \cdot e_3 = e_4,\ e_3 \cdot e_1 = -e_1 \cdot e_3 = e_1$ $e_2 \cdot e_3 = -e_3 \cdot e_2 = -e_2$	$(\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1)^1$	
$\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1$	$[e_2, e_3] = e_1$ $[e_3, e_1] = -e_2$	$e_3 \cdot e_3 = e_4,\ e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$ $e_3 \cdot e_1 = -e_1 \cdot e_3 = -e_2$	$(\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1)^1$	
$\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$	$[e_2, e_3] = e_1 - \alpha e_2$ $[e_3, e_1] = \alpha e_1 - e_2$	$e_3 \cdot e_3 = e_4,\ e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1 - \alpha e_2$ $e_3 \cdot e_1 = -e_1 \cdot e_3 = \alpha e_1 - e_2$	$(\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1)^1$	$\alpha > 0$ $\alpha \neq 1$
$\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$	$[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	$e_3 \cdot e_3 = e_4, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$ $e_3 \cdot e_1 = -e_1 \cdot e_3 = e_2$	$(\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1)^1$	
$\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$	$[e_2, e_3] = e_1 - \alpha e_2$ $[e_3, e_1] = \alpha e_1 + e_2$	$e_3 \cdot e_3 = e_4, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1 - \alpha e_2$ $e_3 \cdot e_1 = -e_1 \cdot e_3 = \alpha e_1 + e_2$	$(\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1)^1$	$\alpha > 0$
$\mathfrak{g}_{4,1}$	$[e_2, e_4] = e_1$ $[e_3, e_4] = e_2$	$e_3 \cdot e_3 = e_1,\ e_4 \cdot e_4 = \varepsilon e_1$ $e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2,\ e_2 \cdot e_4 = -e_4 \cdot e_2 = e_1$	$\mathfrak{g}_{4,1}^1$	$\varepsilon = 0, 1, -1$
		$e_4 \cdot e_4 = e_1,\ e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2$ $e_2 \cdot e_4 = -e_4 \cdot e_2 = e_1$	$\mathfrak{g}_{4,1}^2$	
		$e_4 \cdot e_4 = e_1$ $e_3 \cdot e_4 = e_1 + e_2,\ e_4 \cdot e_3 = e_1 - e_2,\ e_2 \cdot e_4 = -e_4.e_2 = e_1$	$\mathfrak{g}_{4,1}^3$	
		$e_3 \cdot e_3 = e_1, e_4 \cdot e_4 = \mu e_1$ $e_3 \cdot e_4 = e_1 + e_2,\ e_4 \cdot e_3 = e_1 - e_2,\ e_2 \cdot e_4 = -e_4.e_2 = e_1$	$\mathfrak{g}_{4,1}^4$	$\mu \in \mathbb{R}$
$\mathfrak{g}_{4,3}$	$[e_1, e_4] = e_1$ $[e_3, e_4] = e_2$	$e_4 \cdot e_4 = e_2, e_1 \cdot e_4 = -e_4 \cdot e_1 = e_1$ $e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2$	$\mathfrak{g}_{4,3}^1$	
		$e_3 \cdot e_3 = e_2,\ e_4 \cdot e_4 = \mu e_2$ $e_1 \cdot e_4 = -e_4 \cdot e_1 = e_1,\ e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2$	$\mathfrak{g}_{4,3}^2$	$\mu \in \mathbb{R}$
		$e_1 \cdot e_4 = -e_4 \cdot e_1 = e_1, e_3 \cdot e_4 = (\gamma+1)e_2$ $e_4 \cdot e_3 = (\gamma-1)e_2$	$\mathfrak{g}_{4,3}^3$	$\gamma \neq 0$
$\mathfrak{g}_{4,8}^{-1}$	$[e_2, e_3] = e_1$ $[e_3, e_4] = -e_3$ $[e_2, e_4] = e_2$	$e_4 \cdot e_4 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$ $e_3 \cdot e_4 = -e_4 \cdot e_3 = -e_3$ $e_2 \cdot e_4 = -e_4 \cdot e_2 = e_2,$	$(\mathfrak{g}_{4,8}^{-1})^1$	
$\mathfrak{g}_{4,9}^0$	$[e_2, e_3] = e_1$ $[e_2, e_4] = -e_3$ $[e_3, e_4] = e_2$	$e_4 \cdot e_4 = e_1,\ e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1$ $e_2 \cdot e_4 = -e_4 \cdot e_2 = -e_3$ $e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2$	$(\mathfrak{g}_{4,9}^0)^1$	

TABLE 3.1: Symmetric Leibniz algebras of dimension 3 and 4.

3.3.2 Lie racks

In this subsection, we determine by using Method 1 the Lie racks associated to the symmetric Leibniz algebras determined in the last subsection. Then we give the associated topological quandles defined in Proposition 2.3.1. In the last section, we explicit the computations for a particular example.

- $\mathfrak{g}_{3,1}$. The associated simply-connected Lie group is given by

$$G_{3,1} = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, x, y, z, \in \mathbb{R} \right\}.$$

1. The Lie rack structure associated to $\mathfrak{g}_{3,1}^1$ and the associated topological quandle are respectively

$$M(x,y,z) \rhd_1 M(a,b,c) = \begin{bmatrix} 1 & y & \mu yb - bz + cy + zc + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{3,1})^1 = \{M(a,b,c) \in G_{3,1} |\ \mu b^2 + c^2 = 0\}.$$

2. The Lie rack structure associated to $\mathfrak{g}_{3,1}^2$ and the associated topological quandle are respectively

$$M(x,y,z) \rhd_2 M(a,b,c) = \begin{bmatrix} 1 & y & \gamma(bz+cy) - bz + cy + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{3,1})^2 = \{M(a,b,c) \in G_{3,1} |\ bc = 0\}.$$

- $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{2,1} \times \mathbb{R}^2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & e^{-x} & 0 & 0 \\ 0 & 0 & e^y & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.$$

1. The Lie rack structure associated to $(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^1$ and the associated topological quandle are respectively

$$M(x,y,z) \rhd_1 M(a,b,c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^a + e^a w + te^{a-x} & e^{-x} & 0 & 0 \\ 0 & 0 & e^y & 0 \\ 0 & 0 & 0 & e^{ya+xb+z} \end{bmatrix}$$

$$Q(G_{2,1} \times \mathbb{R}^2)^1 = \{M(t,a,b,c) \in G_{2,1} \times \mathbb{R}^2 |\ ab = 0\}.$$

2. The Lie rack structure associated to $(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^2$ and the associated topological quandle are respectively

$$M(w,x,y,z) \rhd_2 M(t,a,b,c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^a + e^a w + te^{a-x} & e^{-x} & 0 & 0 \\ 0 & 0 & e^{ax+y} & 0 \\ 0 & 0 & 0 & e^{-ax+z} \end{bmatrix}$$

$$Q(G_{2,1} \times \mathbb{R}^2)^2 = \{M(t,a,b,c) \in G_{2,1} \times \mathbb{R}^2 |\ a = 0\}.$$

3. The Lie rack structure associated to $(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^3$ and the associated topological quandle are respectively

$$M(w,x,y,z) \rhd_4 M(t,a,b,c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^a + e^a w + te^{a-x} & e^{-x} & 0 & 0 \\ 0 & 0 & e^{xc+za+y} & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}$$

$$Q(G_{2,1} \times \mathbb{R}^2)^3 = \{M(t,a,b,c) \in G_{2,1} \times \mathbb{R}^2 |\ ac = 0\}.$$

4. The Lie rack structure associated to $(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^4$ and the associated topological quandle are respectively

$$M(w,x,y,z) \rhd_4 M(t,a,b,c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^a + e^a w + te^{a-x} & e^{-x} & 0 & 0 \\ 0 & 0 & e^{\varepsilon xc+za+y} & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}$$

$$Q(G_{2,1} \times \mathbb{R}^2)^4 = \{M(t,a,b,c) \in G_{2,1} \times \mathbb{R}^2 |\ \varepsilon ac = 0\}.$$

- $\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,1} \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & x & w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z, \in \mathbb{R} \right\}$$

1. The Lie rack structure associated to $(\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^1$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z)\rhd_1 M(t,\,a,\,b,\,c)=\begin{bmatrix}1 & x & \mu\,yb-ya+xb+zc+w & 0\\ 0 & 1 & y & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \mathrm{e}^z\end{bmatrix}$$

$$Q(G_{3,1}\times\mathbb{R})^1=\{M(t,a,b,c)\in G_{3,1}\times\mathbb{R}|\ \mu b^2+c^2=0\}.$$

2. The Lie rack structure associated to $(\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^2$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z)\rhd_2 M(t,\,a,\,b,\,c)=\begin{bmatrix}1 & x & \gamma\,ay+\gamma\,bx-ya+xb+w & 0\\ 0 & 1 & y & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \mathrm{e}^z\end{bmatrix}$$

$$Q(G_{3,1}\times\mathbb{R})^2=\{M(t,a,b,c)\in G_{3,1}\times\mathbb{R}|\ ab=0\}.$$

- $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,2}\times\mathbb{R}=\left\{\begin{bmatrix}1 & 0 & 0 & 0\\ x & e^y & 0 & 0\\ w & -ye^y & e^y & 0\\ 0 & 0 & 0 & e^z\end{bmatrix},w,x,\,y,z\,\in\mathbb{R}\right\}.$$

The Lie rack structure associated to $(\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1)^1$ and the associated topological quandle are

$$M(w,\,x,\,y,\,z)\rhd M(t,\,a,\,b,\,c)=\begin{bmatrix}1 & 0 & 0 & 0\\ (x-a)\mathrm{e}^{-b}+a\mathrm{e}^{-b+y} & \mathrm{e}^y & 0 & 0\\ (bx-ab)\mathrm{e}^{-b}+\mathrm{e}^{-b+y}(ab-ay) & -y\mathrm{e}^y & \mathrm{e}^y & 0\\ (w-t)\mathrm{e}^{-b}t+t\mathrm{e}^{-b+y} & & & \\ 0 & 0 & 0 & \mathrm{e}^{yb+z}\end{bmatrix}$$

$$Q(G_{3,2}\times\mathbb{R})=\{M(t,a,b,c)\in G_{3,2}\times\mathbb{R}|\ b=0\}.$$

- $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,3}\times\mathbb{R}=\left\{\begin{bmatrix}1 & 0 & 0 & 0\\ x & e^y & 0 & 0\\ w & 0 & e^y & 0\\ 0 & 0 & 0 & e^z\end{bmatrix},w,x,\,y,z\,\in\mathbb{R}\right\}.$$

The Lie rack structure associated to $(\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1)^1$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z)\rhd M(t,\,a,\,b,\,c)=\begin{bmatrix}1 & 0 & 0 & 0\\ -a\mathrm{e}^{-b}+\mathrm{e}^{-b}x+a\mathrm{e}^{-b+y} & \mathrm{e}^y & 0 & 0\\ -t\mathrm{e}^{-b}+\mathrm{e}^{-b}w+t\mathrm{e}^{-b+y} & 0 & \mathrm{e}^y & 0\\ 0 & 0 & 0 & \mathrm{e}^{yb+z}\end{bmatrix}$$

$$Q(G_{3,3}\times\mathbb{R})=\{M(t,a,b,c)\in G_{3,3}\times\mathbb{R}|\ b=0\}.$$

- $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,4}^0\times\mathbb{R}=\left\{\begin{bmatrix}1 & 0 & 0 & 0\\ w & \cosh(y) & -\sinh(y) & 0\\ x & -\sinh(y) & \cosh(y) & 0\\ 0 & 0 & 0 & e^z\end{bmatrix},w,x,\,y,z\,\in\mathbb{R}\right\}.$$

The Lie rack structure associated to $(\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1)^1$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z) \rhd M(t,\,a,\,b,\,c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \begin{matrix}\sinh(y-b)a + \cosh(y-b)t + \\ \sinh(b)(x-a) + \cosh(b)(w-t)\end{matrix} & \cosh(y) & -\sinh(y) & 0 \\ \begin{matrix}\sinh(y-b)t + \cosh(y-b)a + \\ \sinh(b)(w-t) + \cosh(b)(x-a)\end{matrix} & -\sinh(y) & \cosh(y) & 0 \\ 0 & 0 & 0 & e^{yb+z} \end{bmatrix}$$

$$Q(G_{3,4}^0 \times \mathbb{R}) = \{M(t,a,b,c) \in G_{3,4}^0 \times \mathbb{R} |\ b = 0\}.$$

- $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,4}^\alpha \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & e^{\alpha y}\cosh(y) & -e^{\alpha y}\sinh(y) & 0 \\ x & -e^{\alpha y}\sinh(y) & e^{\alpha y}\cosh(y) & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.$$

The Lie rack structure associated to $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z) \rhd M(t,\,a,\,b,\,c) =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \begin{matrix}\sinh(y-b)ae^{\alpha(y-b)} + \cosh(y-b)te^{\alpha(y-b)} + \\ \sinh(b)(x-a)e^{-\alpha b} + \cosh(b)(w-t)e^{-\alpha b}\end{matrix} & e^{\alpha y}\cosh(y) & -e^{\alpha y}\sinh(y) & 0 \\ \begin{matrix}\sinh(y-b)te^{\alpha(y-b)} + \cosh(y-b)ae^{\alpha(y-b)} + \\ \sinh(b)(w-t)e^{-\alpha b} + \cosh(b)(x-a)e^{-\alpha b}\end{matrix} & -e^{\alpha y}\sinh(y) & e^{\alpha y}\cosh(y) & 0 \\ 0 & 0 & 0 & e^{yb+z} \end{bmatrix}$$

$$Q(G_{3,4}^\alpha \times \mathbb{R}) = \{M(t,a,b,c) \in G_{3,4}^\alpha \times \mathbb{R} |\ b = 0\}.$$

- $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,5}^0 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ w & \cos(y) & -\sin(y) & 0 & 0 \\ x & \sin(y) & \cos(y) & 0 & 0 \\ 0 & 0 & 0 & e^y & 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.$$

The Lie rack structure associated to $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$ and the associated topological quandle are respectively

$$M(w,\,x,\,y,\,z) \rhd M(t,\,a,\,b,\,c) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \begin{matrix}\sin(y-b)a + \cos(y-b)t + \\ \sin(b)(x-a) + \cos(b)(w-t)\end{matrix} & \cos(y) & -\sinh(y) & 0 & 0 \\ \begin{matrix}\sin(y-b)t + \cos(y-b)a + \\ \sin(b)(t-w) + \cos(b)(x-a)\end{matrix} & \sin(y) & \cos(y) & 0 & 0 \\ 0 & 0 & 0 & e^y & 0 \\ 0 & 0 & 0 & 0 & e^{yb+z} \end{bmatrix}$$

$$Q(G_{3,5}^0 \times \mathbb{R}) = \{M(t,a,b,c) \in G_{3,5}^0 \times \mathbb{R} |\ b = 0\}.$$

- $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$. The associated simply-connected Lie group is given by

$$G_{3,5}^\alpha \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ w & e^{\alpha y}\cos(y) & -e^{\alpha y}\sin(y) & 0 & 0 \\ x & e^{\alpha y}\sin(y) & e^{\alpha y}\cos(y) & 0 & 0 \\ 0 & 0 & 0 & e^y & 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.$$

The Lie rack structure associated to $\mathfrak{g}_{3,5}^{\alpha} \oplus \mathfrak{g}_1$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd M(t, a, b, c) =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \begin{matrix}\sin(y-b)ae^{\alpha(y-b)} + \cos(y-b)te^{\alpha(y-b)} + \\ \sin(b)(x-a)e^{-\alpha b} + \cos(b)(w-t)e^{-\alpha b}\end{matrix} & e^{\alpha y}\cos(y) & -e^{\alpha y}\sinh(y) & 0 & 0 \\ \begin{matrix}\sin(y-b)te^{\alpha(y-b)} + \cos(y-b)a + \\ \sin(b)(t-w)e^{-\alpha b} + \cos(b)(x-a)e^{-\alpha b}\end{matrix} & e^{\alpha y}\sin(y) & e^{\alpha y}\cos(y) & 0 & 0 \\ 0 & 0 & 0 & e^{y} & 0 \\ 0 & 0 & 0 & 0 & e^{yb+z} \end{bmatrix}$$

$$Q(G_{3,5}^{\alpha} \times \mathbb{R})^3 = \{M(t,a,b,c) \in G_{3,5}^{\alpha} \times \mathbb{R} |\ b = 0\}.$$

$\bullet\mathfrak{g}_{4,1}$. The associated simply-connected Lie group is given by

$$G_{4,1} = \left\{ \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & w \\ 0 & 1 & z & w-x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}, w, x, y, z, \in \mathbb{R} \right\}$$

1. The Lie rack structure associated to $\mathfrak{g}_{4,1}^{1}$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_1 M(t, a, b, c) = \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & \begin{matrix} w + \frac{1}{2}c^2y + \frac{1}{2}bz^2 + (yb + \varepsilon zc)(1+z) \\ -bcz + c(x-w) + z(t-a) \end{matrix} \\ 0 & 1 & z & bz - cy + w - x + (yb + \varepsilon zc) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1})^1 = \{M(t,a,b,c) \in G_{4,1} |\ b^2 + \varepsilon c^2 = 0\}.$$

2. The Lie rack structure associated to $\mathfrak{g}_{4,1}^{2}$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_2 M(t, a, b, c) = \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & \begin{matrix} w + \frac{1}{2}c^2y + \frac{1}{2}bz^2 + zc(1+z) \\ -bcz + c(x-w) + z(t-a) \end{matrix} \\ 0 & 1 & z & bz - cy + w - x + zc \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1})^2 = \{M(t,a,b,c) \in G_{4,1} |\ c = 0\}.$$

3. The Lie rack structure associated to $\mathfrak{g}_{4,1}^{3}$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_3 M(t, a, b, c) = \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & \begin{matrix} w + \frac{1}{2}c^2y + \frac{1}{2}bz^2 + (yc + zb + zc)(1+z) \\ -bcz + c(x-w) + z(t-a) \end{matrix} \\ 0 & 1 & z & bz - cy + w - x + (yc + zb + zc) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1})^3 = \{M(t,a,b,c) \in G_{4,1} |\ 2bc + c^2 = 0\}.$$

4. The Lie rack structure associated to $\mathfrak{g}_{4,1}^{4}$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_4 M(t, a, b, c) = \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & \begin{matrix} w + \frac{1}{2}c^2y + \frac{1}{2}bz^2 + (y(b+c) + z(b+\mu c))(1+z) \\ -bcz + c(x-w) + z(t-a) \end{matrix} \\ 0 & 1 & z & bz - cy + w - x + (y(b+c) + z(b+\mu c)) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1})^4 = \{M(t,a,b,c) \in G_{4,1} |\ b^2 + 2bc + \mu c^2 = 0\}$$

$\bullet\mathfrak{g}_{4,3}$. The associated simply-connected Lie group is given by

$$G_{4,3} = \left\{ \begin{bmatrix} e^{-z} & 0 & 0 & w \\ 0 & 1 & -z & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ w, x, y, z, \in \mathbb{R} \right\}$$

1. The Lie rack structure associated to $\mathfrak{g}_{4,3}^1$ and the associated topological quandle are respectively

$$M(w,\, x,\, y,\, z) \rhd_1 M(t,\, a,\, b,\, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{c-z} + e^c(w-t) \\ 0 & 1 & -z & cy - bz + x + zc \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^1 = \{M(t,a,b,c) \in G_{4,3} |\ c = 0\}.$$

2. The Lie rack structure associated to $\mathfrak{g}_{4,3}^2$ and the associated topological quandle are respectively

$$M(w,\, x,\, y,\, z) \rhd_2 M(t,\, a,\, b,\, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{c-z} + e^c(w-t) \\ 0 & 1 & -z & cy - bz + x + (yb + \mu zc) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^2 = \{M(t,a,b,c) \in G_{4,3} |\ b^2 + \mu c^2 = 0\}.$$

3. The Lie rack structure associated to $\mathfrak{g}_{4,3}^3$ and the associated topological quandle are respectively

$$M(w,\, x,\, y,\, z) \rhd_3 M(t,\, a,\, b,\, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{c-z} + e^c(w-t) \\ 0 & 1 & -z & cy - bz + x + \gamma(yc + zb) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^3 = \{M(t,a,b,c) \in G_{4,3} |\ bc = 0\}.$$

$\bullet\mathfrak{g}_{4,8}^{-1}$. The associated simply-connected Lie group is given by

$$G_{4,8}^{-1} = \left\{ \begin{bmatrix} 1 & x & w \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}, \ w, x, y, z, \in \mathbb{R} \right\}.$$

The Lie rack structure associated to $(\mathfrak{g}_{4,8}^{-1})^1$ and the associated topological quandle are respectively

$$M(w,\, x,\, y,\, z) \rhd M(t,\, a,\, b,\, c) = \begin{bmatrix} 1 & a + xe^c - ae^z & w + zc + ae^{-c}(b + y - be^z)) \\ 0 & e^z & (y-b)e^{-c} + be^{z-c} \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,8}^{-1}) = \{M(t,a,b,c) \in G_{4,8}^{-1} |\ c = 0\}.$$

$\bullet\ \mathfrak{g}_{4,9}^0$. The associated simply-connected Lie group is given by

$$G_{4,9}^0 = \left\{ \begin{bmatrix} 1 & -x\cos(z) - y\sin(z) & y\cos(z) - x\sin(z) & -2w & 0 \\ 0 & \cos(z) & \sin(z) & y & 0 \\ 0 & -\sin(z) & \cos(z) & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}, \ w, x, y, z, \in \mathbb{R} \right\}.$$

The Lie rack structure associated to $(\mathfrak{g}^0_{4,9})^1$ and the associated topological quandle are respectively

$$M(w,\, x,\, y,\, z) \rhd M(t,\, a,\, b,\, c) =$$
$$\begin{bmatrix} 1 & \begin{array}{c} \cos(c+z)(a-x)+ \\ \sin(c+z)(b-y)+ \\ -a\cos(c) - b\sin(c) \end{array} & \begin{array}{c} \cos(c+z)(y-b)+ \\ \sin(c+z)(a-x)+ \\ b\cos(c) - a\sin(c) \end{array} & \begin{array}{c} \cos(z)ay - \cos(z)bx + \sin(z)a^2+ \\ -\sin(z)ax + \sin(z)b^2 - \sin(z)by+ \\ -2zc + ay - bx - 2w \end{array} & 0 \\ 0 & \cos(z) & \sin(z) & \begin{array}{c} \cos(c-z)b + \sin(c-z)a+ \\ \cos(c)(y-b) + \sin(c)(a-x) \end{array} & 0 \\ 0 & -\sin(z) & \cos(z) & \begin{array}{c} \cos(c-z)a - \sin(c-z)b+ \\ \cos(c)(x-a) + \sin(c)(y-b) \end{array} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}$$

$$Q(G^0_{4,9}) = \{M(t,a,b,c) \in G^0_{4,9} |\ c = 0\}.$$

3.4 Some algebraic properties of the obtained topological quandles

Quasi-triviality

We consider $(G, \rhd)$ a Lie rack associated to a symmetric Leibniz algebra. The rack operation is given by

$$h \rhd g \ := \ g^{-1}hg\chi(h,\, g). \tag{3.7}$$

We consider the associated topological quandle

$$Q(G) = \{g \in G, \chi(g,g) = 1\}.$$

$Q(G)$ is quasi-trivial if and only if, for any $g, k \in Q(G)$

$$g \rhd (g \rhd k) = g.$$

Let $g, k \in Q(G)$, we have

$$\begin{aligned} g \rhd (g \rhd k) = g \rhd (k^{-1}gk\chi(g,k)) &= \chi(g,k)^{-1}k^{-1}g^{-1}kgk^{-1}gk\chi(g,k)\chi(g,(k^{-1}gk\chi(g,k)) \\ &= k^{-1}g^{-1}kgk^{-1}gk\chi(g,(k^{-1}gk\chi(g,k)) \\ &= k^{-1}g^{-1}kgk^{-1}gk, \end{aligned}$$

Since

$$\chi(g_1g_2,h) = \chi(h,g_1g_2) = \chi(g_1,h)\chi(g_2,h) \text{ and } \chi(g,\chi(g,k)) = \chi(g,g) = 1.$$

Then $Q(G)$ is quasi-trivial if and only if

$$[g, k^{-1}gk] = 1, \text{ for any } g, k \in Q(G).$$

where $[a,b] = aba^{-1}b^{-1}$.

Note that if the quandle $Conj(G)$ is quasi-trivial, then $Q(G)$ is quasi-trivial.

We use this characterization to obtain the following result by a direct computation.

Proposition 3.4.1. *The following quandles are quasi-trivial:*

1. $Q(G_{3,1})^1$, $Q(G_{3,1})^2$,
2. $Q(G_{2,1} \times \mathbb{R}^2)^2$,
3. $Q(G_{3,1} \times \mathbb{R})^1$, $Q(G_{3,1} \times \mathbb{R})^2$,
4. $Q(G_{3,2} \times \mathbb{R})$, $Q(G_{3,3} \times \mathbb{R})$, $Q(G^0_{3,4} \times \mathbb{R})$, $Q(G^\alpha_{3,4} \times \mathbb{R})$, $Q(G^0_{3,5} \times \mathbb{R})$ *and* $Q(G^\alpha_{3,5} \times \mathbb{R})$,
5. $Q(G_{4,1})^1$, $Q(G_{4,1})^2$, $Q(G_{4,1})^3$, $Q(G_{4,1})^4$,

6. $Q(G_{4,3})^1$,
7. $Q(G_{4,8}^{-1})$ *and* $Q(G_{4,9}^0)$.

It is important to point out that quasi-trivial quandles can be used to obtain link-homotopy invariants (see [30]).

Medial quandles

We prove that all classes of topological quandles $Q(G_{3,1})$ and $Q(G_{3,1} \times \mathbb{R})$ are medial. To do that, we need the following lemma which can be easily shown.

Lemma 3.4.2. *1. For any $M(a,b,c), M(x,y,z),\ M(t,u,w) \in G_{3,1}$, we have*

$$\begin{aligned} M(a,b,c) \rhd [M(x,y,z) \rhd M(t,u,w)] &= M(a,b,c) \rhd M(x,y,z), \\ M(a,b,c) \rhd^{-1} [M(x,y,z) \rhd^{-1} M(t,u,w)] &= M(a,b,c) \rhd^{-1} M(x,y,z), \\ M(a,b,c) \rhd [M(x,y,z) \rhd^{-1} M(t,u,w)] &= M(a,b,c) \rhd M(x,y,z). \end{aligned}$$

2. For any $M(t,a,b,c), M(w,x,y,z), M(s,r,u,v) \in (G_{3,1} \times \mathbb{R})$, we have

$$\begin{aligned} M(t,a,b,c) \rhd [M(w,x,y,z) \rhd M(s,r,u,v)] &= M(t,a,b,c) \rhd M(w,x,y,z), \\ M(t,a,b,c) \rhd^{-1} [M(w,x,y,z) \rhd^{-1} M(s,r,u,v)] &= M(t,a,b,c) \rhd^{-1} M(w,x,y,z), \\ M(t,a,b,c) \rhd [M(w,x,y,z) \rhd^{-1} M(s,r,u,v)] &= M(t,a,b,c) \rhd M(t,a,b,c). \end{aligned}$$

Proof. By straightforward computation. □

Thus, we get

Proposition 3.4.3. *Both classes of quandles $Q(G_{3,1})$ and $Q(G_{3,1} \times \mathbb{R})$ are medial.*

Proof. Let $(M(a_i,b_i,c_i))_{1\leq i\leq 4} \in (G_{3,1})^4$. Due to the above lemma, we have

$$\begin{aligned} (M(a_1,b_1,c_1) &\rhd M(a_2,b_2,c_2)) \rhd (M(a_3,b_3,c_3) \rhd M(a_4,b_4,c_4)) \\ &= (M(a_1,b_1,c_1) \rhd M(a_2,b_2,c_2)) \rhd M(a_3,b_3,c_3), \\ &= (M(a_1,b_1,c_1) \rhd M(a_3,b_3,c_3)) \rhd (M(a_2,b_2,c_2) \rhd M(a_3,b_3,c_3)), \text{ (self-distributivity)} \\ &= (M(a_1,b_1,c_1) \rhd M(a_3,b_3,c_3)) \rhd M(a_2,b_2,c_2), \\ &= (M(a_1,b_1,c_1) \rhd M(a_3,b_3,c_3)) \rhd (M(a_2,b_2,c_2) \rhd M(a_4,b_4,c_4)). \end{aligned}$$

Hence, the quandles $Q(G_{3,1})$ are medial. Similarly, we show that $Q(G_{3,1} \times \mathbb{R})$ are medial. □

3.5 Appendix

In this appendix, we give the details of the computations of the results obtained in Section 4.2. The computation falls in the following steps:

1. We take a Lie algebra $\mathfrak{g}$ with non trivial center in the list of [11]. We denote by G the connected and simply connected Lie group associated to $\mathfrak{g}$. They are 13 Lie algebras, their Lie brackets are given in Table 3.1, the corresponding Lie groups were determined in [11] and we give them in Section 4.2.
2. By using Proposition 3.1.1, we determine the symmetric maps ω satisfying 3.3.
3. By using Proposition 3.1.2, we look for an automorphism $T = (a_{i,j})_{1\leq i,j\leq \dim \mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ in order to reduce the parameters of the obtained ω. Then we get all nonequivalent symmetric Leibniz structures for which $\mathfrak{g}$ is the underlying Lie algebra. We denote them by $\mathfrak{g}^1, \mathfrak{g}^2$ and so on.

4. For each symmetric Leibniz algebra obtained in the item 3, we give the map $\chi : G \times G \longrightarrow G$ given by 3.5 and used to determine the Lie rack structure on the Lie group G. Recall (see Method 1) that χ satisfies:

$$\chi(M(w,x,y,z), M(t,a,b,c)) = \exp(\beta(\kappa(M(w,x,y,z)), \kappa(M(t,a,b,c)))),$$

for any two elements $M(w,x,y,z)$, and $M(t,a,b,c)$ in G.

In the expressions of T and χ, E_{ij} is the matrix with 1 in the i-row and j-column and zero elsewhere.

The Lie algebra $\mathfrak{g}_{3,1}$: $[e_2,e_3] = e_1$, $Z(\mathfrak{g}_{3,1}) = \mathbb{R}e_1$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,1}$:

$$e_2 \bullet e_2 = \alpha e_1, e_3 \bullet e_3 = \beta e_1, e_2 \bullet e_3 = (\gamma+1)e_1, e_2 \bullet e_3 = (\gamma-1)e_1, \quad (\alpha,\ \beta,\ \gamma) \neq 0_{\mathbb{R}^3}.$$

After action by automorphisms of $\mathfrak{g}_{3,1}$, we get two families of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,1}$:

$$\begin{cases} \mathfrak{g}_{3,1}^1 : e_2 \cdot e_2 = \mu e_1, e_3 \cdot e_3 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1, \mu = (\alpha\beta - \gamma^2) \\ \mathfrak{g}_{3,1}^2 : e_2 \circ e_3 = (1+\gamma)e_1,\ e_3 \circ e_2 = (\gamma-1)e_1, \gamma \neq 0. \end{cases}$$

For $\beta \neq 0$, the automorphism $T = \beta(E_{11}+E_{22}) - \gamma E_{32} + E_{33}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$ for any $u, v \in \mathfrak{g}_{3,1}$.
For $\gamma = 0, \alpha \neq 0$, the automorphism $T = \alpha(E_{11} - E_{32}) + E_{23}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$ for any $u, v \in \mathfrak{g}_{3,1}$.
For $\gamma \neq 0, \beta = 0$, the automorphism $T = E_{11} + E_{22} - \frac{\alpha}{2\gamma} E_{32} + E_{33}$ satisfies $T(u \bullet v) = T(u) \circ T(v)$ for any $u, v \in \mathfrak{g}_{3,1}$.
The map χ associated to $(\mathfrak{g}_{3,1}^1, \cdot)$ is given by $\chi = \mathrm{I}_3 + (\mu yb + zc)E_{13}$.
The map χ associated to $(\mathfrak{g}_{3,1}^2, \circ)$ is given by $\chi = \mathrm{I}_3 + \gamma(yc + zb)E_{13}$.

The Lie algebra $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$: $[e_1,e_2] = e_1$, $Z(\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1) = \mathbb{R}e_3 \oplus \mathbb{R}e_4$.

They are two families of general forms of non Lie symmetric Leibniz products on $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$:

$$\begin{cases} e_1 \bullet e_2 = -e_2 \bullet e_1 = e_1, e_2 \bullet e_2 = \frac{\alpha\gamma}{\beta} e_3 + \gamma e_4, e_2 \bullet e_4 = e_4 \bullet e_2 = \alpha e_3 + \beta e_4, \\ e_2 \bullet e_3 = e_3 \bullet e_2 = -\beta e_3 - \frac{\beta^2}{\alpha} e_4, \quad \alpha\beta \neq 0, \\ e_1 \star e_2 = -e_2 \star e_1 = e_1, e_2 \star e_2 = \alpha e_3, e_2 \star e_4 = e_4 \star e_2 = \beta e_3, e_4 \star e_4 = \gamma e_3, (\alpha,\beta,\gamma) \neq (0,0,0). \end{cases}$$

After action by automorphisms of $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$, we get four families of non Lie symmetric Leibniz products on $\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$:

$$\begin{cases} (\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^1 : e_2 \cdot e_3 = e_3 \cdot e_2 = e_4, e_1 \cdot e_2 = -e_2 \cdot e_1 = e_1, \\ (\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^2 : e_2 \circ e_2 = e_3 - e_4, e_1 \circ e_2 = -e_2 \circ e_1 = e_1, \\ (\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^3 : e_2 * e_4 = e_4 * e_2 = e_3, e_1 * e_2 = -e_2 * e_1 = e_1, \\ (\mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1)^4 : e_2 \diamond e_2 = \varepsilon e_3, e_4 \diamond e_4 = e_3, e_1 \diamond e_2 = -e_2 \diamond e_1 = e_1, \varepsilon \in \{0,1,-1\}. \end{cases}$$

The automorphism $T = E_{11} + E_{22} - \frac{\alpha}{\beta} E_{33} + \beta E_{44} + \alpha E_{34} + \frac{\alpha\gamma}{2\beta^2} E_{32}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$, for any $u, v \in \mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$.
For $\alpha \neq 0, \beta = \gamma = 0$, the automorphism $T = E_{11} + E_{22} + \alpha E_{33} + E_{44} + E_{43}$ satisfies $T(u \star v) = T(u) \circ T(v)$, for any $u, v \in \mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$.
For $\beta \neq 0, \gamma = 0$, the automorphism $T = E_{11} + E_{22} + \beta E_{33} + E_{44} - \frac{\alpha}{2\beta} E_{42}$ satisfies $T(u \star v) = T(u) * T(v)$, for any $u, v \in \mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$.
For $\gamma \neq 0$, $\alpha\gamma - \beta^2 = 0$, $\varepsilon = 0$, the automorphism $T = E_{11} + E_{22} + \gamma E_{33} + E_{44} - \frac{\beta}{\gamma} E_{42}$ satisfies $T(u \star v) = T(u) \diamond T(v)$, for any $u, v \in \mathfrak{g}_{2,1} \oplus 2\mathfrak{g}_1$.
For $\gamma \neq 0, \alpha\gamma - \beta^2 \neq 0, \varepsilon$ is the sign of $\alpha\gamma - \beta^2$, the automorphism $T = E_{11} + E_{22} + E_{33} + \frac{\sqrt{\varepsilon(\alpha\gamma-\beta^2)}}{\varepsilon\gamma} E_{44} - \frac{\beta}{\gamma} E_{42}$

satisfies $T(u\star v)=T(u)\diamond T(v)$, for any $u,v\in\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1$. The map χ associated to $((\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1)^1,\cdot)$ is given by : $\chi=E_{11}+E_{22}+E_{33}+e^{xb+ya}E44$.

The map χ associted to $((\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1)^2,\circ)$ is given by : $\chi=E_{11}+E_{22}+e^{xa}E_{33}+e^{-xa}E44$.
The map χ associted to $((\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1)^3,*)$ is given by : $\chi=E_{11}+E_{22}+e^{xc+za}E_{33}+E44$.
The map χ associted to $((\mathfrak{g}_{2,1}\oplus 2\mathfrak{g}_1)^4,\diamond)$ is given by: $\chi=E_{11}+E_{22}+e^{\varepsilon xa+zc}E_{33}+E44$.

The Lie algebra $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$: $[e_2,e_3]=e_1$, $Z(\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)=\mathbb{R}e_1\oplus\mathbb{R}e_4$. The situation is similar to $\mathfrak{g}_{3,1}$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$:

$$e_2\bullet e_2=\alpha e_1, e_3\bullet e_3=\beta e_1, e_2\bullet e_3=(\gamma+1)e_1, e_2\bullet e_3=(\gamma-1)e_1,\quad (\alpha,\beta,\gamma)\neq 0_{\mathbb{R}^3}.$$

After action by automorphisms of $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$, we get two families of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1$:

$$\begin{cases}(\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^1 : e_2\cdot e_2=\mu e_1, e_3\cdot e_3=e_1, e_2\cdot e_3=-e_3\cdot e_2=e_1, \mu=(\alpha\beta-\gamma^2)\\ (\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^2 : e_2\circ e_3=(1+\gamma)e_1,\ e_3\circ e_2=(\gamma-1)e_1, \gamma\neq 0.\end{cases}$$

The map χ associated to $((\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^1,\cdot)$ is given by $\chi=\mathrm{I}_4+(\mu yb+zc)E_{13}$.
The map χ associated to $((\mathfrak{g}_{3,1}\oplus\mathfrak{g}_1)^2,\circ)$ is given by $\chi=\mathrm{I}_4+\gamma(yc+zb)E_{13}$.

The Lie algebra $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$: $[e_2,e_3]=e_1-e_2, [e_3,e_1]=e_1$, $Z(\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1)=\mathbb{R}e_4$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$:

$$e_3\bullet e_3=\alpha e_4, e_2\bullet e_3=-e_3\bullet e_2=e_1-e_2, e_3\bullet e_1=-e_1\bullet e_3=e_1, \alpha\neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$:

$$(\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1)^1 : e_3\cdot e_3=e_4, e_2\cdot e_3=-e_3\cdot e_2=e_1-e_2, e_3\cdot e_1=-e_1\cdot e_3=e_1.$$

The automorphism $T=E_{11}+E_{22}+E_{33}+\alpha E_{44}$ of $\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1$ satisfies $T(u\bullet v)=T(u)\cdot T(v)$.
The map χ associated to $((\mathfrak{g}_{3,2}\oplus\mathfrak{g}_1)^1,\cdot)$ is given by: $\chi=E_{11}+E_{22}+E_{33}+e^{yb}E_{44}$.

The Lie algebra $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$: $[e_2,e_3]=-e_2, [e_3,e_1]=e_1$, $Z(\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1)=\mathbb{R}e_4$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$:

$$e_3\bullet e_3=\alpha e_4, e_2\bullet e_3=-e_3\bullet e_2=-e_2, e_3\bullet e_1=-e_1\bullet e_3=e_1, \alpha\neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$:

$$(\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1)^1 : e_3\cdot e_3=e_4, e_2\cdot e_3=-e_3\cdot e_2=-e_2, e_3\cdot e_1=-e_1\cdot e_3=e_1.$$

The automorphism $T=E_{11}+E_{22}+E_{33}+\alpha E_{44}$ of $\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1$ satisfies $T(u\bullet v)=T(u)\cdot T(v)$.

The map χ associated to $((\mathfrak{g}_{3,3}\oplus\mathfrak{g}_1)^1,\cdot)$ is given by: $\chi=E_{11}+E_{22}+E_{33}+e^{yb}E_{44}$.

The Lie algebra $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$: $[e_2,e_3]=e_1, [e_3,e_1]=-e_2$, $Z(\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1)=\mathbb{R}e_4$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$:

$$e_3\bullet e_3=\alpha e_4, e_2\bullet e_3=-e_3\bullet e_2=e_1, e_3\bullet e_1=-e_1\bullet e_3=-e_2, \alpha\neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$:

$$(\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1)^1 : e_3\cdot e_3=e_4, e_2\cdot e_3=-e_3\cdot e_2=e_1, e_3\cdot e_1=-e_1\cdot e_3=-e_2.$$

The automorphism $T=E_{11}+E_{22}+E_{33}+\alpha E_{44}$ of $\mathfrak{g}_{3,4}^0\oplus\mathfrak{g}_1$ satisfies $T(u\bullet v)=T(u)\cdot T(v)$.

The map χ associated to $((\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1)^1, \cdot)$ is given by: $\chi = E_{11} + E_{22} + E_{33} + e^{yb}E_{44}$.

The Lie algebra $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$: $[e_2, e_3] = e_1 - \alpha e_2, [e_3, e_1] = \alpha e_1 - e_2, \alpha > 0, \alpha \neq 1, Z(\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1) = \mathbb{R}e_4$.
The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$:

$$e_3 \bullet e_3 = \beta e_4, e_2 \bullet e_3 = -e_3 \bullet e_2 = e_1 - \alpha e_2, e_3 \bullet e_1 = -e_1 \bullet e_3 = \alpha e_1 - e_2, \beta \neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$:

$$(\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1)^1 : e_3 \cdot e_3 = e_4, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1 - \alpha e_2, e_3 \cdot e_1 = -e_1 \cdot e_3 = \alpha e_1 - e_2.$$

The automorphism $T = E_{11} + E_{22} + E_{33} + \beta E_{44}$ of $\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
The map χ associated to $((\mathfrak{g}_{3,4}^\alpha \oplus \mathfrak{g}_1)^1, \cdot)$ is given by: $\chi = E_{11} + E_{22} + E_{33} + e^{yb}E_{44}$.

The Lie algebra $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$: $[e_2, e_3] = e_1, [e_3, e_1] = e_2, Z(\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1) = \mathbb{R}e_4$.
The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,4}^0 \oplus \mathfrak{g}_1$:

$$e_3 \bullet e_3 = \alpha e_4, e_2 \bullet e_3 = -e_3 \bullet e_2 = e_1, e_3 \bullet e_1 = -e_1 \bullet e_3 = e_2, \alpha \neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$:

$$(\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1)^1 : e_3 \cdot e_3 = e_4, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1, e_3 \cdot e_1 = -e_1 \cdot e_3 = e_2.$$

The automorphism $T = E_{11} + E_{22} + E_{33} + \alpha E_{44}$ of $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
The map χ associated to $((\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1)^1, \cdot)$ is given by: $\chi = E_{11} + E_{22} + E_{33} + E_{44} + e^{yb}E_{55}$.

The Lie algebra $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$: $[e_2, e_3] = e_1 - \alpha e_2, [e_3, e_1] = \alpha e_1 + e_2, \alpha > 0, Z(\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1) = \mathbb{R}e_4$.
The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$:

$$e_3 \bullet e_3 = \beta e_4, e_2 \bullet e_3 = -e_3 \bullet e_2 = e_1 - \alpha e_2, e_3 \bullet e_1 = -e_1 \bullet e_3 = \alpha e_1 + e_2, \beta \neq 0.$$

After action by automorphisms of $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$:

$$(\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1)^1 : e_3 \cdot e_3 = e_4, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1 - \alpha e_2, e_3 \cdot e_1 = -e_1 \cdot e_3 = \alpha e_1 + e_2.$$

The automorphism $T = E_{11} + E_{22} + E_{33} + \beta E_{44}$ of $\mathfrak{g}_{3,5}^\alpha \oplus \mathfrak{g}_1$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
The map χ associated to $((\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_1)^1, \cdot)$ is given by: $\chi = E_{11} + E_{22} + E_{33} + E_{44} + e^{yb}E_{55}$.

The Lie algebra $\mathfrak{g}_{4,1}$: $[e_2, e_4] = e_1, [e_3, e_4] = e_2, Z(\mathfrak{g}_{4,1}) = \mathbb{R}e_1$.
The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{4,1}$:

$$e_2 \bullet e_4 = -e_4 \bullet e_2 = e_1, e_3 \bullet e_4 = e_2 + \gamma e_1, e_4 \bullet e_3 = \gamma e_1 - e_2, e_3 \bullet e_3 = \alpha e_1, e_4 \bullet e_4 = \beta e_1, (\alpha, \beta, \gamma) \neq (0,0,0).$$

After action by automorphisms of $\mathfrak{g}_{4,1}$, we get four families of non Lie symmetric Leibniz products on $\mathfrak{g}_{4,1}$:

$$\begin{cases} \mathfrak{g}_{4,1}^1 : e_2 \cdot e_4 = -e_4 \cdot e_2 = e_1, e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2, e_3 \cdot e_3 = e_1, e_4 \cdot e_4 = \varepsilon e_1, \varepsilon \in \{0, 1, -1\} \\ \mathfrak{g}_{4,1}^2 : e_2 \circ e_4 = -e_4 \circ e_2 = e_1, e_3 \circ e_4 = -e_4 \circ e_3 = e_2, e_4 \circ e_4 = e_1, \\ \mathfrak{g}_{4,1}^3 : e_2 * e_4 = -e_4 * e_2 = e_1, e_3 * e_4 = e_1 + e_2, e_4 * e_3 = e_1 - e_2, e_4 * e_4 = e_1, \\ \mathfrak{g}_{4,1}^4 : e_2 \diamond e_4 = -e_4 \diamond e_2 = e_1, e_3 \diamond e_4 = e_1 + e_2, e_4 \diamond e_3 = e_1 - e_2, e_3 \diamond e_3 = e_1, e_4 \diamond e_4 = \frac{\alpha\beta}{\gamma^2} e_1 \end{cases}$$

For $\gamma = 0$, $\alpha \neq 0$, ε the sign of $\alpha\beta$, the automorphism $T = \frac{1}{\alpha}(E_{11} + E_{22} + E_{33}) + E_{44}$ of $\mathfrak{g}_{4,1}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.

For $\gamma = 0$, $\alpha = 0$, $\beta \neq 0$, the automorphism $T = \beta(E_{11} + E_{22} + E_{33}) + E_{44}$ of $\mathfrak{g}_{4,1}$ satisfies $T(u \bullet v) = T(u) \circ T(v)$.
For $\gamma \neq 0$, $\alpha = 0$, the automorphism $T = \gamma^2 E_{11} + \gamma E_{22} + E_{33} + \gamma E_{44} + \frac{1}{2}(1-\beta)E_{34}$ of $\mathfrak{g}_{4,1}$ satisfies $T(u \bullet v) = T(u) * T(v)$.
For $\gamma \neq 0, \alpha \neq 0$, the automorphism $T = \frac{\gamma^4}{\alpha} E_{11} - \frac{\gamma^3}{\alpha} E_{22} + \frac{\gamma^2}{\alpha} E_{33} - \gamma E_{44} + \frac{2\gamma^2}{\alpha} E_{34}$ of $\mathfrak{g}_{4,1}$ satisfies $T(u \bullet v) = T(u) \diamond T(v)$.

The map χ associated to $(\mathfrak{g}^1_{4,1}, \cdot)$ is given by: $\chi = \mathrm{I}_4 + (yb + \varepsilon zc)(E_{14} + E_{24})$.
The map χ associated to $(\mathfrak{g}^2_{4,1}, \circ)$ is given by: $\chi = \mathrm{I}_4 + zc(E_{14} + E_{24})$.
The map χ associated to $(\mathfrak{g}^3_{4,1}, *)$ is given by: $\chi = \mathrm{I}_4 + (yc + zb + zc)(E_{14} + E_{24})$.
The map χ associated to $(\mathfrak{g}^4_{4,1}, *)$ is given by: $\chi = \mathrm{I}_4 + (y(b+c) + z(b + \frac{\alpha\beta}{\gamma^2}c))(E_{14} + E_{24})$.

The Lie algebra $\mathfrak{g}_{4,3}$: $[e_1, e_4] = e_1, [e_3, e_4] = e_2$, $Z(\mathfrak{g}_{4,3}) = \mathbb{R}e_2$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}_{4,1}$:

$$e_3 \bullet e_3 = \alpha e_2, e_4 \bullet e_4 = \beta e_2, e_1 \bullet e_4 = -e_4 \bullet e_1 = e_1, e_3 \bullet e_4 = (\gamma+1)e_2, e_4 \bullet e_3 = (\gamma-1)e_2. (\alpha,\beta,\gamma) \neq (0,0,0).$$

After action by automorphisms of $\mathfrak{g}_{4,3}$, we get three family of non Lie symmetric Leibniz products on $\mathfrak{g}_{4,3}$:

$$\begin{cases} \mathfrak{g}^1_{4,3} : e_4 \cdot e_4 = e_2, e_1 \cdot e_4 = -e_4 \cdot e_1 = e_1, e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2, \\ \mathfrak{g}^2_{4,3} : e_4 \circ e_4 = (\alpha\beta - \gamma^2)e_2, e_3 \circ e_3 = e_2, e_1 \circ e_4 = -e_4 \circ e_1 = e_1, e_3 \circ e_4 = -e_4 \circ e_3 = e_2, \\ \mathfrak{g}^3_{4,3} : e_1 * e_4 = -e_4 * e_1 = e_1, e_3 * e_4 = (1+\gamma)e_2, e_4 * e_3 = (\gamma-1)e_2, \quad \gamma \neq 0. \end{cases}$$

For $\alpha = \gamma = 0$, the automorphism $T = E_{11} + \beta(E_{22} + E_{33}) + E_{44}$ of $\mathfrak{g}_{4,3}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$, for any $u, v \in \mathfrak{g}_{4,3}$.
For $\alpha \neq 0$, the automorphism $T = E_{11} + \frac{1}{\alpha}(E_{22} + E_{33}) + E_{44} - \frac{\gamma}{\alpha}E_{34}$ of $\mathfrak{g}_{4,3}$ satisfies $T(u \bullet v) = T(u) \circ T(v)$, for any $u, v \in \mathfrak{g}_{4,3}$.
For $\alpha = 0$ and $\gamma \neq 0$, the automorphism $T = E_{11} + E_{22} + E_{33} + E_{44} - \frac{\beta}{2\gamma}E_{34}$ of $\mathfrak{g}_{4,3}$ satisfies $T(u \bullet v) = T(u) * T(v)$, for any $u, v \in \mathfrak{g}_{4,3}$.
The map χ associated to $(\mathfrak{g}^1_{4,3}, \cdot)$ is given by: $\chi = \mathrm{I}_4 + zcE_{24}$.
The map χ associated to $(\mathfrak{g}^2_{4,3}, \circ)$ is given by: $\chi = \mathrm{I}_4 + (yb + \mu zc)E_{24}$.
The map χ associated to $(\mathfrak{g}^3_{4,3}, *)$ is given by: $\chi = \mathrm{I}_4 + \gamma(yc + zb)E_{24}$.

The Lie algebra $\mathfrak{g}^{-1}_{4,8}$: $[e_2, e_3] = e_1, [e_3, e_4] = -e_3, [e_2, e_4] = e_2$, $Z(\mathfrak{g}^{-1}_{4,8}) = \mathbb{R}e_1$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}^{-1}_{4,8}$:

$$e_4 \bullet e_4 = \alpha e_1, e_2 \bullet e_3 = -e_3 \bullet e_2 = e_1, e_3 \bullet e_4 = -e_4 \bullet e_3 = -e_3, e_2 \bullet e_4 = -e_4 \bullet e_2 = e_2, \quad \alpha \neq 0.$$

After action by automorphisms of $\mathfrak{g}^{-1}_{4,8}$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}^{-1}_{4,8}$:

$$(\mathfrak{g}^{-1}_{4,8})^1 : e_4 \cdot e_4 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1, e_3 \cdot e_4 = -e_4 \cdot e_3 = -e_3, e_2 \cdot e_4 = -e_4 \cdot e_2 = e_2.$$

The automorphism $T = \alpha(E_{11} + E_{33}) + E_{22} + E_{44}$ of $\mathfrak{g}^{-1}_{4,8}$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
The map χ associated to $(\mathfrak{g}^{-1}_{4,8}, \cdot)$ is given by: $\chi = \mathrm{I}_3 + zcE_{13}$.

The Lie algebra $\mathfrak{g}^0_{4,9}$: $[e_2, e_3] = e_1, [e_3, e_4] = e_2, [e_2, e_4] = -e_3$, $Z(\mathfrak{g}^0_{4,9}) = \mathbb{R}e_1$.

The general form of non Lie symmetric Leibniz products on $\mathfrak{g}^0_{4,9}$:

$$e_4 \bullet e_4 = \alpha e_1, e_2 \bullet e_3 = -e_3 \bullet e_2 = e_1, e_3 \bullet e_4 = -e_4 \bullet e_3 = e_2, e_2 \bullet e_4 = -e_4 \bullet e_2 = -e_3, \quad \alpha \neq 0.$$

After action by automorphisms of $\mathfrak{g}_{4,9}^0$, we get a unique non Lie symmetric Leibniz product on $\mathfrak{g}_{4,9}^0$:

$$(\mathfrak{g}_{4,9}^0)^1 : e_4 \cdot e_4 = e_1, e_2 \cdot e_3 = -e_3 \cdot e_2 = e_1, e_3 \cdot e_4 = -e_4 \cdot e_3 = e_2, e_2 \cdot e_4 = -e_4 \cdot e_2 = -e_3.$$

For $\alpha > 0$, the automorphism $T = \alpha E_{11} + \sqrt{\alpha}(E_{22} + E_{33}) + E_{44}$ of $\mathfrak{g}_{4,9}^0$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
For $\alpha < 0$, the automorphism $T = \alpha E_{11} + \sqrt{|\alpha|}(E_{33} - E_{22}) - E_{44}$ of $\mathfrak{g}_{4,9}^0$ satisfies $T(u \bullet v) = T(u) \cdot T(v)$.
The map χ associated to $(\mathfrak{g}_{4,9}^0, \cdot)$ is given by: $\chi = \mathrm{I}_5 - 2zcE_{14}$.

We finish this appendix by giving a particular example to explicit the computation of the map χ[1] obtained in the above computations. We consider the symmetric Leibniz algebra $\mathfrak{g}_{4,1}^1$, and $G_{4,1}$ the connected and simply-connected Lie group associated to the Lie algebra $\mathfrak{g}_{4,1}$. We have $\mathfrak{g}_{4,1}^1 . \mathfrak{g}_{4,1}^1 \simeq \langle e_1, e_2 \rangle$ and the quotient space $\mathfrak{a} = \mathfrak{g}_{4,1}^1 / \mathfrak{g}_{4,1}^1 . \mathfrak{g}_{4,1}^1$ is identified to $\mathbb{R}^2$. So the projection $q : \mathfrak{g}_{4,1}^1 \longrightarrow \mathfrak{a}$ is given by

$$q(w, x, y, z) = (y, z),$$

it follows that the homomorphism $\kappa : G_{4,1} \longrightarrow \mathbb{R}^2$ must be defined by $\kappa(w, x, y, z) = (y, z)$.
Now, $\beta : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathfrak{g}_{4,1}$ is defined by

$$\beta((y, z), (b, c)) = (yb + \varepsilon zc)e_1, \quad \varepsilon = 0, 1, -1$$

and the map $\chi : G_{4,1} \times G_{4,1} \longrightarrow G_{4,1}$ is given by

$$\chi(M(w, x, y, z), M(t, a, b, c)) = \begin{bmatrix} 1 & 0 & 0 & (yb + \varepsilon zc) \\ 0 & 1 & 0 & (yb + \varepsilon zc) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, we get the Lie rack product on $G_{4,1}$

$$M(w, x, y, z) \rhd_1 M(t, a, b, c) = \begin{bmatrix} 1 & z & \frac{1}{2}z^2 & \begin{matrix} w + \frac{1}{2}c^2 y + \frac{1}{2}bz^2 + (yb + \varepsilon zc)(1 + z) \\ -bcz + c(x - w) + z(t - a) \end{matrix} \\ 0 & 1 & z & bz - cy + w - x + (yb + \varepsilon zc) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[1] see Method 1 which explains the result for general case.

Chapter 4

Analytic linear Lie rack structures on Leibniz algebras

This chapter is dedicated to study *linear* Lie rack structures with an emphasis on *analytic linear* Lie rack structures. Throughout this chapter, we will consider the notion of (left) rack[1] with left distributivity.

4.1 Analytic linear Lie rack structures and Rigidity of Leibniz algebras

4.1.1 Analytic linear Lie rack structures

Definition 18. 1. A *linear* Lie rack structure on a finite dimensional vector space V is a Lie rack operation $(x,y) \mapsto x \rhd y$ pointed at 0 and such that for any x, the map $\mathrm{L}_x : y \mapsto x \rhd y$ is linear.

2. A linear Lie rack operation $\rhd$ is called *analytic* if for any $x,y \in V$,

$$x \rhd y = y + \sum_{n=1}^{\infty} A_{n,1}(x,\ldots,x,y), \tag{4.2}$$

where for each n, $A_{n,1} : V \times \ldots \times V \longrightarrow V$ is an $n+1$-multilinear map which is symmetric in the n first arguments. In this case, $A_{1,1}$ is the left Leibniz bracket associated to $\rhd$.

As an immediate example, given a left Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ then the operation $\overset{c}{\rhd}: \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ given by

$$u \overset{c}{\rhd} v = \exp(\mathrm{ad}_u)(v)$$

defines an analytic linear Lie rack structure on $\mathfrak{h}$ such that the associated left Leibniz bracket on $T_0\mathfrak{h} = \mathfrak{h}$ is the initial bracket $[\ ,\ ]$. We call $\overset{c}{\rhd}$ the *canonical* linear Lie rack structure associated to $(\mathfrak{h}, [\ ,\ ])$.

In this section will look more closely at analytic linear Lie rack structures and their characterization. One of our main results is the following theorem.

Theorem 4.1.1. *Let V be a real finite dimensional vector space and $(A_{n,1})_{n\geq 1}$ a sequence of $n+1$-multilinear maps symmetric in the n first arguments. We suppose that the operation $\rhd$ given by*

$$x \rhd y = y + \sum_{n=1}^{\infty} A_{n,1}(x,\ldots,x,y)$$

[1] A (left) rack is a non-empty set X together with a map $\rhd : X \times X \longrightarrow X$, $(a,b) \mapsto a \rhd b$ such that, for any $a,b,c \in X$, the map $\mathrm{L}_a : X \longrightarrow X$, $b \mapsto a \rhd b$ is a bijection and

$$a \rhd (b \rhd c) = (a \rhd b) \rhd (a \rhd c). \tag{4.1}$$

converges. Then $\rhd$ is a Lie rack structure on V if and only if for any $p,q\in\mathbb{N}^*$ and $x,y,z\in V$,

$$A_{p,1}(x,A_{q,1}(y,z)) = \sum_{s_1+\ldots+s_q+k=p} A_{q,1}(A_{s_1,1}(x,y),\ldots,A_{s_q,1}(x,y),A_{k,1}(x,z)), \quad (4.3)$$

where for sake of simplicity $A_{p,1}(x,y):=A_{p,1}(x,\ldots,x,y)$.

In particular, if $p=q=1$ we get that $[\ ,\ ]:=A_{1,1}$ is a left Leibniz bracket which is actually the left Leibniz bracket associated to $(V,\rhd)$.

Remark 3. When $p=1$ and $q\in\mathbb{N}^*$, the relation 4.3 becomes

$$\mathcal{L}_x A_{q,1}(y_1,\ldots,y_{q+1}) := [x,A_{q,1}(y_1,\ldots,y_{q+1})] - \sum_{i=1}^{q+1} A_{q,1}(y_1,\ldots,[x,y_i],\ldots,y_{q+1}) = 0. \quad (4.4)$$

A multilinear map on a left Leibniz algebra satisfying 4.4 will be called *invariant*. Thus Theorem 4.1.1 reduces the study of analytic linear Lie rack structures to the study of the datum of a left Leibniz algebra with a sequence of invariant multilinear maps satisfying a sequence of multilinear equations. Even though equations 4.3 are complicated, we will see in this chapter that they are far more easy to handle than the distributivity condition4.1. **Proof of Theorem 4.1.1.**

Proof. Put $A_{0,1}(x,y)=y$. We have

$$\begin{aligned} x\rhd(y\rhd z) &= \sum_{n\in\mathbb{N}} A_{n,1}(x,\ldots,x,y\rhd z)\\ &= \sum_{n,p\in\mathbb{N}} A_{n,1}(x,\ldots,x,A_{p,1}(y,\ldots,y,z)),\end{aligned}$$

$$\begin{aligned} (x\rhd y)\rhd(x\rhd z) &= \sum_{n=0}^{\infty} A_{n,1}(x\rhd y,\ldots,x\rhd y,x\rhd z)\\ &= \sum_{n,s_1,\ldots,s_n,k} A_{n,1}(A_{s_1,1}(x,y),\ldots,A_{s_n,1}(x,y),A_{k,1}(x,z)).\end{aligned}$$

By identifying the homogeneous component of degree n in x and of degree p in y in both $x\rhd(y\rhd z)$ and $(x\rhd y)\rhd(x\rhd z)$ we get the desired relation. □

The following result is an immediate and important consequence of Theorem 4.1.1.

Corollary 4.1.2. *Let $(\mathfrak{h},[\ ,\ ])$ be a left Leibniz algebra and $\overset{c}{\rhd}$ its canonical linear Lie rack operation. Then*

$$x\overset{c}{\rhd}y=\sum_{n=0}^{\infty}A^0_{n,1}(x,\ldots,x,y)$$

where

$$A^0_{0,1}(x,y)=y \quad \text{and} \quad A^0_{n,1}(x_1,\ldots,x_n,y)=\frac{1}{(n!)^2}\sum_{\sigma\in S_n}\mathrm{ad}_{x_{\sigma(1)}}\circ\ldots\circ\mathrm{ad}_{x_{\sigma(n)}}(y),$$

and S_n is the group of permutations of $\{1,\ldots,n\}$. Furthermore, the $\left(A^0_{n,1}\right)_{n\in\mathbb{N}}$ satisfy the sequence of equations 4.3.

Actually, there is a large class of linear Lie rack structures on a Leibniz algebra $(\mathfrak{h},[\ ,\ ])$ containing the canonical one. Proposition 4.1.3 will ensure this statement. This class was suggested to us by an example sent to us by Martin Bordemann.

Proposition 4.1.3. *Let $(\mathfrak{h},[\ ,\ ])$ be a left Leibniz algebra, $F:\mathbb{R}\longrightarrow\mathbb{R}$ a smooth function and $P:\mathfrak{h}\times\ldots\times\mathfrak{h}\longrightarrow\mathbb{R}$ a symmetric multilinear p-form such that, for any $y,x_1\ldots,x_p\in\mathfrak{h}$,*

$$\sum_{i=1}^{p}P(x_1,\ldots,[y,x_i],\ldots,x_p)=0.$$

Then the operation $\rhd$ given by

$$x \rhd y = \exp(F(P(x, \dots, x))\mathrm{ad}_x)(y)$$

is a linear Lie rack structure on $\mathfrak{h}$ and its associated left Leibniz bracket is $[\ ,\ ]_{\rhd} = F(0)[\ ,\ ]$. Moreover, if F is analytic then $\rhd$ is analytic.

The proof this proposition is a consequence of the following well-known result.

Proposition 4.1.4. *Let $(X, \rhd)$ be a rack and $J : X \longrightarrow X$ a map such that, for any $x, y \in X$, $J(x \rhd y) = x \rhd J(y)$, i.e., $J \circ \mathrm{L}_x = \mathrm{L}_x \circ J$ for any x. Then the operation*

$$x \rhd_J y = J(x) \rhd y$$

defines a rack structure on X.

Proof. We have, for any $x, y, z \in X$,

$$\begin{aligned} x \rhd_J (y \rhd_J z) &= J(x) \rhd (J(y) \rhd z) \\ &= (J(x) \rhd J(y)) \rhd (J(x) \rhd z) \\ &= (J(J(x) \rhd y)) \rhd (x \rhd_J z) \\ &= (x \rhd_J y) \rhd_J (x \rhd_J z). \quad \square \end{aligned}$$

Proof of Proposition 4.1.3.

Proof. We consider the map $J : \mathfrak{h} \longrightarrow \mathfrak{h}$ given by $J(x) = F(P(x, \dots, x))x$. Since P is invariant, we have $P(\exp(\mathrm{ad}_x)(y), \dots, \exp(\mathrm{ad}_x)(y)) = P(y, \dots, y)$ and hence $J(x \overset{c}{\rhd} y) = x \overset{c}{\rhd} J(y)$ and one can apply Proposition 4.1.4 to conclude. $\square$

This proposition shows that a left Leibniz algebra might be associated to many non equivalent pointed Lie rack structures. For instance, if one takes $F(0) = 0$ in Proposition 4.1.3, the two pointed Lie rack structures

$$x \rhd_0 y = y \quad \text{and} \quad x \rhd_1 y = \exp(F(P(x, \dots, x)))\mathrm{ad}_x)(y)$$

are two pointed Lie rack structures on $\mathfrak{h}$ which are not equivalent (even locally near 0) and have the same left Leibniz algebra, namely, the abelian one. This contrasts with the theory of Lie groups where two Lie groups are locally equivalent near their unit elements if and only if they have the same Lie algebra. Moreover, this proposition motivates our study of analytic linear Lie rack structures on Leibniz algebras.

4.1.2 Rigidity of Leibniz algebras

This subsection is an introduction to the rigidity of left Leibniz algebras. Due to Proposition 4.1.3, we have seen that a left Leibniz algebra might be associated to many non equivalent pointed Lie rack structures as shown above. This statement gives a sense of the following definition.

Definition 19. A left Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ is called *rigid* if any analytic linear Lie rack structure $\rhd$ on $\mathfrak{h}$ such that $[\ ,\ ]_{\rhd} = [\ ,\ ]$ is given by

$$x \rhd y = \exp(F(P(x, \dots, x)))\mathrm{ad}_x)(y),$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is analytic with $F(0) = 1$ and $P : \mathfrak{h} \times \dots \times \mathfrak{h} \longrightarrow \mathbb{R}$ is a symmetric multilinear p-form such that, for any $y, x_1 \dots, x_p \in \mathfrak{h}$,

$$\sum_{i=1}^{p} P(x_1, \dots, [y, x_i], \dots, x_p) = 0.$$

Remark 4. We have seen that the abelian left Leibniz algebra is not rigid.

To give more sense of our definition of rigid Leibniz algebras, it is more convenient to show if there are non rigid left Leibniz algebras other than the abelian ones. In fact, the following proposition shows that the class of non rigid left Leibniz algebras is large. Recall that if $\mathfrak{h}$ is a left Leibniz algebra then its center $Z(\mathfrak{h}) = \{a \in \mathfrak{h}, [a,\mathfrak{h}] = [\mathfrak{h},a] = 0\}$.

Proposition 4.1.5. *Let $(\mathfrak{h},[\ ,\ ])$ be a left Leibniz algebra. Choose a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{h}$, $(a_1,b_1,\ldots,a_k,b_k)$ a family of vectors in $[\mathfrak{h},\mathfrak{h}]^\perp \cap Z(\mathfrak{h})^\perp$, $(z_1,\ldots,z_k)$ a family of vector in $Z(\mathfrak{h})$ and $f_1,\ldots,f_k : \mathbb{R} \longrightarrow \mathbb{R}$ with $f_j(0) = 0$ for $j = 1,\ldots,k$. If $\overset{c}{\rhd}$ is the canonical linear Lie rack operation on $\mathfrak{h}$ then*

$$x \rhd y = x \overset{c}{\rhd} y + \sum_{j=1}^{k} \langle y, b_j\rangle f_j(\langle x, a_j\rangle) z_j$$

is a linear Lie rack operation pointed at 0. Moreover, if $f_j'(0) = 0$ for $j = 1,\ldots,k$ then $[\ ,\ ]_\rhd = [\ ,\ ]$.

Proof. Note first that for any $z \in Z(\mathfrak{h})$ and for any $x \in \mathfrak{h}$, $x \rhd z = z$ and $z \rhd x = x$. Moreover, $\langle b_j, x \rhd y\rangle = \langle b_j, x \overset{c}{\rhd} y\rangle = \langle b_j, y\rangle$ and $(x \rhd y) \overset{c}{\rhd} z = (x \overset{c}{\rhd} y) \overset{c}{\rhd} z$ for any $x,y,z \in \mathfrak{h}$. So, for any $x,y,z \in \mathfrak{h}$,

$$\begin{aligned}
x \rhd (y \rhd z) &= x \rhd (y \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle y, a_j\rangle) x \rhd z_j \\
&= x \overset{c}{\rhd} (y \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle y \overset{c}{\rhd} z, b_j\rangle f_j(\langle x, a_j\rangle) z_j + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle y, a_j\rangle) z_j, \\
&= x \overset{c}{\rhd} (y \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle x, a_j\rangle) z_j + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle y, a_j\rangle) z_j, \\
(x \rhd y) \rhd (x \rhd z) &= (x \rhd y) \rhd (x \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle x, a_j\rangle) z_j \\
&= (x \rhd y) \overset{c}{\rhd} (x \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle x \overset{c}{\rhd} z, b_j\rangle f_j(\langle x \rhd y, a_j\rangle) z_j + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle x, a_j\rangle) z_j \\
&= (x \overset{c}{\rhd} y) \overset{c}{\rhd} (x \overset{c}{\rhd} z) + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle y, a_j\rangle) z_j + \sum_{j=1}^{k} \langle z, b_j\rangle f_j(\langle x, a_j\rangle) z_j.
\end{aligned}$$

This proves the proposition. □

Corollary 4.1.6. *1. Let $\mathfrak{h}$ be a left Leibniz algebra which is a Lie algebra such that $[\mathfrak{h},\mathfrak{h}] + Z(\mathfrak{h}) \neq \mathfrak{h}$, $Z(\mathfrak{h}) \neq \{0\}$. Then $\mathfrak{h}$ is not rigid.*

2. Let $\mathfrak{h}$ be a left Leibniz algebra such that $[\mathfrak{h},\mathfrak{h}] + Z(\mathfrak{h}) \neq \mathfrak{h}$ and $Z(\mathfrak{h})$ is not contained in $[\mathfrak{h},\mathfrak{h}]$. Then $\mathfrak{h}$ is not rigid.

Proof. 1. By virtue of Definition 19, if $\mathfrak{h}$ is rigid then any linear analytic rack structure $\rhd$ on $\mathfrak{h}$ satisfies $x \rhd x = x$ for any $x \in \mathfrak{h}$. Choose $z \in Z(\mathfrak{h}) \setminus \{0\}$, a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{h}$ and $a \in [\mathfrak{h},\mathfrak{h}]^\perp \cap Z(\mathfrak{h})^\perp$ with $a \neq 0$. According to Proposition 4.1.5, the operation

$$x \rhd y = x \overset{c}{\rhd} y + \langle x, a\rangle^2 \langle y, a\rangle z$$

is an analytic linear Lie rack structure on $\mathfrak{h}$ satisfying $[\ ,\ ]_\rhd = [\ ,\ ]$. However, this operation satisfies $a \rhd a = a + |a|^6 z \neq a$ and hence $\mathfrak{h}$ is not rigid.

2. We have also that if $\mathfrak{h}$ is rigid then any linear analytic rack structure $\rhd$ on $\mathfrak{h}$ satisfies $x \rhd y - x \overset{c}{\rhd} y \in [\mathfrak{h},\mathfrak{h}]$. We proceed as the first case and we consider the same Lie rack operation on $\mathfrak{h}$ with $a \in Z(\mathfrak{h})$ and $a \notin [\mathfrak{h},\mathfrak{h}]$ and we get a contradiction. □

Remark 5. There is a large class of left Leibniz algebras satisfying the hypothesis of Corollary 4.1.6, for instance, any 2-step nilpotent Lie algebra belongs to this class.

4.2 Analytic linear Lie racks structures over left Leibniz algebras with trivial 0-cohomology and 1-cohomology

In this section, we will give an important expression of the $A_{n,1}$ defining an analytic linear Lie rack structure on a left Leibniz algebra $\mathfrak{h}$ when $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$.

Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra. First we recall, again, the definition of the cohomology of a left Leibniz algebra [20]. For any $n \geq 0$, the operator $\delta : Hom(\otimes^n \mathfrak{h}, \mathfrak{h}) \longrightarrow Hom(\otimes^{n+1}\mathfrak{h}, \mathfrak{h})$ given by

$$\begin{aligned}\delta(\omega)(x_0, \ldots, x_n) &= \sum_{i=0}^{n-1} [x_i, \omega(x_0, \ldots, \hat{x}_i, \ldots, x_n)] + (-1)^{n-1}[\omega(x_0, \ldots, x_{n-1}), x_n] \\ &\quad + \sum_{i<j} (-1)^{i+1} \omega(x_0, \ldots, \hat{x}_i, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_n),\end{aligned}$$

satisfies $\delta^2 = 0$ and then defines a cohomology $H^p(\mathfrak{h})$ for $p \in \mathbb{N}$. For any $x \in \mathfrak{h}$ and $F, G : \mathfrak{h} \longrightarrow \mathfrak{h}$, we have

$$\delta(x)(m) = -[x, m] \quad \text{and} \quad \delta(F)(y, z) = [y, F(z)] + [F(y), z] - F([y, z])$$

and one can see easily that

$$\delta(F \circ G)(y, z) = \delta(F)(y, G(z)) + \delta(F)(G(y), z) + F \circ \delta(G)(y, z) - [F(y), G(z)] - [G(y), F(z)]. \tag{4.5}$$

Remark 6. Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra which is a Lie algebra. The cohomology of $\mathfrak{h}$ as left Leibniz algebra is different from its cohomology as a Lie algebra, however H^0 and H^1 are the same for both cohomologies.

Now let's take a closer look to equations 4.3 when $q = 1$. Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra and $(A_{n,1})_{p \in \mathbb{N}}$ a sequence of $(n+1)$-multilinear maps on $\mathfrak{h}$ with values in $\mathfrak{h}$ symmetric in the n first arguments and such that $A_{0,1}(x, y) = y$ and $A_{1,1}(x, y) = [x, y]$. For sake of simplicity we write $A_{n,1}(x, y) = A_{n,1}(x, \ldots, x, y)$.
Equation 4.3 for $q = 1$ can be written for any $x, y, z \in \mathfrak{h}$,

$$A_{p,1}(x, [y, z]) = [y, A_{p,1}(x, z)] + [A_{p,1}(x, y), z] + \sum_{r=1}^{p-1} [A_{r,1}(x, y), A_{p-r,1}(x, z)].$$

Thus

$$\delta(i_x \ldots i_x A_{p,1})(y, z) = -\sum_{r=1}^{p-1} [A_{r,1}(x, y), A_{p-r,1}(x, z)], \tag{4.6}$$

where $i_x \ldots i_x A_{p,1} : \mathfrak{h} \longrightarrow \mathfrak{h}, y \mapsto A_{p,1}(x, \ldots, x, y)$.

On the other hand, the sequence $(A^0_{n,1})_{n \in \mathbb{N}}$ defining the canonical linear Lie rack structure of $\mathfrak{h}$ (see Corollary 4.1.2) satisfies 4.3 and hence

$$\delta(i_x \ldots i_x A^0_{p,1})(y, z) = -\sum_{r=1}^{p-1} [A^0_{r,1}(x, y), A^0_{p-r,1}(x, z)]. \tag{4.7}$$

If $p = 2$, since $A_{0,1} = A^0_{0,1}$ and $A_{1,1} = A^0_{1,1}$, Equations 4.6 and 4.7 implies that, for any $x \in \mathfrak{h}$,

$$\delta(i_x i_x A_{2,1} - i_x i_x A^0_{2,1}) = 0.$$

Since $A_{2,1}$ and $A^0_{2,1}$ are symmetric in the two first arguments this is equivalent to

$$\delta(i_x i_y A_{2,1} - i_x i_y A^0_{2,1}) = 0, \quad \text{for any } x, y \in \mathfrak{h}.$$

This is a cohomological equation and if $H^1(\mathfrak{h}) = 0$ then there exists $B_2 : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ such that, for any $x, y, z \in \mathfrak{h}$,

$$A_{2,1}(x, y, z) = A^0_{2,1}(x, y, z) + [B_2(x, y), z]. \tag{4.8}$$

Moreover, if $H^0(\mathfrak{h}) = 0$ then B_2 is unique and symmetric and one can check easily that $A_{2,1}$ is invariant if and only if B_2 is invariant.

We have triggered an induction process and, under the same hypothesis, the $(A_{p,1})_{p\geq 2}$ satisfy a similar formula as 4.8. This is the purpose of the following theorem.

Theorem 4.2.1. *Let $(\mathfrak{h}, [\ ,\ ])$ be a left Leibniz algebra such that $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$. Let $(A_{n,1})_{n\geq 0}$ be a sequence where $A_{0,1}(x,y) = y$ and $A_{1,1}(x,y) = [x,y]$ and, for any $n \geq 2$, $A_{n,1} : \mathfrak{h}\times\ldots\times\mathfrak{h} \longrightarrow \mathfrak{h}$ is multilinear invariant and symmetric in the n first arguments. We suppose that the $A_{n,1}$ satisfy 4.6. Then there exists a unique sequence $(B_n)_{n\geq 2}$ of invariant symmetric multilinear maps $B_n : \mathfrak{h}\times\ldots\times\mathfrak{h} \longrightarrow \mathfrak{h}$ such that, for any $x, y \in \mathfrak{h}$,*

$$A_{n,1}(x,y) = A^0_{n,1}(x,y) + \sum_{\substack{1\leq k\leq \left[\frac{n}{2}\right] \\ s = l_1+\ldots+l_k\leq n}} A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x), A^0_{n-s,1}(x,y)), \quad (4.9)$$

where $A_{p,1}(x,y) = A_{p,1}(x,\ldots,x,y)$ and $B_l(x) = B_l(x,\ldots,x)$.

Remark 7. Formula 4.9 deserves some explications. As any formula depending inductively on n, to find the general form one needs to check it for the first values of n and it is what we have done. There are the formulas we found directly and which helped us to establish the expression 4.9.

$$\begin{aligned}
A_{3,1}(x,y) &= A^0_{3,1}(x,y) + [B_2(x), A^0_{1,1}(x,y)] + [B_3(x), A^0_{0,1}(x,y)] \\
&= A^0_{3,1}(x,y) + A^0_{1,1}(B_2(x), A^0_{1,1}(x,y)) + A^0_{1,1}(B_3(x), A^0_{0,1}(x,y)), \\
A_{4,1}(x,y) &= A^0_{4,1}(x,y) + [B_4(x), A^0_{0,1}(x,y)] + [B_3(x), A^0_{1,1}(x,y)] + [B_2(x), A^0_{2,1}(x,y)] \\
&\quad + \frac{1}{2}[B_2(x), [B_2(x), A^0_{0,1}(x,y)]], \\
A_{5,1}(x,y) &= A^0_{5,1}(x,y) + [B_5(x), y] + [B_4(x), A^0_{1,1}(x,y)] + [B_3(x), A^0_{2,1}(x,y)] + \\
&\quad [B_2(x), A^0_{3,1}(x,y)] + \frac{1}{2}\left([B_2(x), [B_3(x), y]] + [B_3(x), [B_2(x), y]]\right) + \\
&\quad \frac{1}{2}[B_2(x), [B_2(x), A^0_{1,1}(x,y)]].
\end{aligned}$$

To prove Theorem 4.2.1, we will proceed by induction. The proof is rather technical and needs some preliminary formulas.

Fix $n \geq 2$ and $x \in \mathfrak{h}$. For any $1 \leq k \leq \left[\frac{n+1}{2}\right]$ and $s = l_1 + \ldots + l_k \leq n+1$, in the proof of Theorem 4.2.1, we will need to compute $\delta(F_k \circ G_s)$ where $F_k, G_s : \mathfrak{h} \longrightarrow \mathfrak{h}$ are given by

$$F_k(y) = A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x), y), \quad G_s(y) = A^0_{n+1-s,1}(x,y).$$

This is straightforward from 4.5 and the formula

$$\delta(i_{x_1}\ldots i_{x_k} A^0_{k,1})(y,z) = -\frac{1}{k!}\sum_{p=1}^{k-1}\sum_{\sigma\in S_k}[A^0_{p,1}(x_{\sigma(1)},\ldots,x_{\sigma(p)},y), A^0_{k-p,1}(x_{\sigma(p+1)},\ldots,x_{\sigma(k)},z)]$$

whose polar form is 4.7. We use here the well-know fact that two symmetric multilinear forms are equal if and only if their polar forms are equal. For sake of simplicity put $Q(k,s) = \delta(F_k \circ G_s)(y,z)$.

Proposition 4.2.2. *We have*

$$\begin{aligned}
Q(1,s) &= -\sum_{r=0}^{n-s}[A^0_{1,1}(B_s(x),A^0_{r,1}(x,y)),A^0_{n+1-s-r,1}(x,z)]\\
&\quad -\sum_{r=1}^{n+1-s}[A^0_{r,1}(x,y),A^0_{1,1}(B_s(x),A^0_{n+1-s-r,1}(x,z))],\quad s\leq n,\\
Q(k,n+1) &= -\frac{1}{k!}\sum_{h=1}^{k-1}\sum_{\sigma\in S_k}[A^0_{h,1}(B_{l_{\sigma(1)}}(x),\ldots,B_{l_{\sigma(h)}}(x),y),A^0_{k-h,1}(B_{l_{\sigma(h+1)}}(x),\ldots,B_{l_{\sigma(k)}}(x),z)],\quad k\geq 2,\\
Q(k,s) &= -\frac{1}{k!}\sum_{r=0}^{n+1-s}\sum_{p=1}^{k-1}\sum_{\sigma\in S_k}[A^0_{p,1}(B_{l_{\sigma(1)}}(x),\ldots,B_{l_{\sigma(p)}}(x),A^0_{r,1}(x,y)),\\
&\quad A^0_{k-p,1}(B_{l_{\sigma(p+1)}}(x),\ldots,B_{l_{\sigma(k)}}(x),A^0_{n+1-s-r,1}(x,z))],\\
&\quad -\sum_{r=1}^{n+1-s}[A^0_{r,1}(x,y),A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-s-r,1}(x,z))]\\
&\quad -\sum_{r=0}^{n-s}[A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{r,1}(x,y)),A^0_{n+1-s-r,1}(x,z)],\\
&\quad k\geq 2, s\leq n.
\end{aligned}$$

Proof of Theorem 4.2.1.

Proof. We prove the formula by induction on n. For $n=2$, the formula has been established in 4.8.

Suppose that there exists a family $(B_2,\ldots,B_n)$ where B_k is an invariant symmetric k-from on $\mathfrak{h}$ with values in $\mathfrak{h}$ such that for any $2\leq r\leq n$,

$$A_{r,1}(x,y) = A^0_{r,1}(x,y)+\sum_{\substack{1\leq k\leq\left[\frac{r}{2}\right]\\ l_1+\ldots+l_k=s\leq r}} A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{r-s,1}(x,y)). \tag{4.10}$$

We look for $B_{n+1}:\mathfrak{h}\times\ldots\times\mathfrak{h}\longrightarrow\mathfrak{h}$ symmetric and invariant such that

$$\begin{aligned}
A_{n+1,1}(x,y) &= A^0_{n+1,1}(x,y)+\sum_{\substack{1\leq k\leq\left[\frac{n+1}{2}\right]\\ l_1+\ldots+l_k=s\leq n+1}} A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-s,1}(x,y))\\
&= [B_{n+1}(x),y]+A^0_{n+1,1}(x,y)+\sum_{\substack{1\leq k\leq\left[\frac{n+1}{2}\right]\\ l_1+\ldots+l_k=s\leq n+1\\ l_1,\ldots,l_k\leq n}} A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-s,1}(x,y))\\
&= [B_{n+1}(x),y]+R(x)(y),
\end{aligned}$$

where $R(x)$ depends only on $(B_2,\ldots,B_n)$.

The idea of the proof is to show that, for any $x\in\mathfrak{h}$, $\delta(D(x))=0$ where $D(x):\mathfrak{h}\longrightarrow\mathfrak{h}$ is given by $D(x)(y)=A_{n+1,1}(x,y)-R(x)(y)$. Then since $H^0(\mathfrak{h})=H^1(\mathfrak{h})=0$ there exists a unique B_{n+1} satisfying $D(x)(y)=[B_{n+1}(x),y]$. By using the fact that $D(x)$ is the polar form of a symmetric form and $H^0(\mathfrak{h})=0$ one can see that $B_{n+1}(x)$ is the polar form of a symmetric form which is also invariant.

Let us compute now $\delta(D(x))$. According to 4.6, we have

$$\delta(i_x\ldots i_xA_{n+1,1})(y,z) = -\sum_{r=1}^{n}[A_{r,1}(x,\ldots,x,y),A_{n+1-r,1}(x,\ldots,x,z)].$$

By expanding this relation using our induction hypothesis given in 4.10, we get that

$$\delta(i_x\ldots i_xA_{n+1,1})(y,z)=\delta(i_x\ldots i_xA^0_{n+1,1})(y,z)+S+T+U,$$

where

$$
\begin{aligned}
S &= -\sum_{r=1}^{n-1} \sum_{\substack{1 \leq k \leq \left[\frac{n+1-r}{2}\right] \\ s=l_1+\ldots+l_k \leq n+1-r}} [A^0_{r,1}(x,\ldots,x,y), A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-r-s,1}(x,z))],\\
T &= -\sum_{r=2}^{n} \sum_{\substack{1 \leq k \leq \left[\frac{r}{2}\right] \\ s=l_1+\ldots+l_k \leq r}} [A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{r-s,1}(x,y)), A^0_{n+1-r,1}(x,\ldots,x,z)],\\
U &= -\sum_{r=2}^{n-1} \sum_{\substack{1 \leq k \leq \left[\frac{r}{2}\right] \\ s_1=l_1+\ldots+l_k \leq r}} \sum_{\substack{1 \leq h \leq \left[\frac{n+1-r}{2}\right] \\ s_2=p_1+\ldots+p_h \leq n+1-r}}\\
&\quad [A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{r-s_1,1}(x,y)), A^0_{h,1}(B_{p_1}(x),\ldots,B_{p_h}(x),A^0_{n+1-r-s_2,1}(x,z))].
\end{aligned}
$$

On the other hand, if we denote

$$D_{k,s}(x)(y) = A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-s,1}(x,y)),$$

we remark that the computation of $\delta(R(x))$ is based on Proposition 4.2.2 where we have computed the $\delta(D_{k,s}(x))$.

To conclude, we need to show that

$$S+T+U = \sum_{\substack{1 \leq k \leq \left[\frac{n+1}{2}\right] \\ l_1+\ldots+l_k = s \leq n+1 \\ l_1,\ldots,l_k \leq n}} Q(k,s),$$

where $Q(k,s)$ is given in Proposition 4.2.2. Let S_1 and T_1 be the terms in S and T corresponding to $k=1$. We have

$$
\begin{aligned}
S_1 &= -\sum_{r=1}^{n-1} \sum_{2 \leq s \leq n+1-r} [A^0_{r,1}(x,\ldots,x,y), A^0_{1,1}(B_s(x),A^0_{n+1-r-s,1}(x,z))],\\
T_1 &= -\sum_{r=2}^{n} \sum_{2 \leq s \leq r} [A^0_{1,1}(B_s(x),A^0_{r-s,1}(x,y)), A^0_{n+1-r,1}(x,\ldots,x,z)].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sum_{2\leq s\leq n} Q(1,s) &= -\sum_{2\leq s\leq n}\sum_{r=0}^{n-s}[A^0_{1,1}(B_s(x),A^0_{r,1}(x,y)), A^0_{n+1-s-r,1}(x,z)]\\
&\quad -\sum_{2\leq s\leq n}\sum_{r=1}^{n+1-s}[A^0_{r,1}(x,y), A^0_{1,1}(B_s(x),A^0_{n+1-s-r,1}(x,z))].
\end{aligned}
$$

Since

$$
\begin{aligned}
\{(r,s), 1\leq r\leq n-1, 2\leq s\leq n+1-r\} &= \{(r,s), 1\leq r\leq n+1-s, 2\leq s\leq n\},\\
\{(r-s,s), 2\leq r\leq n, 2\leq s\leq r\} &= \{(r,s), 0\leq r\leq n-s, 2\leq s\leq n\},
\end{aligned}
$$

$S_1+T_1=\sum_{2\leq s\leq n}Q(1,s)$. In the same way, one can see easily that

$$\begin{aligned} S-S_1+T-T_1 &= -\sum_{\substack{2\leq k\leq\left[\frac{n+1}{2}\right]\\ l_1+\ldots+l_k=s\leq n+1}} \sum_{r=1}^{n+1-s}[A^0_{r,1}(x,y),A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{n+1-s-r,1}(x,z))] \\ &\quad -\sum_{r=0}^{n-s}[A^0_{k,1}(B_{l_1}(x),\ldots,B_{l_k}(x),A^0_{r,1}(x,y)),A^0_{n+1-s-r,1}(x,z)]. \end{aligned}$$

To conclude, we must show that $Q_1=Q_2$ where

$$\begin{aligned} Q_1 &= -\sum_{\substack{2\leq k\leq\left[\frac{n+1}{2}\right]\\ l_1+\ldots+l_k=s\leq n+1}}\frac{1}{k!}\sum_{r=0}^{n+1-s}\sum_{p=1}^{k-1}\sum_{\sigma\in S_k} \\ &\quad [A^0_{p,1}(B_{l_{\sigma(1)}}(x),\ldots,B_{l_{\sigma(p)}}(x),A^0_{r,1}(x,y)),A^0_{k-p,1}(B_{l_{\sigma(p+1)}}(x),\ldots,B_{l_{\sigma(k)}}(x),A^0_{n+1-s-r,1}(x,z))]. \end{aligned}$$

$$\begin{aligned} Q_2 &= -\sum_{r=2}^{n-1}\sum_{\substack{1\leq p\leq\left[\frac{r}{2}\right]\\ s_1=l_1+\ldots+l_p\leq r}}\sum_{\substack{1\leq h\leq\left[\frac{n+1-r}{2}\right]\\ s_2=m_1+\ldots+m_h\leq n+1-r}} \\ &\quad [A^0_{p,1}(B_{l_1}(x),\ldots,B_{l_p}(x),A^0_{r-s_1,1}(x,y)),A^0_{h,1}(B_{m_1}(x),\ldots,B_{m_h}(x),A^0_{n+1-r-s_2,1}(x,z))]. \end{aligned}$$

Denote by $\mathbb{N}_2=\{2,3,\ldots\}$, for any $l=(l_1,\ldots,l_k)\in\mathbb{N}^k$, $|l|=l_1+\ldots+l_k$, for any $\sigma\in S_k$, $l^\sigma=(l_{\sigma(1)},\ldots,l_{\sigma(k)})$ and for $k\in\left\{1,\ldots,\left[\frac{n+1}{2}\right]\right\}$ and $s\leq n+1$ put

$$\mathcal{S}(k,s)=\left\{(l,\sigma,r,p)\in\mathbb{N}_2^k\times S_k\times\mathbb{N}\times\mathbb{N},|l|=s,0\leq r\leq n+1-s,1\leq p\leq k-1\right\}$$

and for any $(l,\sigma,r,p)\in\mathcal{S}(k,s)$ put

$$\Phi(l,\sigma,r,p)=[A^0_{p,1}(B_{l_{\sigma(1)}}(x),\ldots,B_{l_{\sigma(p)}}(x),A^0_{r,1}(x,y)),A^0_{k-p,1}(B_{l_{\sigma(p+1)}}(x),\ldots,B_{l_{\sigma(k)}}(x),A^0_{n+1-s-r,1}(x,z))].$$

Thus

$$Q_1=-\sum_{2\leq k\leq\left[\frac{n+1}{2}\right],s\leq n+1}\left[\frac{1}{k!}\sum_{(l,\sigma,r,p)\in\mathcal{S}(k,s)}\Phi(l,\sigma,r,p)\right].$$

The map $S_k\times\mathcal{S}(k,s)\longrightarrow\mathcal{S}(k,s)$, $(\mu,(l,\sigma,r,p))\mapsto(l^\mu,\sigma\circ\mu^{-1},r,p)$ defines a free action of S_k and the map $[l,\sigma,r,p]\mapsto(l^\sigma,r,p)$ identifies the quotient $\widetilde{\mathcal{S}(k,s)}$ to

$$\widetilde{\mathcal{S}(k,s)}=\left\{(l,r,p)\in\mathbb{N}_2^k\times\mathbb{N}\times\mathbb{N},|l|=s,0\leq r\leq n+1-s,1\leq p\leq k-1\right\}.$$

Moreover, $\Phi(l,\sigma,r,p)=\Phi(l^\mu,\sigma\circ\mu^{-1},r,p)$ so

$$Q_1=-\sum_{2\leq k\leq\left[\frac{n+1}{2}\right],s\leq n+1}\left[\sum_{(l,r,p)\in\widetilde{\mathcal{S}(k,s)}}\Phi(l,\mathrm{Id},r,p)\right].$$

On the other hand, put

$$\begin{aligned} T &= \Big\{(l,m,p,q,r)\in\mathbb{N}_2^p\times\mathbb{N}_2^q\times\mathbb{N}\times\mathbb{N}\times\mathbb{N},2\leq r\leq n-1,|l|\leq r, \\ &\qquad |m|\leq n+1-r,1\leq p\leq\left[\frac{r}{2}\right],1\leq q\leq\left[\frac{n+1-r}{2}\right]\Big\}. \end{aligned}$$

We have
$$Q_2 = \sum_{(l,m,p,q,r)\in T} \Psi(l,m,p,q,r),$$
where
$$\Psi(l,m,p,q,r) = [A^0_{p,1}(B_{l_1}(x),\ldots,B_{l_p}(x),A^0_{r-|l|,1}(x,y)),A^0_{h,1}(B_{m_1}(x),\ldots,B_{m_h}(x),A^0_{n+1-r-|m|,1}(x,z))].$$
We consider now
$$J : T \longrightarrow \bigcup_{2\leq k\leq\left[\frac{n+1}{2}\right], s\leq n+1} \widetilde{\mathcal{S}(k,s)}$$
given by
$$J(l,m,p,q,r) = ((l,m),r-|l|,p) \in \mathcal{S}(p+\widetilde{q,|l|}+|m|).$$
Indeed, it is obvious that $2 \leq p+q \leq \left[\frac{n+1}{2}\right]$, $1 \leq p \leq p+q-1$. Moreover, since $|l| \leq r$ and $|m| \leq n+1-r$ then $0 \leq r-|l| \leq n+1-(|l|+|m|)$ and hence $((l,m),r-|l|,p) \in \mathcal{S}(p+\widetilde{q,|l|}+|m|)$..
J is a bijection since we have
$$J^{-1}(l,p,r) = ((l_1,\ldots,l_p),(l_{p+1},\ldots,l_k),p,k-p,s=r+l_1+\ldots+l_p) \in T.$$
Indeed, we have
$$2\leq k\leq \left[\frac{n+1}{2}\right],\ 1\leq p\leq k-1 \quad\text{and}\quad 0\leq r\leq n+1-|l|.$$
This implies obviously that $1 \leq p$ and $1 \leq k-p$. Now $2p \leq l_1+\ldots+l_p$ and hence $p \leq \left[\frac{s}{2}\right]$. This with $k \leq \left[\frac{n+1}{2}\right]$ imply that $k-p \leq \left[\frac{n+1-s}{2}\right]$. It is obvious that $s \geq 2$ and from $0 \leq r \leq n+1-|l|$, we get
$$s \leq n+1-(l_{p+1}+\ldots l_k) \leq n-1 \quad\text{and}\quad l_{p+1}+\ldots+l_k \leq n+1-s.$$
This completes the proof. □

4.3 Analytic linear Lie rack structures on $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$

We denote by $\mathfrak{sl}_2(\mathbb{R})$ the Lie algebra of traceless real 2×2-matrices and by $\mathfrak{so}(3)$ the Lie algebra of skew-symmetric real 3×3-matrices. We consider them as left Leibniz algebras and the purpose of this section is to prove that they are rigid in the sense of Definition 19. Namely, we will prove the following theorem.

Theorem 4.3.1. *Let $\mathfrak{h}$ be either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3)$ and $\rhd$ an analytic linear Lie rack structure on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd$ is the Lie algebra bracket of $\mathfrak{h}$. Then there exists an analytic function $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by*
$$F(u) = 1+\sum_{k=1}^{\infty} a_k u^k$$
such that, for any $x,y \in \mathfrak{h}$,
$$x \rhd y = \exp(F(\langle x,x\rangle)\mathrm{ad}_x)(y),$$
where $\langle x,x\rangle = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x\circ\mathrm{ad}_x)$. So $\mathfrak{h}$ is rigid.

The proof of this theorem is based on Theorem 4.2.1. So the first step is the determination of symmetric invariant multilinear maps on $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$. To achieve that, we use the Chevalley restriction theorem for vector-valued functions proved in [33]. We recall its statement as explained in [5].

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, G the connected and simply connected Lie group of $\mathfrak{g}$ and H the maximal torus in G generated by $\exp_G(\mathfrak{h})$. We denote by $N_G(H)$ the normalizer of H in G. Note that for any $a \in N_G(H)$, Ad_a leaves $\mathfrak{h}$ invariant and $W = \{\mathrm{Ad}_a|\mathfrak{h}, a \in N_G(H)\}$ is the Weyl group of $\mathfrak{h}$. Let $B : \mathfrak{g}\times\ldots\times\mathfrak{g} \longrightarrow \mathfrak{g}$ be a symmetric

n-multilinear map which is $\mathfrak{g}$-invariant, i.e., for any $a \in G$ and any $x_1, \ldots, x_n \in \mathfrak{g}$,

$$B(\mathrm{Ad}_a x_1, \ldots, \mathrm{Ad}_a x_n) = \mathrm{Ad}_a B(x_1, \ldots, x_n). \tag{4.11}$$

This is equivalent to

$$[y, B(x_1, \ldots, x_n)] = \sum_{i=1}^{n} B(x_1, \ldots, [y, x_i], \ldots, x_n), \quad y, x_1, \ldots, x_n \in \mathfrak{g}. \tag{4.12}$$

We denote by $S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ the vector space of $\mathfrak{g}$-invariant n-multilinear symmetric forms on $\mathfrak{g}$ with values in $\mathfrak{g}$.

Let $B \in S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ and we denote by $\widetilde{B}$ its restriction to $\mathfrak{h}$. From 4.12, we get that for any $y, x_1, \ldots, x_n \in \mathfrak{h}$,

$$[y, \widetilde{B}(x_1, \ldots, x_n)] = 0$$

and hence $\widetilde{B}(x_1, \ldots, x_n) \in \mathfrak{h}$ (since $\mathfrak{h}$ is a maximal abelian subalgebra). So $\widetilde{B}$ defines a n-multilinear symmetric map $\widetilde{B} : \mathfrak{h} \times \ldots \times \mathfrak{h} \longrightarrow \mathfrak{h}$ which is W-invariant. If we denote by $S_n^W(\mathfrak{h}, \mathfrak{h})$ the vector space of G-invariant n-multilinear symmetric forms on $\mathfrak{h}$ with values in $\mathfrak{h}$, we get a map $\mathrm{Res} : S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \longrightarrow S_n^W(\mathfrak{h}, \mathfrak{h})$.

Theorem 4.3.2 ([33])**.** Res *is injective.*

Let $\mathfrak{g}$ be a real semi-simple Lie algebra. The definition of $S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ is similar to the complex case. The complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of $\mathfrak{g}$ is also semi-simple and we have an injective map $S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \longrightarrow S_n^{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$, which assigns to each $\mathfrak{g}$-invariant n-multilinear invariant form B on $\mathfrak{g}$ the unique $\mathbb{C}$-multilinear map $B^{\mathbb{C}}$ from $\mathfrak{g}^{\mathbb{C}} \times \ldots \times \mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{g}^{\mathbb{C}}$ whose restriction to $\mathfrak{g}$ is B. By using 4.12 one can see easily that since B is $\mathfrak{g}$-invariant then $B^{\mathbb{C}}$ is $\mathfrak{g}^{\mathbb{C}}$-invariant.

We will now apply Theorem 4.3.2 and the embedding above to compute $S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ for any $n \in \mathbb{N}^*$ when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$.

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$. For any $n \in \mathbb{N}^*$, we define $P : \mathfrak{g}^{2n} \longrightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) by

$$P_n(x_1, \ldots, x_{2n}) = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} \langle x_{\sigma(1)}, x_{\sigma(2)} \rangle \ldots \langle x_{\sigma(2n-1)}, x_{\sigma(2n)} \rangle \quad \text{and} \quad P_0 = 1,$$

where $\langle x, x \rangle = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x^2)$. This defines a symmetric invariant form on $\mathfrak{g}$ and the map $B_n^{\mathfrak{g}} : \mathfrak{g}^{2n+1} \longrightarrow \mathfrak{g}$ given by

$$B_n^{\mathfrak{g}}(x_1, \ldots, x_{2n+1}) = \sum_{k=1}^{2n+1} P_n(x_1, \ldots, \hat{x}_k, \ldots, x_{2n+1}) x_k$$

is symmetric and invariant.

Theorem 4.3.3. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then, for any $n \in \mathbb{N}^*$, we have*

$$S_{2n}^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) = 0 \quad \text{and} \quad S_{2n+1}^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) = \mathbb{C} B_n^{\mathfrak{g}}.$$

Proof. A Cartan subalgebra of $\mathfrak{g}$ is $\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is one dimensional and hence, for any $n \in \mathbb{N}^*$, the dimension of $S_n^W(\mathfrak{h}, \mathfrak{h})$ is less than or equal to ≤ 1. By virtue of Theorem 4.3.2 we get $\dim S_n^{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \leq 1$. Moreover, the associated Lie group to $\mathfrak{h}$ is $H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, z \in \mathbb{C}^* \right\}$ and one can see easily that $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_G(H)$ and hence $Ad_a|\mathfrak{h} \in W$. Now

$$Ad_a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a^{-1} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus for any $B \in S_n^W(\mathfrak{h}, \mathfrak{h})$, the invariance by Ad_a implies that, for any $x_1, \ldots, x_n \in \mathfrak{h}$,

$$(-1)^n B(x_1, \ldots, x_n) = -B(x_1, \ldots, x_n).$$

So if n is even, then $B = 0$ and hence $S_n^{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) = 0$. If $n = 2p+1$ is odd, the restriction theorem shows that $\dim S_{2p+1}^{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) \leq 1$ and since $B_p^{\mathfrak{g}} \in S_{2p+1}^{\mathfrak{g}}(\mathfrak{g},\mathfrak{g})$ we get the result. □

If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$ then $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and since the invariants of $\mathfrak{g}$ are embedded in the invariants of $\mathfrak{g}^{\mathbb{C}}$ we get the following corollary.

Corollary 4.3.4. *If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(3)$ then, for any $n \in \mathbb{N}^*$, we have*

$$S_{2n}^{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) = 0 \quad \textit{and} \quad S_{2n+1}^{\mathfrak{g}}(\mathfrak{g},\mathfrak{g}) = \mathbb{R}B_n^{\mathfrak{g}}.$$

Let us pursue our preparation of the proof of Theorem 4.3.1. Let $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3,\mathbb{R})$ and $x \in \mathfrak{h}$. Then

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \text{or} \quad x = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

Put

$$\langle x,x\rangle = \begin{cases} \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x^2) = 2\mathrm{tr}(x^2) = 4(a^2+bc) & \text{if} \quad \mathfrak{h} = \mathfrak{sl}_2(\mathbb{R}), \\ \frac{1}{2}\mathrm{tr}(\mathrm{ad}_x^2) = \frac{1}{2}\mathrm{tr}(x^2) = -a^2-b^2-c^2 & \text{if} \quad \mathfrak{h} = \mathfrak{so}(3). \end{cases}$$

The following formula which is true in both $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$ is easy to check and will play a crucial role in the proof of Theorem 4.3.1. Indeed, for any $x, y \in \mathfrak{h}$,

$$\mathrm{ad}_x \circ \mathrm{ad}_x(z) = -\langle x,z\rangle x + \langle x,x\rangle z. \tag{4.13}$$

This implies easily, by virtue of Corollary 4.1.2, that

$$\begin{cases} A_{2n,1}^0(x,y) = \dfrac{\langle x,x\rangle^{n-1}}{(2n)!}\mathrm{ad}_x^2(y) = \dfrac{\langle x,x\rangle^{n}}{(2n)!}y - \dfrac{\langle x,x\rangle^{n-1}\langle x,y\rangle}{(2n)!}x, \ n \geq 1, \\ A_{2n+1,1}^0(x,y) = \dfrac{\langle x,x\rangle^{n}}{(2n+1)!}[x,y], \quad n \geq 0. \end{cases} \tag{4.14}$$

Proposition 4.3.5. *Let $\mathfrak{h}$ be either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{so}(3)$ and $\rhd$ an analytic linear Lie rack product on $\mathfrak{h}$ such that $[\ ,\ ]_\rhd$ is the Lie algebra bracket of $\mathfrak{h}$. Then there exists a sequence $(U_n)_{n\in\mathbb{N}^*}$ with $U_1 = 1$, $U_2 = \frac{1}{2}$, for any $x, y \in \mathfrak{h}$,*

$$x \rhd y = y + \left(\sum_{n=0}^{\infty} U_{2n+1}\langle x,x\rangle^n\right)[x,y] + \left(\sum_{n=1}^{\infty} U_{2n}\langle x,x\rangle^{n-1}\right)\mathrm{ad}_x^2(y)$$

and for any $n \in \mathbb{N}^*$,

$$U_{2n} = \frac{1}{2}\left[\sum_{r=0}^{n-1} U_{2r+1}U_{2(n-r)-1} - \sum_{r=1}^{n-1} U_{2r}U_{2(n-r)}\right].$$

Proof. By virtue of Theorem 4.1.1, $x \rhd y = \sum_{n=0}^{\infty} A_{n,1}(x,y)$ where the sequence $(A_{n,1})$ satisfies (4.3). Moreover, since $\mathfrak{h}$ is simple $H^0(\mathfrak{h}) = H^1(\mathfrak{h}) = 0$ and we can apply Theorem 4.2.1. Thus

$$A_{n,1}(x,y) = A_{n,1}^0(x,y) + \sum_{\substack{2k \leq s = l_1+\ldots+l_k \leq n \\ 1 \leq k \leq [\frac{n}{2}]}} A_{k,1}^0(B_{l_1}(x),\ldots,B_{l_k}(x),A_{n-s,1}^0(x,y)),$$

and the B_l are symmetric invariant. By virtue of Corollary 4.3.4,

$$B_{2l} = 0 \quad \text{and} \quad B_{2l+1}(x) = c_l\langle x,x\rangle^l x.$$

Thus

$$A_{n,1}(x,y) \quad = \quad A_{n,1}^0(x,y) + \sum_{\substack{2k \leq s = 2l_1+\ldots+2l_k+k \leq n \\ 1 \leq k \leq [\frac{n}{2}]}} c_{l_1}\ldots c_{l_k}\langle x,x\rangle^{l_1+\ldots+l_k}A_{k,1}^0(x,A_{n-s,1}^0(x,y)).$$

But by using Corollary 4.1.2, we have $A^0_{k,1}(x, A^0_{n-s,1}(x,y)) = \dfrac{(n+k-s)!}{k!(n-s)!} A^0_{n-2(l_1+\ldots+l_k)}(x,y)$ and hence we can write

$$A_{n,1}(x,y) = \sum_{l=0}^{[\frac{n-1}{2}]} K_{n,l} \langle x,x \rangle^l A^0_{n-2l}(x,y),$$

where $K_{n,l}$ are constant such that $K_{n,0} = 1$. Note that in particular $A_{2,1}(x,y) = A^0_{2,1}(x,y)$. Now by using 4.14, we get

$$\begin{aligned} A_{2n+1,1}(x,y) &= \sum_{l=0}^{n} K_{2n+1,l} \langle x,x \rangle^l A^0_{2(n-l)+1}(x,y) \\ &= \sum_{l=0}^{n} K_{2n+1,l} \langle x,x \rangle^l \frac{1}{(2(n-l)+1)!} \langle x,x \rangle^{n-l} [x,y] \\ &= U_{2n+1} \langle x,x \rangle^n [x,y] = C_{2n+1} A^0_{2n+1,1}(x,y), \end{aligned}$$

where $U_{2n+1} = \dfrac{C_{2n+1}}{(2n+1)!}$ are constant. In the same way, one can show that there exists constants $U_{2n} = \dfrac{C_{2n}}{(2n)!}$ such that

$$A_{2n,1}(x,y) = U_{2n} \langle x,x \rangle^{n-1} \mathrm{ad}_x^2(y) = C_{2n} A^0_{2n,1}(x,y),$$

and get the desired expression of $x \rhd y$.

On the other hand, the equation 4.3 for $q = 1$ and $p = 2n$ holds for both the A_n and the A^0_n so we get

$$\begin{aligned} A_{2n,1}(x,[y,z]) &= C_{2n} A^0_{2n,1}(x,[y,z]) \\ &= C_{2n}[y, A^0_{2n,1}(x,z)] + C_{2n}[A^0_{2n,1}(x,y),z] + C_{2n} \sum_{r=1}^{2n-1} [A^0_{r,1}(x,y), A^0_{2n-r,1}(x,z)] \\ &= C_{2n}[y, A^0_{2n,1}(x,z)] + C_{2n}[A^0_{2n,1}(x,y),z] + \sum_{r=1}^{2n-1} C_r C_{2n-r} [A^0_{r,1}(x,y), A^0_{2n-r,1}(x,z)]. \end{aligned}$$

Thus

$$\sum_{r=1}^{2n-1} (C_{2n} - C_r C_{2n-r}) [A^0_{r,1}(x,y), A^0_{2n-r,1}(x,z)] = 0$$

and hence

$$\begin{aligned} 0 &= \sum_{r=1}^{n-1} (C_{2n} - C_{2r} C_{2(n-r)}) [A^0_{2r,1}(x,y), A^0_{2(n-r),1}(x,z)] \\ &\quad + \sum_{r=0}^{n-1} (C_{2n} - C_{2r+1} C_{2(n-r)-1}) [A^0_{2r+1,1}(x,y), A^0_{2(n-r-1)+1,1}(x,z)]. \end{aligned}$$

By using 4.14 we get

$$\begin{aligned} 0 &= \sum_{r=1}^{n-1} \frac{(C_{2n} - C_{2r} C_{2(n-r)}) \langle x,x \rangle^{n-2}}{(2r)!(2(n-r)!)} [\mathrm{ad}_x^2(y), \mathrm{ad}_x^2(z)] \\ &\quad + \sum_{r=0}^{n-1} \frac{(C_{2n} - C_{2r+1} C_{2(n-r)-1}) \langle x,x \rangle^{n-1}}{(2r+1)!(2(n-r)-1)!} [[x,y],[x,y]]. \end{aligned}$$

One can show easily by using 4.13 that

$$[\mathrm{ad}_x^2(y), \mathrm{ad}_x^2(z)] + \langle x,x \rangle [[x,y],[x,z]] = 0$$

and deduce that

$$\sum_{r=1}^{n-1} \frac{1}{(2r)!(2(n-r)!)}(C_{2n} - C_{2r}C_{2(n-r)}) = \sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!}(C_{2n} - C_{2r+1}C_{2(n-r)-1}).$$

On the other hand

$$0 = (1-1)^{2n} = \sum_{r=0}^{n} \frac{(2n)!}{(2r)!(2(n-r))!} - \sum_{r=0}^{n-1} \frac{(2n)!}{(2r+1)!(2(n-r)-1)!},$$

and finally,

$$\frac{C_{2n}}{(2n)!} = \frac{1}{2}\left[\sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!}C_{2r+1}C_{2(n-r)-1} - \sum_{r=1}^{n-1} \frac{1}{(2r)!(2(n-r)!)}C_{2r}C_{2(n-r)}\right].$$

To complete the proof, it suffices to replace $\frac{C_r}{r!}$ by U_r. □

Proof of Theorem 4.3.1.

Proof. According to Proposition 4.3.5, there exists a sequence $(U_n)_{n\in\mathbb{N}^*}$ with $U_1 = 1$, $U_2 = \frac{1}{2}$, for any $x, y \in \mathfrak{h}$,

$$x \rhd y = y + \left(\sum_{n=0}^{\infty} U_{2n+1}\langle x, x\rangle^n\right)[x, y] + \left(\sum_{n=1}^{\infty} U_{2n}\langle x, x\rangle^{n-1}\right)\mathrm{ad}_x^2(y)$$

and for any $n \in \mathbb{N}^*$,

$$U_{2n} = \frac{1}{2}\left[\sum_{r=0}^{n-1} U_{2r+1}U_{2(n-r)-1} - \sum_{r=1}^{n-1} U_{2r}U_{2(n-r)}\right]. \tag{4.15}$$

We will show that there exists a unique sequence $(a_n)_{n\geq 1}$ such that the function $F(t) = 1 + \sum_{t=1}^{\infty} a_n t^n$ satisfies

$$x \rhd y = \exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y) = y + \sum_{n=0}^{\infty} F(\langle x, x\rangle)^{2n+1}A^0_{2n+1,1}(x, y) + \sum_{n=1}^{\infty} F(\langle x, x\rangle)^{2n}A^0_{2n,1}(x, y).$$

Thus

$$\exp(F(\langle x, x\rangle)\mathrm{ad}_x)(y) \;=\; y + \left(\sum_{n=0}^{\infty} \frac{[F(\langle x, x\rangle)]^{2n+1}\langle x, x\rangle^n}{(2n+1)!}\right)[x, y] + \left(\sum_{n=1}^{\infty} \frac{[F(\langle x, x\rangle)]^{2n}\langle x, x\rangle^{n-1}}{(2n)!}\right)\mathrm{ad}_x^2(y).$$

Put $[F(\langle x, x\rangle)]^n = \sum_{m=0}^{\infty} B_{n,m}\langle x, x\rangle^m$ and compute the coefficients $B_{n,m}$. Indeed,

$$\begin{aligned}[F(\langle x, x\rangle)]^n &= \left(1 + a_1\langle x, x\rangle + a_2\langle x, x\rangle^2 + \ldots + a_m\langle x, x\rangle^m + R\right)^n \\ &= \left(1 + a_1\langle x, x\rangle + a_2\langle x, x\rangle^2 + \ldots + a_m\langle x, x\rangle^m\right)^n + P,\end{aligned}$$

where P contains terms of degree $\geq m+1$. The multinomial theorem gives

$$\left(1 + a_1\langle x, x\rangle + a_2\langle x, x\rangle^2 + \ldots + a_m\langle x, x\rangle^m\right)^n = \sum_{k_0+\ldots+k_m=n} \frac{n!}{k_0!k_1!\ldots k_m!}a_1^{k_1}\ldots a_m^{k_m}\langle x, x\rangle^{k_1+2k_2+\ldots+mk_m}.$$

Thus

$$B_{n,0} = 1 \quad \text{and} \quad B_{n,m} = \sum_{\substack{k_1+2k_2+\ldots+mk_m=m,\\ k_0+k_1+\ldots+k_m=n}} \frac{n!}{k_0!k_1!\ldots k_m!}a_1^{k_1}\ldots a_m^{k_m} \text{ for } m \geq 1.$$

So

$$\sum_{n=0}^{\infty}\frac{F(\langle x,x\rangle)^{2n+1}\langle x,x\rangle^n}{(2n+1)!} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{B_{2n+1,m}\langle x,x\rangle^{m+n}}{(2n+1)!} = \sum_{n=0}^{\infty}\left(\sum_{p=0}^{n}\frac{B_{2p+1,n-p}}{(2p+1)!}\right)\langle x,x\rangle^n,$$

$$\sum_{n=1}^{\infty}\frac{F(\langle x,x\rangle)^{2n}\langle x,x\rangle^{n-1}}{(2n)!} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{B_{2n,m}\langle x,x\rangle^{m+n-1}}{(2n)!} = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{n}\frac{B_{2p,n-p}}{(2p)!}\right)\langle x,x\rangle^{n-1}.$$

For the sake of simplicity and clarity, put

$$V_{n,m}(a_1,\ldots,a_m) = \frac{B_{n,m}}{n!} = \sum_{\substack{k_1+2k_2+\ldots+mk_m=m,\\ k_0+k_1+\ldots+k_m=n}}\frac{a_1^{k_1}\ldots a_m^{k_m}}{k_0!k_1!\ldots k_m!}.$$

To prove the theorem we need to show that there exists a unique sequence $(a_n)_{n\geq 1}$ such that

$$U_{2n+1} = \sum_{p=0}^{n} V_{2p+1,n-p}(a_1,\ldots,a_{n-p}), \quad n\geq 1, \tag{4.16}$$

$$U_{2n} = \sum_{p=1}^{n} V_{2p,n-p}(a_1,\ldots,a_{n-p}), \quad n\geq 1. \tag{4.17}$$

Note first that the relation 4.15 and the fact that $U_2 = \frac{1}{2}$ defines the sequence $(U_{2n})_{n\geq 1}$ entirely in function of the sequence $(U_{2n+1})_{n\geq 0}$. On the other hand, since $V_{1,n}(a_1,\ldots,a_n) = a_n$ and $U_1 = 1$ then

$$U_3 = a_1 + \frac{1}{3!} \quad \text{and} \quad U_{2n+1} = a_n + \sum_{p=1}^{n} V_{2p+1,n-p}(a_1,\ldots,a_{n-p}), \quad n\geq 2.$$

Since the quantity $\sum_{p=1}^{n} V_{2p+1,n-p}(a_1,\ldots,a_{n-p})$ depends only on $(a_1,\ldots,a_{n-1})$, these relations define inductively and uniquely the sequence $(a_n)_{n\geq 1}$ in function of $(U_{2n+1})_{n\geq 0}$. To achieve the proof we need to prove 4.17. We will proceed by induction and we will use the following relation

$$\frac{\partial V_{n,m}}{\partial a_l}(a_1,\ldots,a_m) = V_{n-1,m-l}(a_1,\ldots,a_{m-l}), \quad l=1,\ldots,m. \tag{4.18}$$

Indeed,

$$\begin{aligned}\frac{\partial V_{n,m}}{\partial a_l}(a_1,\ldots,a_m) &= \sum_{\substack{k_1+2k_2+\ldots+mk_m=m,\\ k_0+k_1+\ldots+k_m=n,k_l\geq 1}}\frac{a_1^{k_1}\ldots a_l^{k_l-1}\ldots a_m^{k_m}}{k_0!k_1!\ldots(k_l-1)!\ldots k_m!}\\ &\overset{k_l'=k_l-1}{=} \sum_{\substack{k_1+2k_2+\ldots+lk_l'+\ldots+mk_m=m-l,\\ k_0+k_1+\ldots+k_l'+\ldots+k_m=n-1}}\frac{a_1^{k_1}\ldots a_l^{k_l'}\ldots a_m^{k_m}}{k_0!k_1!\ldots(k_l')!\ldots k_m!}.\end{aligned}$$

To conclude, it suffices to remark that in the relation

$$k_1+2k_2+\ldots+lk_l'+\ldots+mk_m = m-l$$

the left side is a sum of nonnegative numbers and the right side is nonnegative so

$$(m-l+1)k_{m-l+1} = \ldots = mk_m = 0$$

and hence the relation is equivalent to

$$k_1+2k_2+\ldots+(m-l)k_{m-l} = m-l.$$

Now, we are able to prove 4.17. We proceed by induction. For $n = 1$, we have $U_2 = \frac{1}{2}$ and $V_{2,0} = \frac{1}{2}$. Suppose that the relation holds from 1 to $n-1$. By virtue of 4.15, we have

$$U_{2n} = \frac{1}{2}\left[\sum_{r=0}^{n-1} U_{2r+1}U_{2(n-r)-1} - \sum_{r=1}^{n-1} U_{2r}U_{2(n-r)}\right]$$

and all the U_r appearing in this formula are given by 4.16 and 4.17 this implies that U_{2n} is a function of $(a_1, \dots, a_{n-1})$ and we can put $U_{2n} = H(a_1, \dots, a_{n-1})$. We can also put

$$\sum_{p=1}^{n} V_{2p,n-p}(a_1, \dots, a_{n-p}) = G(a_1, \dots, a_{n-1}).$$

To show that U_{2n} satisfies 4.17 is equivalent to showing

$$H(0) = G(0) \quad \text{and} \quad \frac{\partial H}{\partial a_l} = \frac{\partial G}{\partial a_l}, \quad l = 1, \dots n-1.$$

But $V_{n,m}(0) = 0$ if $m \geq 1$ and $V_{n,0}(0) = \frac{1}{n!}$. Hence

$$\begin{aligned}
H(0) &= \frac{1}{2}\left(\sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!} - \sum_{r=1}^{n-1} \frac{1}{(2r)!(2(n-r))!}\right) \\
&= \frac{1}{2}\left(\sum_{r=0}^{n-1} \frac{1}{(2r+1)!(2(n-r)-1)!} - \sum_{r=0}^{n} \frac{1}{(2r)!(2(n-r))!}\right) + \frac{1}{(2n)!} \\
&= -\frac{1}{2}(1-1)^{2n} + \frac{1}{(2n)!} = \frac{1}{(2n)!}, \\
G(0) &= V_{2n,0}(0) = \frac{1}{(2n)!} = H(0).
\end{aligned}$$

For $r = 0, \dots, n-1$, by induction hypothesis U_{2r+1} is given by 4.16 and by using 4.18 one can see easily that $\frac{\partial U_{2r+1}}{\partial a_l} = U_{2(r-l)}$ if $l = 1, \dots, r$ and 0 if $l \geq r+1$. Similarly, we have $\frac{\partial U_{2r}}{\partial a_l} = U_{2(r-l)-1}$ if $l = 1, \dots, r-1$ and 0 if $l \geq r$. For the sake of simplicity, we put

$$\frac{\partial U_{2r+1}}{\partial a_l} = U_{2(r-l)} \quad \text{and} \quad \frac{\partial U_{2r}}{\partial a_l} = U_{2(r-l)-1}$$

with the convention $U_0 = 1$ and $U_s = 0$ if s is negative. Then, for $l = 1, \dots, n-1$, we have

$$\begin{aligned}
\frac{\partial H}{\partial a_l} &= \frac{1}{2}\left[\sum_{r=0}^{n-1}\left(\frac{\partial U_{2r+1}}{\partial a_l}U_{2(n-r)-1} + \frac{\partial U_{2(n-r)-1}}{\partial a_l}U_{2r+1}\right) - \sum_{r=1}^{n-1}\left(\frac{\partial U_{2r}}{\partial a_l}U_{2(n-r)} + \frac{\partial U_{2(n-r)}}{\partial a_l}U_{2r}\right)\right] \\
&= \frac{1}{2}\left[\sum_{r=0}^{n-1}\left(U_{2(r-l)}U_{2(n-r)-1} + U_{2(n-r-l-1)}U_{2r+1}\right) - \sum_{r=1}^{n-1}\left(U_{2(r-l)-1}U_{2(n-r)} + U_{2(n-r-l)-1}U_{2r}\right)\right] \\
&= \frac{1}{2}\sum_{r=0}^{n-1-l} U_{2r}U_{2(n-r-l)-1} + \frac{1}{2}\sum_{r=0}^{n-1} U_{2(n-r-l-1)}U_{2r+1} - \frac{1}{2}\sum_{r=0}^{n-l-2} U_{2r+1}U_{2(n-r-l-1)} - \frac{1}{2}\sum_{r=1}^{n-1} U_{2(n-r-l)-1}U_{2r} \\
&= \frac{1}{2}U_{2(n-l)-1} + \frac{1}{2}\sum_{r=n-l-1}^{n-1} U_{2(n-r-l-1)}U_{2r+1} - \frac{1}{2}\sum_{r=n-l}^{n-1} U_{2(n-r-l)-1}U_{2r} \\
&= U_{2(n-l)-1}.
\end{aligned}$$

This completes the proof. $\square$

Appendices

Appendix A

Lie Theory

A.1 Structure Theory of Lie Algebras

In this section, Lie algebras are considered from a purely algebraical point of view, without reference to Lie groups or differential geometry.

A.1.1 Definitions and examples

Definition 20. A Lie algebra over a field $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ with a $\mathbb{K}$-bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is skew-symmetric and satisfies Jacobi identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{A.1}$$

for $x, y, z \in \mathfrak{g}$.

This map is called a **Lie bracket** on $\mathfrak{g}$. If $\mathfrak{g}$ is a finite-dimensional real or complex Lie algebra and $e_1, \ldots, e_n$ is a basis for $\mathfrak{g}$ as a vector space. By bilinearity the $[\ ,\ ]$-operation in $\mathfrak{g}$ is completely determined once the values $[e_i, e_j]$ are known. We know then by writing them as a linear combinations of e_i, that is, the Lie bracket of basis elements is uniquely given by

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k, \quad \forall\, i, j = 1, \ldots, n.$$

The constants c_{ij}^k are called the **structure constants** of $\mathfrak{g}$ with respect to the chosen basis. From the skew-symmetry of the Lie bracket, and the Jacobi identity, the structure constants satisfy the following two conditions:

$$\begin{aligned} c_{ij}^k + c_{ji}^k &= 0, \\ \sum_m c_{ij}^m c_{mk}^l + c_{jk}^m c_{mi}^l + c_{ki}^m c_{mj}^l &= 0 \end{aligned}$$

for all i, j, k, l.

Example 7. A vector space $\mathfrak{A}$ with a bilinear map $. : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$ is called an *associative* algebra if

$$a.(b.c) = (a.b).c \quad \text{for} \quad a, b, c \in \mathfrak{A}. \tag{A.2}$$

The commutator

$$[a,b] := a.b - b.a$$

defines a Lie bracket on $\mathfrak{A}$. The Lie algebra $(\mathfrak{A}, [\ ,\ ])$ is noted $\mathfrak{A}_L$.

Examples 2. (i) Let V be a vector space and $\mathrm{End}(V)$ be the associative algebra of linear endomorphisms of V. We write $\mathfrak{gl}(V) := \mathrm{End}(V)_L$ for the corresponding Lie algebra.

(ii) The space $M_n(\mathbb{K})$ of $(n \times n)$-matrix with entries in $\mathbb{K}$ is an associative algebra with respect to matrix multiplication. The corresponding Lie algebra $M_n(\mathbb{K})_L$ is noted $\mathfrak{gl}_n(\mathbb{K})$.

Definition 21. (a) A homomorphism of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a $\mathbb{K}$-linear map $\alpha : \mathfrak{g} \longrightarrow \mathfrak{h}$ which preserves the Lie brackets,i.e.

$$\alpha([x,y]) = [\alpha(x), \alpha(y)] \quad \text{for} \quad x, y \in \mathfrak{g}.$$

If, in addition, α is bijective, then α is called a *Lie algebra isomorphism*. A Lie algebra isomorphism of a Lie algebra with itself is called a *Lie algebra automorphism*.

A representation of a Lie algebra $\mathfrak{g}$ on the vector space V is a homomorphism of Lie algebras $\alpha : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. We write (α, V) for a representation of $\mathfrak{g}$ on V.

(b) Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a *Lie subalgebra* if it is closed under bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If we even have $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, we call $\mathfrak{h}$ an *ideal* of $\mathfrak{g}$.

(c) The Lie algebra $\mathfrak{g}$ is called abelian if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, which means that all brackets vanish.

Remark 8. It is easy to see that the image of a homomorphism $\alpha : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ of Lie algebras is a subalgebra of $\mathfrak{g}_2$. Furthermore, if $\mathfrak{h}$ is an ideal of $\mathfrak{g}_2$, then $\alpha^{-1}(\mathfrak{h})$ is an ideal, and $\alpha^{-1}(\mathfrak{h})$ is a subalgebra if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}_2$. In particular, the *kernel* $\ker\alpha$ of a Lie algebra homomorphism is always an ideal.

Examples 3. 1. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is defined by

$$\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x,y] = 0 \; \forall y \in \mathfrak{g}\}.$$

Obviously, $\mathfrak{z}(\mathfrak{g})$ is an ideal in $\mathfrak{g}$.

2. For each Lie algebra $\mathfrak{g}$, the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$.

3. The *orthogonal* Lie algebra

$$\mathfrak{o}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x^T = -x\}$$

is a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

4. The *unitary* Lie algebra

$$\mathfrak{u}_n(\mathbb{C}) := \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x^* = -x\}$$

is a real subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

5. The *special linear* Lie algebra

$$\mathfrak{sl}_n(\mathbb{K}) := \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid tr(x) = 0\}$$

is an ideal in $\mathfrak{gl}_n(\mathbb{K})$, where $tr(x)$ denotes the trace of x.

For more examples see [26, 28]. Ado's theorem stats immediately the following theorem:

Theorem A.1.1. *Every finite-dimensional Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is isomorphic to a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.*

Definition 22. Let $\mathfrak{g}$ be a Lie algebra. A derivation of $\mathfrak{g}$ (the Lie derivative) is a linear operator δ on $\mathfrak{g}$ satisfying the Leibniz law with respect to the bracket:

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)], \quad \forall \, x, y \in \mathfrak{g}. \tag{A.3}$$

The set of all derivations is denoted by $\mathrm{der}(\mathfrak{g})$.

For example, $\mathfrak{sl}_n(\mathbb{K}) = \mathrm{der}(\mathfrak{gl}_n(\mathbb{K}))$.

Definition 23. For any $x \in \mathfrak{g}$, an inner derivation (also referred as adjoint homomorphism, adjoint representation or the adjoint of x) of the Lie algebra $\mathfrak{g}$ is the operator ad_x defined as follows:

$$\mathrm{ad}_x(y) = [x,y], \forall \; y \in \mathfrak{g}.$$

It is clear that the inner derivation holds the following properties:

(a) $\mathrm{ad}_x([y,z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]$.

(b) $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$; that is, $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism.

(c) $\ker(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$.

(d) $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g}) \Longleftrightarrow \mathrm{ad}_x \circ \mathrm{ad}_y = \mathrm{ad}_{[x,y]}$.

For $x, y, z \in \mathfrak{g}$. It is obvious, that $\mathrm{ad}_{\mathfrak{g}}$ is an ideal of $\mathrm{der}(\mathfrak{g})$. In particular,

$$[\delta, \mathrm{ad}_x] = ad_{\delta x} \quad \text{for } \delta \in \mathrm{der}(\mathfrak{g}), x \in \mathfrak{g}.$$

A.1.2 Quotient and Semidirect sum

Proposition A.1.2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ be an ideal in $\mathfrak{g}$. Then the quotient space $\mathfrak{g}/\mathfrak{i} := \{x + \mathfrak{i} : \ x \in \mathfrak{g}\}$ is a Lie algebra with respect to the bracket*

$$[x + \mathfrak{i}, y + \mathfrak{i}] := [x, y] + \mathfrak{i}. \tag{A.4}$$

The quotient map $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$ is a surjective homomorphism of Lie algebras with kernel $\mathfrak{i}$.

Proof. From the definition of an ideal, the bracket A.4 is well defined. Thanks to the properties of the Lie bracket on $\mathfrak{g}$, the bracket A.4 defines a Lie bracket on $\mathfrak{g}/\mathfrak{i}$. □

Theorem A.1.3 ([28])**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras.*

(i) If $\alpha : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a homomorphism, then

$$\alpha(\mathfrak{g}) \cong \mathfrak{g}/ker\alpha.$$

For any ideal $\mathfrak{i}$ in $\mathfrak{g}$ with $\mathfrak{i} \subseteq ker\alpha$, there is exactly one homomorphism $\beta : \mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{h}$ such that $\beta \circ \pi = \alpha$, where $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$ is the quotient map.

(ii) If $\mathfrak{i}$, $\mathfrak{j}$ are two ideals in $\mathfrak{g}$ with $\mathfrak{i} \subseteq \mathfrak{j}$, then $\mathfrak{j}/\mathfrak{i}$ is an ideal in $\mathfrak{g}/\mathfrak{i}$ and $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$.

(iii) If $\mathfrak{i}$, $\mathfrak{j}$ are two ideals, then $\mathfrak{i} + \mathfrak{j}$ and $\mathfrak{i} \cap \mathfrak{j}$ are ideals of $\mathfrak{g}$ and

$$\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$$

Lie algebras can be constructed in many various ways: via linear space or linear maps (endomorphisms) on a vector space, via vector fields, via Lie groups, via a set of structure constants, or via differential operators (see [28, 43] The proposition A.1.2 describes an important construction: for each ideal $\mathfrak{i}$, we obtain a Lie algebra $\mathfrak{g}/\mathfrak{i}$, so the Lie algebra $\mathfrak{g}$ can be decomposed into two Lie algebra $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$. So we may ask how can build a Lie algebra $\mathfrak{g}$ from two Lie algebras $\mathfrak{i}$ and $\mathfrak{h}$ such that $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{i} \cong \mathfrak{h}$. The semidirect sum of Lie algebras give rise to such construction.

Definition 24. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and $\alpha : \mathfrak{h} \longrightarrow \mathrm{der}(\mathfrak{g})$ be a homomorphism. Then the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ in the vector space sense is a Lie algebra with respect to the following bracket

$$[(x_1, x_2), (y_1, y_2)] := (\alpha(x_2)y_1 - \alpha(y_2)x_1 + [x_1, y_1], [x_2, y_2]) \tag{A.5}$$

for $x_1, y_1 \in \mathfrak{g}$, $x_2, y_2 \in \mathfrak{h}$. This Lie algebra is called the *semidirect sum* with respect to α of $\mathfrak{g}$ and $\mathfrak{h}$. It is denoted by $\mathfrak{g} \rtimes_\alpha \mathfrak{h}$. In a particular case, when $\alpha = 0$, then the semidirect sum reduces to the direct sum of Lie algebras, and it is denoted by $\mathfrak{g} \oplus \mathfrak{h}$, where

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2]), \quad \text{for } x_1, y_1 \in \mathfrak{g}, \ x_2, y_2 \in \mathfrak{h}. \tag{A.6}$$

The subspace $\{(x, 0) \in \mathfrak{g} \rtimes_\alpha \mathfrak{h}\}$ is an ideal in $\mathfrak{g} \rtimes_\alpha \mathfrak{h}$ isomorphic to $\mathfrak{g}$ and $\{(0, y) \in \mathfrak{g} \rtimes_\alpha \mathfrak{h}\}$ is a subalgebra of $\mathfrak{g} \rtimes_\alpha \mathfrak{h}$ isomorphic to $\mathfrak{h}$. For a derivation $D \in \mathrm{der}(\mathfrak{g})$, we simply write $\mathfrak{g} \rtimes_D \mathbb{K}$ for the semidirect sum defined by $\alpha(t) := tD$.

Example 8. Let V be a vector space over a field $\mathbb{K}$ with $\dim V = n$. We endow V with the abelian Lie bracket, that is $[u,v] := 0$ for any $u,v \in V$. We consider the Lie algebra $\mathfrak{gl}_n(\mathbb{K})$ together with its induced action on V which is given by

$$\alpha(A)v := Av, \quad \forall\, A \in \mathfrak{gl}_n(\mathbb{K}), v \in V.$$

It is easy to check that for all $A \in \mathfrak{gl}_n(\mathbb{K})$, $u,v \in V$ the Leibniz condition is satisfied, that is

$$A[u,v] = [Au,v] + [u,Av]$$

which can be rewritten as

$$\alpha(A)[u,v] = [\alpha(A)u,v] + [u,\alpha(A)v].$$

It means that the algebra $\mathfrak{gl}_n(\mathbb{K})$ stands for the derivation algebra of the Lie algebra $(V,[\ ,\ ])$. Formally, $\mathfrak{gl}_n(\mathbb{K}) = \mathrm{der}(V)$. In addition, the semidirect sum $V \rtimes_\alpha \mathfrak{gl}_n(\mathbb{K})$ is well defined with respect to the bracket

$$[(u,A),(v,B)] := (Au - Bv, [A,B]) \quad \text{for all } A,\ B \in \mathfrak{gl}_n(\mathbb{K}),\ u,\ v \in V.$$

A.1.3 Complexification and Real form

Definition 25. If V is a finite-dimensional real vector space, then the *complexification* of V, denoted $V^{\mathbb{C}}$, is the space of formal linear combinations

$$v_1 + iv_2$$

with $v_1, v_2 \in V$. This becomes a real vector space in the obvious way and becomes a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1$$

We could regard the complexification $V^{\mathbb{C}}$ as the tensor product $\mathbb{C} \otimes_{\mathbb{R}} V$ which a complex vector space with respect to the scalar multiplication $\lambda.(z \otimes v) := \lambda z \otimes v$. We identify V with the subspace $1 \otimes V$ of $V^{\mathbb{C}}$. Furthermore, if $\{v_1, \ldots, v_n\}$ a real basis of V, then $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a complex basis of $V^{\mathbb{C}}$.

Proposition A.1.4. *Let $\mathfrak{g}$ be a real Lie algebra.*

(i) $\mathfrak{g}^{\mathbb{C}}$ *is a complex Lie algebra with respect to the complex bilinear Lie bracket, defined by*

$$[x_1 + iy_1, x_2 + iy_2] := ([x_1,x_2] - [y_1,y_2]) + i([x_1,y_2] + [y_1,x_2]).$$

(ii) $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}] \cong [\mathfrak{g},\mathfrak{g}]^{\mathbb{C}}$ *as a complex Lie algebra.*

Example 9. The Lie algebras $\mathfrak{so}_3(\mathbb{R}) = \mathfrak{o}_3(\mathbb{R}) \cap \mathfrak{sl}_3(\mathbb{R}))$ and $\mathfrak{su}(2)$ are isomorphic. This latter is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, even though $\mathfrak{so}_3(\mathbb{R})^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{R})^{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$. For more details see [26, 28]

It is important to notice that the complexification of real Lie algebra satisfy the **universal property**: let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ its complexification, and $\mathfrak{h}$ an arbitrary complex Lie algebra. Every real Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{h}$ extends uniquely to a complex Lie algebra homomorphism of $\mathfrak{g}^{\mathbb{C}}$ into $\mathfrak{h}$.

A.1.4 Representations and Modules

In this subsection, we shortly introduce the concept of representations and modules of Lie algebras [26].

Definition 26. A representation (π, V) of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, where V is a vector space. If π is a one-to-one homomorphism, then the representation is called **faithful**.

Definition 27. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module is a vector space V together with a bilinear map

$$\mathfrak{g} \times V \longrightarrow V, \quad (x,v) \longmapsto x.v$$

satisfying

$$[x,y].v = x.(y.v) - y.(x.v) \quad \text{for } x,y \in \mathfrak{g},\ v \in V.$$

Definition 28. (a) Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. A subspace $W \subseteq V$ is called a $\mathfrak{g}$-submodule if $\mathfrak{g}.W \subseteq W$.

(b) A $\mathfrak{g}$-module V is called *simple* if it is nonzero and there are no submodules except $\{0\}$ and V. It is called *semisimple*, if V is the direct sum of simple submodules.

(c) If V and W are $\mathfrak{g}$-modules, then a linear map $\phi : V \longrightarrow W$ is called a homomorphism of $\mathfrak{g}$-modules if for all $x \in \mathfrak{g}$ and all $v \in V$,

$$\phi(x.v) = x.\phi(v).$$

We note $\mathrm{Hom}_{\mathfrak{g}}(V,\ W)$ for the vector space of all $\mathfrak{g}$-module homomorphisms from V to W and note that the set $\mathrm{End}_{\mathfrak{g}}(V) := \mathrm{Hom}_{\mathfrak{g}}(V,\ V)$ of module endomorphisms of V is an associative subalgebra of $\mathrm{End}(V)$. If $\phi \in \mathrm{Hom}_{\mathfrak{g}}(V,\ W)$ is bijective, then the inverse map $\varphi : W \longrightarrow V$ is also a homomorphism of g-modules satisfying $\phi \circ \varphi = id_W$ and $\varphi \circ \phi = \mathrm{id}_V$. Therefore, ϕ is called an isomorphism of $\mathfrak{g}$-modules. The set of isomorphisms $V \longrightarrow V$ is the group $\mathrm{Aut}_{\mathfrak{g}}(V) := \mathrm{End}_{\mathfrak{g}}(V)^{\times}$ of units in the algebra $\mathrm{End}_{\mathfrak{g}}(V)$.

Example 10. We easily show that any Lie algebra $\mathfrak{g}$ carries a natural $\mathfrak{g}$-module structure defined by the adjoint representation $x.y := [x,y]$. From the above definition, the $\mathfrak{g}$-submodules of $\mathfrak{g}$ are precisely the ideals.

Remark 9. It is important to notice that the representations of $\mathfrak{g}$ and $\mathfrak{g}$-modules are equivalent concepts. Indeed, if $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation, then a $\mathfrak{g}$-module structure on V is defined by $x.v = \pi(x)v$. Conversely, for every $\mathfrak{g}$-module V, the map $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ defined by $\pi(x)v = x.v$ is a representation.

A.1.5 Cohomology of Lie algebras

Definition 29. Let V and W be a vector spaces and $n \in \mathbb{N}$. A multilinear map $f : W^n \longrightarrow V$ is called alternating if

$$f(w_{\sigma_1}, \ldots, w_{\sigma_n}) = sgn(\sigma) f(w_1, \ldots, w_n)$$

for $w_i \in W$ and $sgn(\sigma)$ is the sign of the permutation $\sigma \in S_n$.

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. For any non-negative integer n, we set

$$C^n(\mathfrak{g}, V) := \mathrm{Hom}(\wedge^n \mathfrak{g}, V)$$

the space of alternating n-linear mappings $\mathfrak{g}^n \longrightarrow V$ (the n-cochains), in particular we put $C^0(\mathfrak{g}, V) := V$, and we consider the differential operator $d^n : C^n(\mathfrak{g}, V) \longrightarrow C^{n+1}(\mathfrak{g}, V)$ given by

$$\begin{aligned} d^n(\omega)(x_0, \ldots, x_n) \quad = \quad & \sum_{i=0}^{n-1} [x_i, \omega(x_0, \ldots, \hat{x}_i, \ldots, x_n)] \\ & + \sum_{i<j} (-1)^{i+1} \omega(x_0, \ldots, x_{j-1}, [x_i, x_j], x_{j+1}, \ldots, x_n), \end{aligned}$$

where $\hat{x}_j$ means that x_j is omitted. It is proved in ...that $\{C^n(\mathfrak{g}, V), d^n\}_{n \geq 0}$ is a cochain of complex, i.e, $d^{n+1} \circ d^n = 0$.

Definition 30. Let $\mathfrak{g}$ be a Lie algebra and let V be a $\mathfrak{g}$-module.

(i) The nth cohomology space of $\mathfrak{g}$ with coefficients in V is the nth cohomology space of the cochain complex $\{C^n(\mathfrak{g}, V), d^n\}_{n \geq 0}$ defined by

$$H^n(\mathfrak{g}, V) := H(\{C^n(\mathfrak{g}, V), d^n\}_{n \geq 0}) := \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$$

where the subspace

$$Z^n(\mathfrak{g}, V) := \ker(d^n)$$

is the space of the n-cocycles, and the subspace

$$B^n(\mathfrak{g}, V) := \mathrm{Im}(d^{n-1}), \quad B^0(\mathfrak{g}, V) := \{0\}$$

is the space of the n-coboundaries with $B^n(\mathfrak{g}, V) \subseteq Z^n(\mathfrak{g}, V)$ for all $n \geq 0$.

(ii) for each $x \in \mathfrak{g}$ and $n > 0$ the insertion map or contraction

$$i_x : C^n(\mathfrak{g}, V) \longrightarrow C^{n-1}(\mathfrak{g}, V), \quad i_x(\omega)(x_1, \ldots, x_{n-1}) = \omega(x, x_1, \ldots, x_{n-1}).$$

We further define i_x to be 0 on $C^0(\mathfrak{g}, V)$.

For more details concerning cohomology of Lie algebras one can see [28].

A.1.6 Some classes of Lie algebras

In this subsection we define some classes of Lie algebras, namely, nilpotent, solvable and semi-simple Lie algebras.

Nilpotent Lie algebras

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra. Define its *lower central* series recursively by

$$C^0(\mathfrak{g}) := \mathfrak{g} \quad \text{and} \quad C^n(\mathfrak{g}) := [\mathfrak{g}, C^{n-1}(\mathfrak{g})]$$

for any $n \geq 0$. In particular, $C^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is the commutator algebra of $\mathfrak{g}$. By induction, we show immediately the following result.

Proposition A.1.5. *Each $C^n(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, and $C^{n+1}(\mathfrak{g}) \subseteq C^n(\mathfrak{g})$ for any $n \geq 1$.*

Definition 31.
- The Lie algebra $\mathfrak{g}$ is called nilpotent, if $C^d(\mathfrak{g}) = \{0\}$ for some d. If d is minimal with this property, then it is called the *nilpotence degree* of $\mathfrak{g}$.
- The Lie algebra $\mathfrak{g}$ is called abelian, if $C^2(\mathfrak{g}) = \{0\}$.

Example 11. (i) The Heisenberg Lie algebra

$$\mathfrak{h}_3 \simeq \left\{ \begin{pmatrix} 0 & 0 & 0 \\ z & 0 & 0 \\ x & y & 0 \end{pmatrix}, x, y, z, \in \mathbb{R} \right\}$$

is nilpotent.

(ii) Every abelian Lie algebra is nilpotent.

Proposition A.1.6. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) If $\mathfrak{g}$ is nilpotent, then all subalgebras and all homomorphic images of $\mathfrak{g}$ are nilpotent.

(ii) If $\mathfrak{a}$ is a subalgebra of $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}/\mathfrak{a}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.

(iii) If $\mathfrak{g} \neq \{0\}$ is nilpotent, then $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$.

(iv) If $\mathfrak{g}$ is nilpotent, then there is an $n \in \mathbb{N}$ with $\mathrm{ad}(x)^n = 0$ for all $x \in \mathfrak{g}$, i.e., the $\mathrm{ad}(x)$ are nilpotent as linear maps.

(v) If $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, then all the spaces $C^n(\mathfrak{i})$ are ideals of $\mathfrak{g}$.

Theorem A.1.7 (Engel's Characterization Theorem for Nilpotent Lie Algebras)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if for each $x \in \mathfrak{g}$ the operator $\mathrm{ad}(x)$ is nilpotent.*

Solvable Lie algebras

The *derived* series of a Lie algebra $\mathfrak{g}$ is defined inductively by

$$D^0(\mathfrak{g}) := \mathfrak{g} \quad \text{and} \quad D^n := [D^{n-1}(\mathfrak{g}), D^{n-1}(\mathfrak{g})]$$

for $n \in \mathbb{N}$. From $D^1(\mathfrak{g}) \subset \mathfrak{g}$, we inductively conclude that $D^n(\mathfrak{g})D^{n-1}(\mathfrak{g})$.

Proposition A.1.8. *Each $D^n(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.*

Definition 32. The Lie algebra $\mathfrak{g}$ is said to be solvable, if there exists an $n \in \mathbb{N}$ with $D^n(\mathfrak{g}) = \{0\}$.

Examples 4. 1. The oscillator algebra[1] is solvable, but not nilpotent.

2. Every nilpotent Lie algebra is solvable because $D^n(\mathfrak{g}) \subseteq C^{n+1}(\mathfrak{g})$ follows easily by induction. (iii) Consider $\mathbb{R}$ and $\mathbb{C}$ as abelian real Lie algebras and write $I \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ for the multiplication with i. Then the Lie algebra $\mathbb{C} \rtimes_I \mathbb{R}$ is solvable, but not nilpotent. It is isomorphic to $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, where $\mathfrak{g}$ is the oscillator algebra.

Proposition A.1.9. *Let $\mathfrak{g}$ be a Lie algebra.*

(i) If g is solvable, then all subalgebras and homomorphic images of g are solvable.

(ii) If $\mathfrak{i}$ is a solvable ideal of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{i}$ is solvable, then $\mathfrak{g}$ is solvable.

(iii) If $\mathfrak{i}$ and $\mathfrak{j}$ are solvable ideals of $\mathfrak{g}$, then the ideal $\mathfrak{i} + \mathfrak{j}$ is solvable.

(iv) If $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, then the $D^n(\mathfrak{i})$ are ideals in $\mathfrak{g}$.

Definition 33. Every finite-dimensional Lie algebra $\mathfrak{g}$ contains a unique maximal solvable ideal $\mathfrak{rad}(\mathfrak{g})$. This ideal is called the radical of $\mathfrak{g}$.

Lie's theorem:
Let V be a vector space. A tuple $\mathcal{F} = (V_0, ldots, V_n)$ of subspaces with

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V$$

is called a *flag* in V . Then

$$\mathfrak{g}(\mathcal{F}) := \{x \in \mathfrak{gl}(V) : \ (\forall j) x V_j \subseteq V_j\}$$

a Lie subalgebra of $\mathfrak{gl}(V) = \mathrm{End}(V)_L$.

$$\mathfrak{g}_n(\mathcal{F}) := \{x \in \mathfrak{gl}(V) : \ (\forall j > 0) x V_j \subseteq V_{j-1}\}$$

is an ideal of $\mathfrak{g}(\mathcal{F})$. Here the n in $\mathfrak{g}_n(\mathcal{F})$ stands for "nilpotent".

Definition 34. Let V be an n-dimensional vector space. A complete flag in V is a flag $(V_0, \ldots, V_n)$ with $\dim V_k = k$ for each k.

Theorem A.1.10 (Lie's theorem for solvable Lie algebras)**.** *Let V be a finite-dimensional complex vector space and $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a complete $\mathfrak{g}$-invariant[2] flag in V.*

Related to the notion of representation of Lie algebras, we have, as consequence of the above theorem, the following results.

Corollary A.1.11. *Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of the solvable Lie algebra $\mathfrak{g}$. Then the restriction to $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent representation[3].*

Corollary A.1.12. *A Lie algebra $\mathfrak{g}$ is solvable if and only if its commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Cartan's Solvability Criterion:

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[1] The oscillator algebra is given by $\mathfrak{h}_3 \rtimes_D \mathbb{R}$, where D is a dervation of the Heisenberg Lie algebra $\mathfrak{h}_3$ [28]

[2] $\mathfrak{g}(V_i) \subseteq V_i$ for any i

[3] A representation (π, V) is said to be nilpotent if there exists an $n \in \mathbb{N}$ with $\pi(\mathfrak{g})^n = \{0\}$

Theorem A.1.13 (**[herm]**). *Let V be a finite dimensional vector space and $\mathfrak{g}$ a subalgebra of $\mathfrak{gl}(V)$. Then the following are equivalent*

(i) $\mathfrak{g}$ *is solvable.*

(ii) $\mathrm{tr}(xy) = 0$ *for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and* $y \in \mathfrak{g}$.

As a direct consequence, we have the following corollary.

Corollary A.1.14. *Let $\mathfrak{g}$ be a Lie algebra. Then the following statements are equivalent*

(i) $\mathfrak{g}$ *is solvable.*

(ii) $\mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y) = 0$ *for all for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and* $y \in \mathfrak{g}$.

Semisimple Lie algebras

Definition 35. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is called semisimple if its radical is trivial, i.e., $\mathfrak{rad}(\mathfrak{g}) = \{0\}$. The Lie algebra $\mathfrak{g}$ is called simple if it is not abelian and it contains no ideals other than $\mathfrak{g}$ and $\{0\}$.

Notice that this definition is equivalent to say a *semisimple* Lie algebra $\mathfrak{g}$ is a Lie alegbra that contains no nonzero solvable ideals. It implies in particular that the center $\mathfrak{z}(\mathfrak{g}) = 0$.

The condition that a simple Lie algebra $\mathfrak{g}$ should not be abelian is included to rule out one-dimensional Lie algebra: there are many reasons not to include it in the class of simple Lie algebras. One of these reasons is the following proposition.

Proposition A.1.15. *Any simple Lie algebra is semisimple.*

Example 12. The classical Lie algebras such as $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{su}(n)$, $\mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$ are semisimple.

Cartan's semisimplicity Criterion:
Let $\mathfrak{g}$ be a Lie algebra. We define the *Cartan Killing form* as follows:

$$\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}, \quad \kappa_{\mathfrak{g}}(x, y) := \mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y).$$

In connection with the Cartan criterion for solvable Lie algebras, we have seen that the bilinear form on finite-dimensional Lie algebra is of interest. This symmetric form has a compatibility with the Lie algebra structure expressed by its invariance

$$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]), \text{for} \quad x, y, z \in \mathfrak{g}.$$

It is worth pointing out that the Cartan–Killing form of a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ cannot be calculated in terms of the Cartan–Killing form of $\mathfrak{g}$, but for ideals we have:

Lemma A.1.16. *For any ideal $\mathfrak{i}$ of $\mathfrak{g}$, $\kappa_{\mathfrak{i}} := \kappa_{\mathfrak{g}}|_{\mathfrak{i} \times \mathfrak{i}}$.*

The following theorem characterizes semisimplicity in terms of the Cartan–Killing form.

Theorem A.1.17 ([28]). *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\kappa_{\mathfrak{g}}$ is nondegenerate, i.e., $\mathfrak{rad}(\kappa_{\mathfrak{g}}) = \{0\}$*

As a connection between the structure of a Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}^{\mathbb{C}}$, we have: since $\mathfrak{rad}(\kappa_{\mathfrak{g}})^{\mathbb{C}} = \mathfrak{rad}(\kappa_{\mathfrak{g}^{\mathbb{C}}})$. Theorem A.1.17 shows that a real Lie algebra $\mathfrak{g}$ is semisimple if and only if its complexification $\mathfrak{g}^{\mathbb{C}}$ is semisimple.

There is another characterization of semisimple Lie algebras using direct sum of simple ideals and Lie algebras.

Proposition A.1.18. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there are simple ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ of $\mathfrak{g}$ with*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \mathfrak{g}_k.$$

Every ideal $\mathfrak{i}$ of $\mathfrak{g}$ is semisimple and a direct sum $\mathfrak{i} = \oplus_{j \in I} \mathfrak{g}_j$ for some subset $I \in \{1, \ldots, k\}$. Conversely, each direct sum of simple Lie algebras is semisimple.

Levi's decomposition:
For a general Lie algebra $\mathfrak{g}$, which is neither semisimple nor solvable, we can try to **separate** the solvable and semisimple parts. In fact, we have the following stronger result.

Theorem A.1.19 ([35]). *Any Lie algebra can be written as a direct sum*

$$g = rad(g) \oplus \mathfrak{g}_{ss}$$

where $\mathfrak{g}_{ss}$ is a semisimple subalgebra (not an ideal!) in $\mathfrak{g}$. Such a decomposition is called Levi decomposition for $\mathfrak{g}$.

A.2 Structure Theory of Lie groups

We first fix some notations of differential geometry: For a manifold M and a point $m \in M$, we denote by T_mM the tangent space to M at point m, and by TM the tangent bundle to M. A vector fields on M is a smooth map $X : M \longrightarrow TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$ that is a global sections of TM. The vector space of of all vector fields is denoted by $\mathcal{V}(M)$. For a morphism $f : X \longrightarrow Y$ and a point $x \in X$, we denote by $T_xf := f_*(x) : T_xX \longrightarrow T_{f(x)}Y$ the corresponding map of tangent spaces. Recall that a morphism $f : X \longrightarrow Y$ is called an immersion if $\mathrm{rank} f_* = dim(X)$ for every point $x \in X$; in this case, one can choose local coordinates in a neighbourhood of $x \in X$ and in a neighbourhood of $f(x) \in Y$ such that f is given by $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$. An immersed submanifold in a manifold M is a subset $N \subset M$ with a structure of a manifold not necessarily the one inherited from M such that inclusion map $i : N \hookrightarrow M$ is an immersion. Note also that for any point $p \in$ N, the tangent space to N is naturally a subspace of tangent space to M: $T_pN \subset T_p$M. An embedded submanifold $N \subset M$ is an immersed submanifold such that the inclusion map $i : N \hookrightarrow M$ is a homeomorphism. In this case the smooth structure on N is uniquely determined by the smooth structure on N. In this section, we briefly review some of the standard facts on Lie groups [28, 41].

Definition 36. A (real) Lie group G is a smooth manifold together with a smooth multiplication $m_G : G \times G \longrightarrow G$ that makes G into a group and the inverse map $g \longrightarrow g^{-1}$ is a smooth map of G into itself.

It is worth pointing out that this definition has a complex analogue, in which smooth manifolds are replaced by complex analytic manifolds and smooth maps by holomorphic maps (see for instance [43]). The class of groups endowed with a complex analytic manifold structure compatible with the group structure are so-called complex Lie groups. When talking about complex Lie groups, "submanifold" will mean "complex analytic submanifold", tangent spaces will be considered as complex vector spaces, all morphisms between manifolds will be assumed holomorphic, etc.

Throughout this section, most results apply to real and complex Lie groups. Therefore, we frequently omit the words "real and complex" and we simply refer to "real or complex" Lie groups as Lie groups. Unless one wants to stress the difference between real and complex case, it is common to indicate which class will be considered.

Now, let G be a Lie group with neutral element 1 and g be an element of G. We denote respectively $\mathrm{L}_g : G \longrightarrow G$, $x \longmapsto gx$ and $\mathrm{R}_g : G \longrightarrow G$, $x \longmapsto xg$ the left and right multiplication by g which are smooth since the multiplication map is assumed smooth. We further write $c_g : G \longrightarrow G$, $x \longmapsto gxg^{-1}$ for the conjugation map with g. It is, again, a smooth map. Furthermore, all maps L_g, R_g and c_g are bijective with inverse $\mathrm{L}_{g^{-1}}$, $\mathrm{R}_{g^{-1}}$ and $c_{g^{-1}}$ respectively. Thus they are diffeomorphisms of G onto itself.

Examples 5. 1. $(\mathbb{R}^n, +)$ is a Lie group.

2. The usual groups of linear algebra, such as

   - the group $\mathrm{GL}_n(\mathbb{K})$ of invertible $(n \times n)$-matrices,
   - the special linear group: $\mathrm{SL}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K}) \mid det(g) = 1\}$,
   - the orthogonal group: $\mathrm{O}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K}) \mid g^T = g^{-1}\}$,
   - The special orthogonal group: $\mathrm{SO}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{O}_n(\mathbb{K})$,

- the unitary group: $\mathrm{U}_n(\mathbb{K}) := \{g \in \mathrm{GL}_n(\mathbb{K}) \mid g^* = g^{-1}\}$. Note that $\mathrm{U}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R})$, but $\mathrm{U}_n(\mathbb{C}) \neq \mathrm{O}_n(\mathbb{C})$,
- the special unitary group: $\mathrm{SU}_n(\mathbb{K}) := \mathrm{SL}_n(\mathbb{K}) \cap \mathrm{U}_n(\mathbb{K})$ are Lie groups.

3. The unit circle $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ in $\mathbb{C}$ is a Lie group with group product

$$(x,y).(x',y') := (xx' - yy', xy' + x'y), \quad (x,y)^{-1} = (x, -y).$$

Definition 37. Let G be a Lie group. A closed subgroup H that is also a submanifold is called Lie subgroup.

A.2.1 Representations and smooth actions

Definition 38. Let M be a smooth manifold and G a Lie group. A left (smooth) action of G on M is a smooth map

$$\begin{array}{rccc} \sigma: & G \times M & \longrightarrow & M, \\ & (g,m) & \longmapsto & \sigma(g,m) := g.m \end{array}$$

with the following properties:

(a) $1.m = m$ for all $m \in M$.

(b) $g_1.(g_2.m) = (g_1g_2).m \quad$ for $g_1, g_2 \in G,\ m \in M$.

For each smooth action, the map

$$\begin{array}{rcc} \hat{\sigma} : G & \longrightarrow & \mathrm{Diff}(M), \\ g & \longmapsto & \sigma_g := \sigma(g,.) \end{array}$$

is a group homomorphism. Conversely, any homomorphism $\gamma : G \longrightarrow \mathrm{Diff}(M)$ gives rise to a smooth action of G on M:

$$\begin{array}{rccc} \sigma: & G \times M & \longrightarrow & M, \\ & (g,m) & \longmapsto & \sigma(g,m) := \gamma(g)(m). \end{array}$$

. Similarly, we define a right action as a smooth map $\sigma_R : G \times M \longrightarrow M$, $(m,g) \longmapsto \sigma_R(m,g) := m.g$ with

$$m.1 = m, \quad (m.g_1).g_2 = m.(g_1g_2) \quad \text{for } g_1, g_2 \in G,\ m \in M.$$

Furthermore, the concepts of left and right actions are analogues. One can pass from one type of action to the other: if σ_L is a smooth left action, then

$$\sigma_R(m,g) := \sigma_L(g^{-1}, m)$$

defines a smooth right action of G on M. Conversely, if σ_R is a smooth right action, then

$$\sigma_L(g,m) := \sigma_R(m, g^{-1})$$

defines a smooth left action.

Definition 39. A representation of Lie group (π, V) is a smooth homomorphism of Lie groups $\pi : G \longrightarrow \mathrm{GL}(V)$ where V is a finite dimensional vector space.

It is clear that any representation defines a smooth action of G on V via

$$\sigma(g,v) := \pi(g)(v).$$

Thus, representations are the same as linear actions, i.e., actions on vector spaces for which the σ_g are linear.

A.3 Lie groups and their Lie algebras

The Lie functor assigns a Lie algebra to each Lie group and a Lie algebra homomorphism to each morphism of Lie groups. It is the key tool to translate Lie group problems into problems in linear algebra. The basic point of Lie functor construction is that the vector space of left invariant vector fields on a Lie group defined below inherits a structure of Lie algebra as a Lie subalgebra of $(\mathcal{V}(G), [\ ,\ ])$: we recall that for any $X,\ Y \in \mathcal{V}(G)$, and for each (φ, U) chart of G, there exists a vector field $[X, Y] \in \mathcal{V}(G)$ which is uniquely determined by

$$[X, Y]_\varphi(p) := [X_\varphi, Y_\varphi](p) := \mathrm{d}Y_\varphi(p)X_\varphi(p) - \mathrm{d}X_\varphi(p)Y_\varphi(p). \tag{A.7}$$

for $p \in \varphi(U)$ with $X_\varphi := T(\varphi) \circ X \circ \varphi^{-1}$ is the φ-related vector field to X on the open subset $\varphi(U) \subseteq \mathbb{R}^n$. The bracket A.7 turns $\mathcal{V}(G)$ into a Lie algebra. For the proof one can see [28]. Let us define the Lie algebra of a Lie group G.

Definition 40. A vector field $X \in \mathcal{V}(G)$ is called left invariant if

$$X = (\mathrm{L}_g)_* X := T(\mathrm{L}_g) \circ X \circ \mathrm{L}_g^{-1}$$

for any $g \in G$. We write $\mathcal{V}(G)^l$ for the set of left invariant vector fields in $\mathcal{V}(G)$.

From the definition, we show that left invariance of a vector field X is equivalent to L_g-relatedness of to itself for each $g \in G$. Due to **Related Vector Field Lemma** (see for instance [28]) one can show that the space $\mathcal{V}(G)^l$ is a Lie algebra (as a Lie subalgebra of $\mathcal{V}(G)$). Now, for any $X \in \mathcal{V}(G)^l$, we have $X(g) = T(\mathrm{L}_g) \circ X(1) := g.X(1), \quad \forall g \in G$, so that any vector field is completely determined by its values at the identity $X(1) \in T_1(G)$. Conversely, for each $x \in T_1(G)$, we associate a left vector field $x_l \in \mathcal{V}(G)^l$ given by $x_l(1) = x$ and $x_l(g) = T(\mathrm{L}_g)(x) = g.x$. Thus

$$T_1(G) \longrightarrow \mathcal{V}(G)^l, \quad x \longmapsto x_l$$

is a linear bijective map. So the tangent space $T_1(G)$ inherits a structure of Lie bracket with respect the following bracket

$$[x, y]_l := [x_l, y_l] \quad \text{for all} \quad x, y \in T_1(G).$$

We write

$$L(G) := (T_1(G), [\ ,\ ]) \cong \mathcal{V}(G)^l$$

for the Lie algebra of G. Furthermore, for a smooth homomorphism $\varphi : G_1 \longrightarrow G_2$ of Lie groups, the tangent map

$$L(\varphi) := T_1\varphi : L(G_1) \longrightarrow L(G_2)$$

is a homomorphism of Lie algebras. These results mean that the assignments $G \longmapsto L(G)$ and $\alpha \longmapsto L(\alpha)$ define a functor, called the **Lie functor**,

$$L : \mathrm{LieGrp} \longrightarrow \mathrm{LieAlg}$$

from the category LieGrp of Lie groups to the category LieAlg of Lie algebras. Now, we describe a fundamental example of Lie functor.

Example 13. Let G be a Lie group and $L(G)$ its Lie algebra. For any $g \in G$, we denote by $\mathrm{Ad}_g : L(G) \longrightarrow L(G)$ the differential of conjugation automorphism c_g at the identity element 1. We have then $\mathrm{Ad}_g \in \mathrm{Aut}(L(G))$. By Chain rule formula, we get

$$\mathrm{Ad}_{g_1g_2} := L(c_{g_1g_2}) = L(c_{g_1}) \circ L(c_{g_2}) = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}, \forall g_1, g_2 \in G.$$

Therefore, $\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(L(G))$ is group homomorphism which is so-called the **Adjoint representation**. It remains to show that Ad is smooth. Indeed, let $x \in L(G)$, we have

$$\mathrm{Ad}_g(x) = T_1(L_g \circ R_{g^{-1}})(x) = T_{g^{-1}}(L_g)T_1(R_{g^{-1}})(x)$$

in the Lie group $T(G)$ (the tangent bundle of G). Since the multiplication $T(m_G)$ in $T(G)$ is smooth, the representation Ad of G on $L(G)$ is smooth, so that

$$L(\mathrm{Ad}) : L(G) \longrightarrow \mathfrak{gl}(L(G))$$

is a representation of $L(G)$ on $L(G)$. Moreover,

$$L(\mathrm{Ad}) = \mathrm{ad}, \text{ i.e., } L(\mathrm{Ad})(x)(y) = [x,y] \quad \text{for all} \quad x, y \in L(G).$$

Next, let introduce an important tool of Lie theory which allows us to describe the inverse of the Lie functor: the exponential map $\exp_G : L(G) \longrightarrow G$ which generalizes the matrix exponential map obtained for $G = \mathrm{GL}_n(\mathbb{R})$ and its Lie algebra $L(G) = \mathfrak{gl}_n(\mathbb{R})$.

Definition 41. The exponential function is defined by

$$\exp_G : L(G) \longrightarrow G, \quad \exp_G(x) := \gamma_x(1),$$

where $\gamma_x : \mathbb{R} \longrightarrow G$ is the unique solution of the initial value problem

$$\gamma(0) = 1, \quad \gamma'(t) = x_l(\gamma(t)) = \gamma(t).x \quad \text{for all } t \in \mathbb{R}.$$

That is γ_x is the unique maximal integral curve of the left invariant field x_l. It is obviously to show that the exponential function is well defined since each left invariant vector field is complete (defined on the whole space $\mathbb{R}$) of Lie groups satisfies many properties but here we restrict briefly to give the immediate one:

Proposition A.3.1. *For a Lie group G, the exponential function*

$$\exp_G : L(G) \longrightarrow G$$

is a smooth map and satisfies

$$T_0(\exp_G) = \mathrm{id}_{L(G)}.$$

In particular, $\exp_G$ *is a local diffeomorphism at* 0 *in the sense that it maps some 0-neighborhood in* $L(G)$ *diffeomorphically onto some* 1*-neighborhood in G.*

Now, let us summarize some main results concerning the **correspondence between Lie groups and Lie algebras**:

- for each smooth homomorphism $\varphi : G_1 \longrightarrow G_2$ of Lie groups, the tangent map

  $$L(\varphi) := T_1\varphi : L(G_1) \longrightarrow L(G_2)$$

  is a homomorphism of Lie algebras for which the diagram

  $$\begin{array}{ccc} & L(\varphi) & \\ L(G_1) & \longrightarrow & L(G_2) \\ \exp_{G_1} \uparrow & & \uparrow \exp_{G_2} \\ G_1 & \longrightarrow & G_2 \\ & \varphi & \end{array}$$

  commutes, that is

  $$\exp_{G_2} \circ L(\varphi) = \varphi \circ \exp_{G_1}.$$

  For example, for each Lie group G, we have the following crucial formula:

  $$\mathrm{Ad} \circ \exp_G = e^{\mathrm{ad}}.$$

- **Theorem A.3.2 (Lie's Third Theorem).** *For a (finite dimensional) Lie algebra* $\mathfrak{g}$*, there is, up to isomorphism, a unique* 1*-connected Lie group with Lie algebra* $\mathfrak{g}$*.*

- Let $\mathfrak{h}$ be a Lie algebra and $\beta : \mathfrak{h} \longrightarrow L(G)$ be a homomorphism into the Lie algebra of a Lie group G. Then there is a unique morphism $\alpha : H \longrightarrow G$ such that the diagram

$$\begin{array}{ccc} & \beta = L(\alpha) & \\ \mathfrak{h} & \longrightarrow & L(G) \\ \exp_H \uparrow & & \uparrow \exp_G \\ H & \longrightarrow & G \\ & \alpha & \end{array}$$

  commutes, where H is the connected Lie group such that $\mathfrak{h} = L(H)$.

- The homomorphic images of connected Lie groups H in a Lie group G are the subgroups of the form $< \exp_G(\mathfrak{h}) >$ for a Lie subalgebra $\mathfrak{h} \subseteq L(G)$, i.e., the integral subgroups.

Appendix B

Racks and knots

The combinatorial diagrammatic viewpoint suggests a method for deriving an algebraic structure determined by Reidemeister equivalence of knot diagrams[1]: start by labeling sections of a knot diagram with generators of an algebraic structure and defining operations where the arcs meet at crossings. The Reidemeister moves then determine axioms for suggested algebraic structures namely keis, quandles and racks [24, 32]. In the way that the axioms of a rack correspond to *blackboard-framed* isotopy moves on link diagrams. Quandles axioms correspond to the three Reidemeister moves which combinatorially encode ambient isotopy of oriented link. Omit the orientation of a link, kei axioms describe the three Reidemeister moves for links.

Kei and Reidemeister moves

Let X be a kei and let D be a link digram. An arc coloring of D by the kei X is a map

$$\{\text{arcs of } D\} \longrightarrow X$$

which assigns to each arc a color in X such that at each crossing the arc colored by x passing under another arc colored y gets the color $x \triangleright y$. Now, let describe kei axioms and corresponding Reidemeister moves:

- Idempotency condition corresponds to the type I Reidemeister move.
- Own right-inverse corresponds to the type II Reidemeister move.
- Right self-distributivty corresponds to the type III Reidemeister move.

These facts are visualized in the following figure.

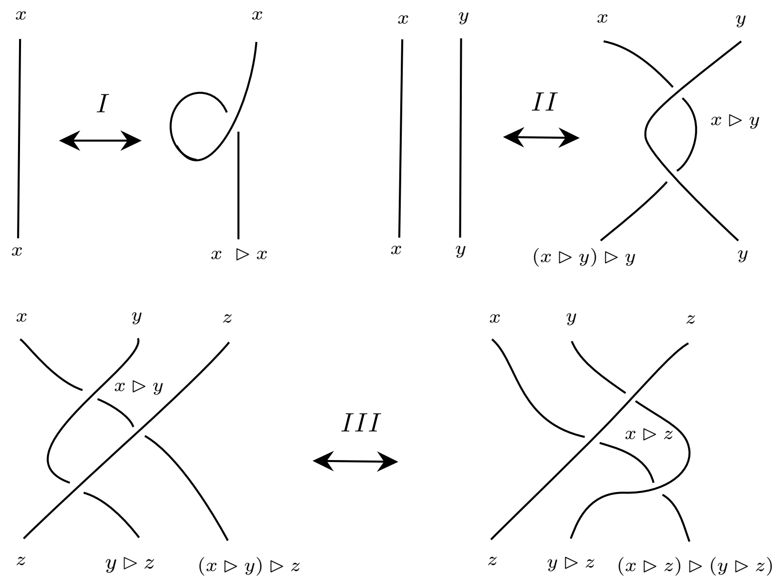

FIGURE B.1: Kei axioms and Reidemeister moves

Quandle and Reidemeister moves

We first fix a convention of signs of crossings which depends on whether the understrand is directed right-to-left or left-to-right when viewed from the overstrand.

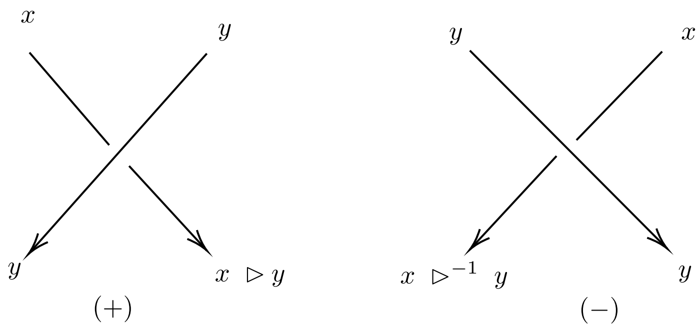

FIGURE B.2: Crossing sings and quandle operation

Analogously to keis, quandle axioms encode Reidemeister moves of oriented diagrams as depicted in Figure B.3.

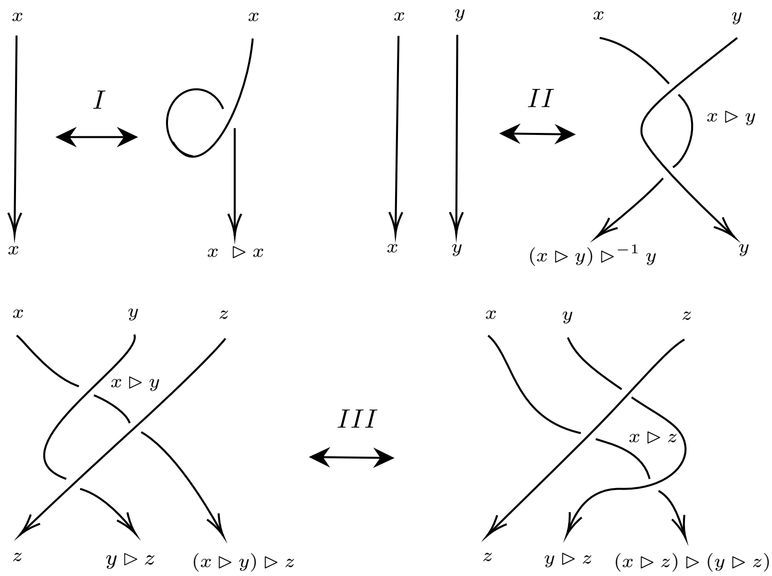

FIGURE B.3: Quandle axioms and Reidemeister moves

The **rack axioms** encode the Reidemeister moves of type II and III.

One of the main concerns of knot theory is to provide tools enabling one to decide if two given knot diagrams determine the same knot. In practice, guessing the Reidemeister moves relating two equivalent diagrams is not always an easy task.

Usually, to distinguish two knot diagrams we often use **knot invariants**. There are mathematical objects (numerical value, polynomial, group,...) assigned to knots (or knot diagrams). Although, knot invariants are not usually complete, they are so powerful for proving inequivalence of knots. Here, the word "complete" means that two knots are equivalents if and only if they have the same considered knot invariant. Many examples of elementary invariants as crossing number, linking number, knot group, Fox-coloring are introduced in [21, 22]. We have mentioned that quandles provide many knot invariants. The fundamental quandle or knot quandle was introduced by Joyce and Matveev. Joyce showed that it is a complete invariant of a knot (up to a weak equivalence) [32].

In 2007, Rubinsztein introduced the notion of topological quandles [40]. Using a particular action of the braid group B_n on the Cartesian product of n copies of a topological quandle $(Q, \rhd)$, he associated the space $J_Q(L)$ which the set of fixed points under the action of the braid $\sigma \in B_n$ for a braid σ whose closure is the oriented link L. The main result of the paper was that the space $J_Q(L)$ depends only on the isotopy class of the oriented link L. During our investigations, we have used some obtained topological quandles mentioned in Subsection 3.3.2, to determine the corresponding Rubinsztein spaces of some particular knots and links. For example, by using the quasi-trivial quandle $Q(G_{3,1})$, we have found by straightforward computation the following results concerning Hopf, Whitehead, and Borremean links:

- Let L be the Hopf link.

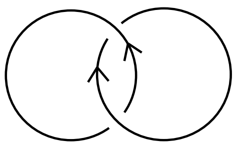

FIGURE B.4: Hopf link

It is the closure of $\sigma_1^{-2} \in B_2$.

$$J_{Q(G_{3,1})}(L) = \{((a,b,c),(x,y,z)) \in \Pi^2 Q(G_{3,1}) \mid bz = cy\}.$$

- Consider the Whitehead link L. It is the closure of the braid

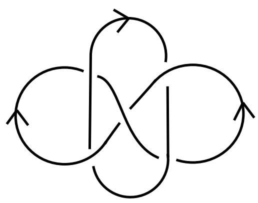

FIGURE B.5: Whitehead link

$$\sigma = \sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1} \in B_3$$

$$J_{Q(G_{3,1})}(L) = \{((a,b,c),(x,y,z),(x,y,z)) \mid ((a,b,c),(x,y,z)) \in \Pi^2 Q(G_{3,1})\}.$$

- Consider the Borromean rings $L6a4$. It is the closure of the braid

$$\sigma = (\sigma_1\sigma_2)^3 \in B_3.$$

Let $((a,b,c),(x,y,z),(t,u,w)) \in \Pi^3 Q(G_{3,1})$, then

$$\sigma((a,b,c),(x,y,z),(t,u,w)) = ((a,b,c),(x,y,z),(t,u,w))$$

that is equivalent to the following system

$$\begin{cases} [(a,b,c) \rhd (t,u,w)] \rhd (x,y,z) & = (a,b,c) \\ [(x,y,z) \rhd (t,u,w)] \rhd (a,b,c) & = (x,y,z) \\ [(t,u,w) \rhd (x,y,z)] \rhd (a,b,c) & = (t,u,w) \end{cases}$$

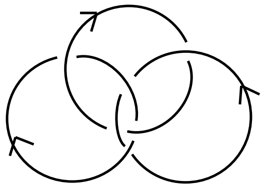

FIGURE B.6: Borromean rings

One can easily check that $((a,b,c),(a,b,c),(a,b,c))$ and $((a,b,c),(a,b,c),(x,b,c))$ with $a \neq x$ are two different solutions of the above system and therefore $J_{Q(G_{3,1})}(L6a4)$ contains at least two different elements.

From the above computations, we immediately deduce that the Rubinsztein space $J_{Q(G_{3,1})}$ distinguishes those three links. However, surprisingly, we have showed that the Rubinsztein space $J_{Q(G_{3,1})}$ does not distinguish many knots (one component).

Index

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